JUAS
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Introduction to MAGNETS II

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Fundamentals 1 : Maxwell

CURL / ROTOR

The speed of water ***S*** is rotational around an axis determinined by a driving force ***F***, its amplitude depends on the distance from the axis and on the driving force ***.***
Put in mathematics we get:

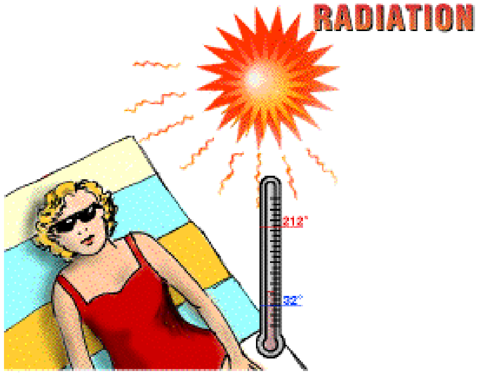
$$\nabla x\vec{S} = k\vec{F}$$

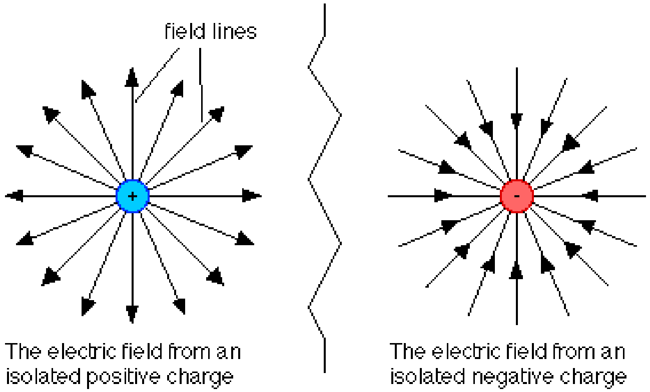
Remark: a whirlpool is turbulent, the analogy is for didactics purposes only

DIVERGENCE

The Divergence of a vector is the amount of flux of vector entering or leaving a point.

$$\nabla \cdot \vec{F} = \lim_{vol \to 0} \frac{Flux\vec{\mathbf{F}}}{vol}$$





$$\nabla \cdot \vec{Q} \neq 0$$

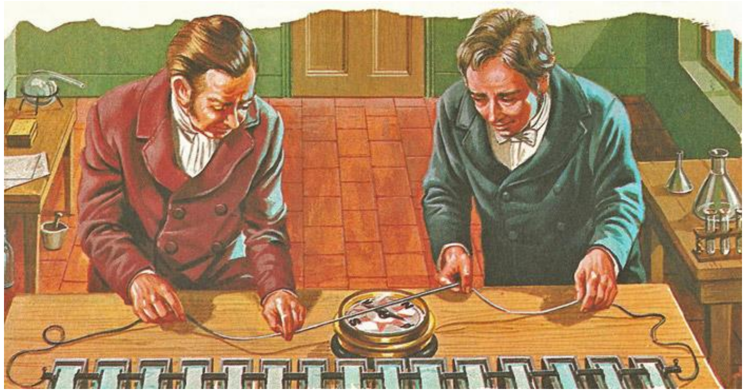
$$\nabla \cdot \vec{D} = \rho$$

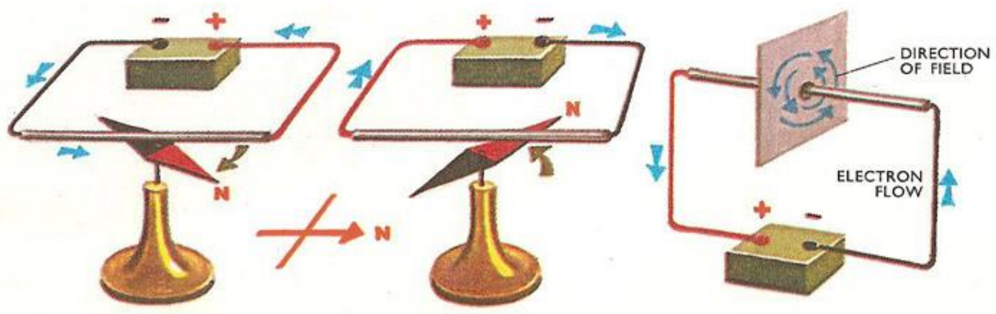
This is the 1st Maxwell equation, corresponding to the Gauss Law.

