Non-linear effects

Hannes BARTOSIK and Yannis PAPAPHILIPPOU
Accelerator and Beam Physics group
Beams Department
CERN

Joint University Accelerator School

Archamps, FRANCE

5 February 2017

Bibliography

- Books on non-linear dynamical systems
  - M. Tabor, Chaos and Integrability in Nonlinear Dynamics, An Introduction, Willey, 1989.
  - A.J Lichtenberg and M.A. Lieberman, Regular and Chaotic Dynamics, 2nd edition, Springer 1992.
- Books on beam dynamics
  - E. Forest, Beam Dynamics - A New Attitude and Framework, Harwood Academic Publishers, 1998.
  - H. Wiedemann, Particle accelerator physics, 3rd edition, Springer 2007.
- Lectures on non-linear beam dynamics
  - A. Chao, Advanced topics in Accelerator Physics, USPAS, 2000.
  - A. Wolski, Lectures on Non-linear dynamics in accelerators, Cockroft Institute 2008.
  - W. Herr, Lectures on Mathematical and Numerical Methods for Non-linear Beam Dynamics in Rings, CAS 2013.
  - L. Nadolski, Lectures on Non-linear beam dynamics, Master NPAC, LAL, Orsay 2013.

Contents of the 1st lecture

- Accelerator performance parameters and non-linear effects
- Linear and non-linear oscillators
  - Integral and frequency of motion
  - The pendulum
- Phase space dynamics
  - Fixed point analysis
- Non-autonomous systems
  - Driven (damped) harmonic oscillator, resonance conditions
- Linear equations with periodic coefficients – Hill's equations
  - Floquet solutions and normalized coordinates
- Perturbation theory
  - Non-linear oscillator
  - Perturbation by periodic function – single dipole perturbation
  - Application to single multipole – resonance conditions
  - Examples: single quadrupole, sextupole, octupole perturbation
  - General multi-pole perturbation– example: linear coupling
  - Resonance conditions and working point choice
- Summary
  - Appendix I: Damped harmonic oscillator

Contents of the 1st lecture

- Accelerator performance parameters and non-linear effects
- Linear and non-linear oscillators
  - Integral and frequency of motion
  - The pendulum
- Phase space dynamics
  - Fixed point analysis
- Non-autonomous systems
  - Driven (damped) harmonic oscillator, resonance conditions
- Linear equations with periodic coefficients – Hill's equations
  - Floquet solutions and normalized coordinates
- Perturbation theory
  - Non-linear oscillator
  - Perturbation by periodic function – single dipole perturbation
  - Application to single multipole – resonance conditions
  - Examples: single quadrupole, sextupole, octupole perturbation
  - General multi-pole perturbation– example: linear coupling
  - Resonance conditions and working point choice
- Summary
- Appendix I: Damped harmonic oscillator

Accelerator performance parameters

$$L = \frac{N_b^2 k_b f_{\text{rev}} \gamma}{4\pi\epsilon\beta^*}\mathcal{R}(\phi)$$

- **Colliders**
  - **Luminosity** (i.e. rate of particle production)
    - N_b bunch population
    - k_b number of bunches
    - f_{rev} the revolution frequency
    - γ relativistic reduced energy
    - ε_n normalized emittance
    - β^* "betatron" amplitude function at collision point
    - $R(\varphi)$ geometric reduction factor due to crossing angle

$$\bar{P} = \bar{I}E = f_N N e E$$

- **High intensity** rings
  - **Average beam power**
    - $\bar{I}$ mean current intensity
    - E energy
    - f_N repetition rate
    - N number of particles/pulse

$$B = \frac{N_p}{4\pi^2\epsilon_x\epsilon_y}$$

- **X-ray (low emittance) rings**
  - **Brightness** (photon density in phase space)
    - N_p number of photons
    - $\varepsilon_{x,,y}$ transverse emittances

- **Non-linear effects** limit **performance** of particle accelerators but impact also design **cost**

Non-linear effects in colliders

$$L = \frac{N_b^2 k_b f_{\mathrm{rev}} \gamma}{4\pi\epsilon\beta^*}\mathcal{R}(\phi)$$

- At **injection**
  - Non-linear magnets (sextupoles, octupoles)
  - Magnet imperfections and misalignments
  - Power supply ripple
  - Ground motion (for e+/e-)
  - Electron (Ion) cloud
- At **collision**
  - Non-linear magnets (sextupoles and octupoles
  - Field imperfections in the insertion quadrupoles
  - Magnets in experimental areas (solenoids, dipoles)
  - “Incoherent” Beam-beam effects (head on and long range)
  - “Incoherent” E-cloud effect

Non-linear effects in colliders

$$L = \frac{N_b^2 k_b f_{\mathrm{rev}} \gamma}{4\pi\epsilon\beta^*}\mathcal{R}(\phi)$$

- At **injection**
  - Non-linear magnets (sextupoles, octupoles)
  - Magnet imperfections and misalignments
  - Power supply ripple
  - Ground motion (for e+/e-)
  - Electron (Ion) cloud
- At **collision**
  - Non-linear magnets (sextupoles and octupoles
  - Field imperfections in the insertion quadrupoles
  - Magnets in experimental areas (solenoids, dipoles)
  - "Incoherent" Beam-beam effects (head on and long range)
  - "Incoherent" E-cloud effect

- Limitations affecting (integrated) **luminosity**
  - **Particle losses** causing
    - Reduced lifetime
    - Radio-activation (super-conducting magnet quench)
    - Reduced machine availability
  - Emittance **blow-up**
  - Reduced **number** of **bunches** (either due to electron cloud or long-range beam-beam)
  - Increased **crossing angle**
  - Reduced **intensity**
- **Cost** issues
  - Number of magnet **correctors** and **families** (power convertors)
  - Magnetic **field** and **alignment tolerances**
  - Design of the **collimation** system

$$\bar{P} = \bar{I}E = f_N N e E$$

- Non-linear magnets (sextupoles, octupoles)
- Magnet imperfections and misalignments
- Injection chicane
- Magnet fringe fields
- Space-charge effect

$$\bar{P} = \bar{I}E = f_N N e E$$

- Non-linear magnets (sextupoles, octupoles)
- Magnet imperfections and misalignments
- Injection chicane
- Magnet fringe fields
- Space-charge effect

- Limitations affecting beam **power**
  - Particle **losses** causing
    - Reduced intensity
    - Radio-activation (hands-on maintenance)
    - Reduced machine availability
  - Emittance **blow-up** which can lead to particle loss
- **Cost** issues
  - Number of magnet correctors and families (power convertors)
  - Magnetic field and alignment tolerances
  - Design of the **collimation** system

$$B = \frac{N_p}{4\pi^2 \epsilon_x \epsilon_y}$$

- Chromaticity sextupoles
- Magnet imperfections and misalignments
- Insertion devices (wigglers, undulators)
- Injection elements
- Ground motion
- Magnet fringe fields
- Space-charge effect (in the vertical plane for damping rings)
- Electron cloud (Ion) effects

$$B