Basic Principles

A «magnetic field strength» $\vec{H}$ is produced by electrical currents

1820 Hans Christian Ørsted





An electrical current produces a circular magnetic field around the wire

This discovery pushed scientists to understand the mathematics behind this evidence

Generating a magnetic field strength

$$\nabla x \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

Ampère

Maxwell

1826 André-Marie Ampère

1861 James Clerk Maxwell

Let’s use this formula !

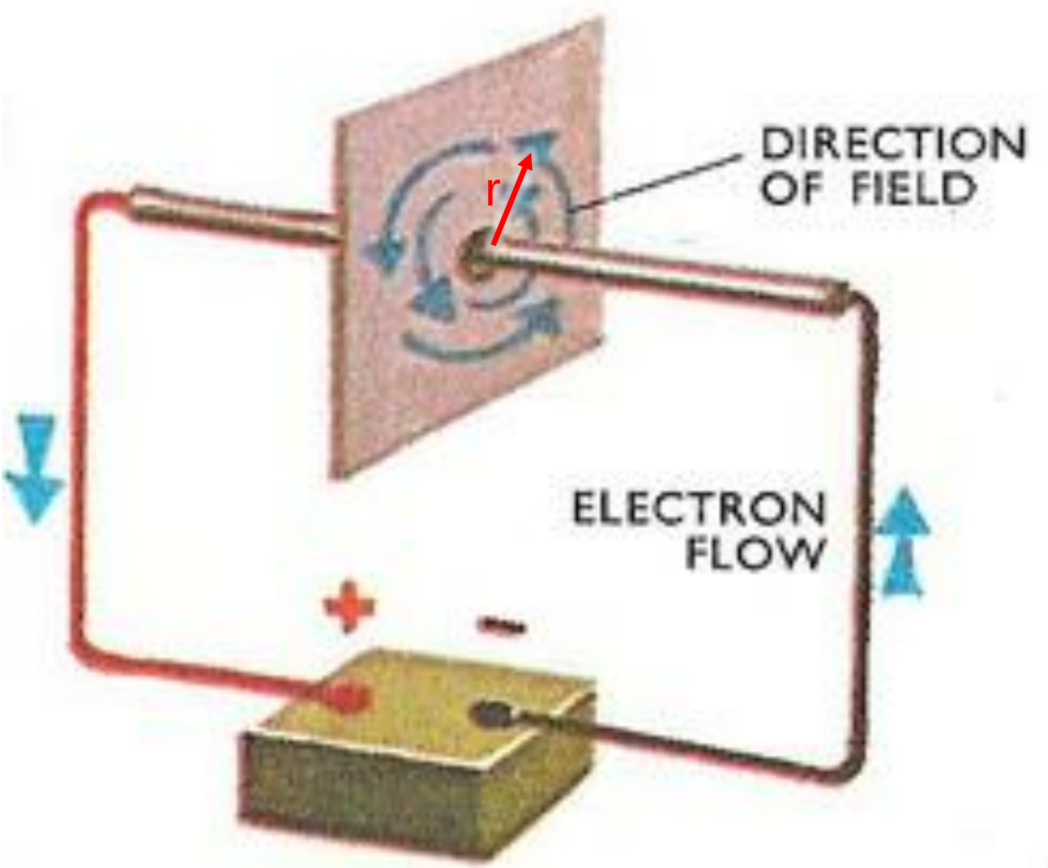
We consider $\frac{\partial \boldsymbol{D}}{\partial t} = 0$

$$\nabla x \overrightarrow{\boldsymbol{H}} = \overrightarrow{\boldsymbol{J}}$$

We recall that, thanks to the Kelvin – Stokes theorem the surface integral of the curl of a vector field over a surface $\boldsymbol{S}$ is equal to the line integral of the vector field along its boundary $\partial \boldsymbol{S}$:

$$\iint_S \nabla x \overrightarrow{\boldsymbol{H}} \cdot d\overrightarrow{\boldsymbol{S}} = \oint_{\partial S} \overrightarrow{\boldsymbol{H}} \cdot d\vec{\boldsymbol{l}} = \oint_{\partial S} \vec{\boldsymbol{J}} \cdot d\vec{\boldsymbol{l}} = \mathbf{I}$$

Let's use this formula ! *cont*



$$\oint_{\partial S} \vec{H} \cdot d\vec{l} = \mathrm{I}$$

We consider the boundary along the circumference at radius «r». Keeping the same radius, due to symmetry, *H* remains constant.

$$\mathrm{H} \cdot 2 \cdot \pi \cdot r = \mathrm{I}$$

$$\mathrm{H} = \frac{I}{2 \cdot \pi \cdot r}$$

The Magnetic Field Induction

What produces the «effect» is the «magnetic field induction» $\vec{B}$

$$\vec{F} = q\vec{v}x\vec{B}$$

The «magnetic field induction» is created by the «magnetic field strength»
In certain materials (ferromagnetic) we just need a small strength to produce a large induction, in most materials we need a large strength to produce a large induction. We define the following constitutive equation:

$$\vec{B} = \mu_0\mu_r\vec{H}$$

$$\mu_0 = 4\pi \cdot 10^{-7}\ H/m$$

The Fantastic Four

$$\nabla \cdot \overrightarrow{\boldsymbol{D}} = \rho$$

Gauss law for electricity

$$\nabla \cdot \overrightarrow{\boldsymbol{B}} = 0$$

Gauss law for magnetism

$$\nabla \boldsymbol{x} \overrightarrow{\boldsymbol{E}} = -\frac{\partial \overrightarrow{\boldsymbol{B}}}{\partial t}$$

Faraday-Lenz law of induction

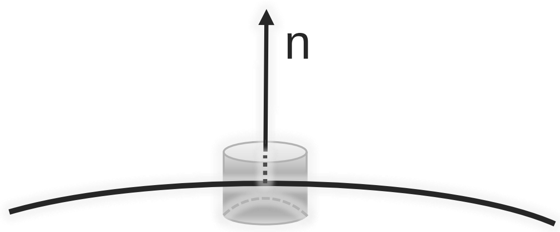
$$\nabla \boldsymbol{x} \overrightarrow{\boldsymbol{H}} = \vec{J} + \frac{\partial \overrightarrow{\boldsymbol{D}}}{\partial t}$$

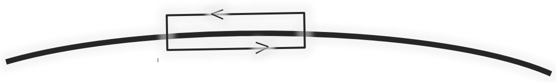
Ampère law with correction

Continuity conditions

$$\nabla \cdot \vec{\boldsymbol{B}} = 0$$

$$\nabla x \vec{\boldsymbol{H}} = 0$$





The flux which enters shall be equal to the flux which exits

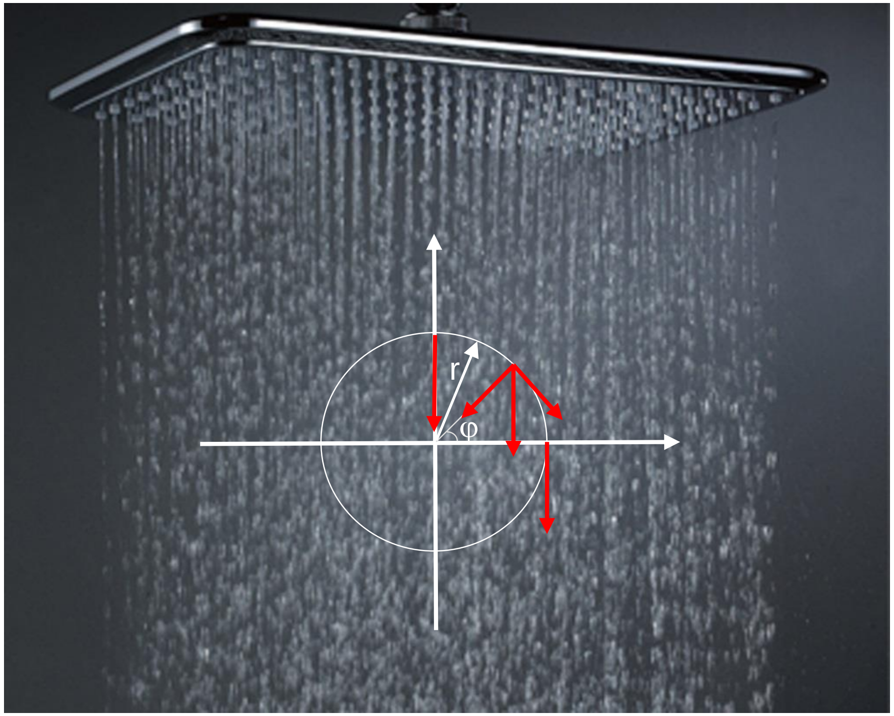
$$B_{\perp} = constant$$

The integral of the field strength

$$H_{\parallel} = constant$$

Fundamentals 2 : Field Harmonics

Decomposition of magnetic field



In cartesian coordinates

$$V_x = 0\ ;\ V_y = -k$$

$$\dot{V} = V_y + iV_x = -k + i0$$

In polar coordinates

$$V_\varphi = -kcos\varphi;\ V_r = -ksin\varphi$$

In case of a combination of uniform vertical **k** and uniform horizontal field **h** we have:

$$\dot{V} = V_y + iV_x = (k + i0) + (0 + ih) = k + ih$$

$$V_\varphi = -kcos\varphi + h\sin\varphi\ ;\ V_r = -ksin\varphi + hcos\varphi$$

The coefficients caracterizing the vertical field (producing horizontal beam deflection) are called «*normal*», the ones caracterizing the horizontal field are called «*skew*».

To go ahead we need … The Potential

Since the divergence of a curl is zero, we can define a vector potential $\vec{A}$ such that:

$$\nabla \cdot \overrightarrow{\boldsymbol{B}} = \nabla \cdot (\nabla \boldsymbol{x} \overrightarrow{\boldsymbol{A}}) = \mathbf{0}$$

With then:

$$\overrightarrow{\boldsymbol{B}} = \nabla \boldsymbol{x} \overrightarrow{\boldsymbol{A}}$$

In air, as $\vec{\boldsymbol{B}} = \mu_0 \vec{\boldsymbol{H}}$:

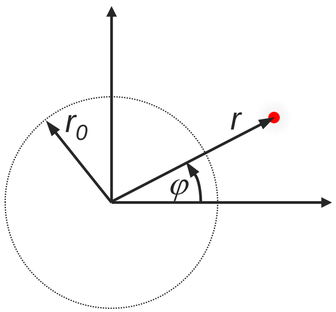
$$\mu_0 \vec{\boldsymbol{J}} = \nabla \boldsymbol{x} \overrightarrow{\boldsymbol{B}} = \nabla \boldsymbol{x}\left(\nabla \boldsymbol{x} \overrightarrow{\boldsymbol{A}}\right) = -\nabla^2 \overrightarrow{\boldsymbol{A}}$$

In air, in a volume with no currents:

$$\nabla^2 \overrightarrow{\boldsymbol{A}} = 0$$

Why we need the (vector) potential

As $\vec{\boldsymbol{B}} = \nabla x \vec{\boldsymbol{A}}$, in a 2D case (plane geometry) the only component of $\vec{\boldsymbol{A}}$ is A_z



$\nabla^2 A_z = 0$ in polar coordinates becomes:

$$r^2 \frac{\partial^2 A_z}{\partial r^2} + r \frac{\partial A_z}{\partial r} + \frac{\partial^2 A_z}{\partial \varphi^2} = 0$$

The solution of this equation is:

$$A_z(r, \varphi) = \sum_{n=1}^{\infty} r^n (C_n \sin n\varphi + D_n \cos n\varphi)$$

… and the field components are :

$$B_r(r, \varphi) = \frac{1}{r} \frac{\partial A_z}{\partial \varphi} = \sum_{n=1}^{\infty} n r^{n-1} (C_n \cos n\varphi - D_n \sin n\varphi)$$

$$B_\varphi(r, \varphi) = -\frac{\partial A_z}{\partial r} = -\sum_{n=1}^{\infty} n r^{n-1} (C_n \sin n\varphi + D_n \cos n\varphi)$$

For a further insight I recommend checking the Feynman Lectures on Physics, now «free to read online» at http://www.feynmanlectures.caltech.edu

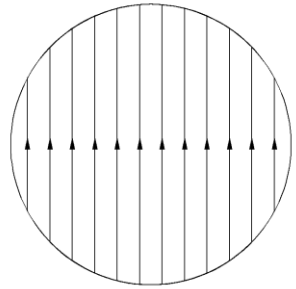
Field Harmonics

$n = 1$: *Dipole*

$$B_r(r,\varphi) = C_1\cos\varphi - D_1\sin\varphi$$
$$= A_1\cos\varphi - B_1\sin\varphi$$

$$B_\varphi(r,\varphi) = -(C_1\sin n\varphi + D_1\cos n\varphi)$$
$$= -(A_1\sin n\varphi + B_1\cos n\varphi)$$

«normal» dipole component



$$B_r(r,\varphi) = -B_1\sin\varphi$$

$$B_\varphi(r,\varphi) = -B_1\cos n\varphi$$

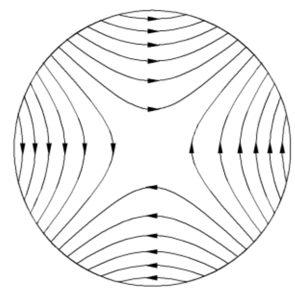
$$B_{mod}(r,\varphi) = B_1$$

$n = 2$: Quadrupole

$$B_r(r,\varphi) = 2r^1(C_2\cos 2\varphi - D_2\sin 2\varphi)$$
$$= \left(\frac{r}{r_0}\right)^1 (A_2\cos 2\varphi - B_2\sin 2\varphi)$$

$$B_\varphi(r,\varphi) = -2r^1(C_2\sin 2\varphi + D_2\cos 2\varphi)$$
$$= -\left(\frac{r}{r_0}\right)^1 (A_2\sin 2\varphi + B_2\, cos\, 2\varphi)$$

«normal» quadrupole component normalized at r_0



We define $B_2@r_0 = r_0 2D_2$

$$B_r(r,\varphi) = -\frac{r}{r_0}B_2\sin 2\varphi$$

$$B_\varphi(r,\varphi) = -\frac{r}{r_0}B_2\cos 2\varphi$$

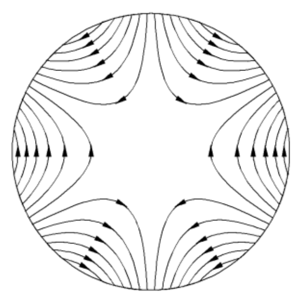
*We also define** $G = \frac{|B|}{r} = \frac{|B_2|}{r_0}$

$n = 3$: *Sextupole*

$$B_r(r,\varphi) = 3r^2(C_3\cos 3\varphi - D_3\sin 3\varphi)$$
$$= \left(\frac{r}{r_0}\right)^2 (A_3\cos 3\varphi - B_3\sin 3\varphi)$$

$$B_\varphi(r,\varphi) = -3r^2(C_3\sin 3\varphi + D_3\cos 3\varphi)$$
$$= -\left(\frac{r}{r_0}\right)^2 (A_3\sin 3\varphi + B_3\cos 3\varphi)$$

«normal» sextupole component normalized at r_0



We define $B_3@r_0 = r_0^2 3D_3$

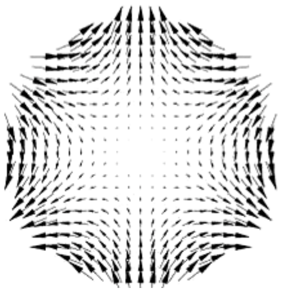
$$B_r(r,\varphi) = -\frac{r^2}{r_0^2}B_3\sin 2\varphi$$

$$B_\varphi(r,\varphi) = -\frac{r^2}{r_0^2}B_3\cos 2\varphi$$

*We also define** $S = \frac{|B|}{r^2} = \frac{|B_3|}{r_0^2}$

* for practical reasons I prefer a definition with no sign to avoid misunderstanding

Warning on the Sextupole Component



$We\ define\ B_3@r_0 = r_0^2 3D_3$

$$B_r(r,\varphi) = -\frac{r^2}{r_0^2} B_3 \sin 2\varphi$$

$$B_\varphi(r,\varphi) = -\frac{r^2}{r_0^2} B_3 \cos 2\varphi$$

$We\ also\ define\ S = \frac{|B|}{r^2} = \frac{|B_3|}{r_0^2}$

When you reconstruct the field amplitude as a function of the radius you obtain:

For a quadrupole, using the gradient «G»: $|B| = Gr$

For a sextupole, using «S» : $|B| = Sr^2$

However, if you express the magnetic field by a polynomial expansion:

For a quadrupole : $|B| = \frac{\partial B}{\partial r} r = Gr$, so there is no uncertitude of what is G

For a sextupole : $|B| = \frac{\partial^2 B}{\partial r^2} r^2 = 2Sr^2 \neq Sr^2$

Above the quadrupole, always cross check the definition: best is to specify the the field amplitude at a refence radius (which means you specify B_{3r_0})

Relative Field Harmonics

When you have a real magnet of a specific type, you wish it produces that given type of harmonic only, but you will get also other harmonics, more or less large.
We define «relative field harmonic» the ratio between that field harmonic and the reference field harmonic expressed in units of 10^{-4} of the main harmonic, at a reference radius r_0.

$$b_i = \frac{B_i}{B_{ref}} 10^4$$

Exercice 1: on a «normal» quadrupole with gradient $G = 50\ T/m$, we measure at a radius of $r_0 = 10\ mm$ a «skew» dipole field of $10\ Gauss$ and a «normal» sextupole field of $25\ Gauss$
Compute the relevant field harmonics in units of 10^{-4}

Solution :

$$B_2 = Gr_0 = 0.5\ T = 5000\ Gauss$$

$$a_1 = \frac{A_1}{B_2} 10^4 = 20\ units$$

$$b_3 = \frac{B_3}{B_2} 10^4 = 50\ units$$

Scaling of Relative Field Harmonics

$$B_r(r,\varphi) = \frac{1}{r}\frac{\partial A_z}{\partial \varphi} = \sum_{n=1}^{\infty} n r^{n-1}(C_n \cos n\varphi - D_n \sin n\varphi) \qquad B_\varphi(r,\varphi) = -\frac{\partial A_z}{\partial r} = -\sum_{n=1}^{\infty} n r^{n-1}(C_n \sin n\varphi + D_n \cos n\varphi)$$

Let's consider the dependency vs radius of a given field harmonic amplitude, does not matter normal or skew

$$H_n = k r^{n-1}$$

The field harmonic relative to a «reference order m» scales as:

$$h_n(r) = h_n(r_0)\frac{\left(\frac{r}{r_0}\right)^{n-1}}{\left(\frac{r}{r_0}\right)^{m-1}} = h_n(r_0)\left(\frac{r}{r_0}\right)^{n-m}$$

Exercice 2: scale the field harmonics of Exercice 1 to a radius of $r = 20\,mm$

Solution :

$$a_1(20\,mm) = 20\,units\;x\;(\frac{20}{10})^{1-2} = 10\,units$$

$$b_3(20\,mm) = 50\,units\;x\;(\frac{20}{10})^{3-2} = 100\,units$$

Thanks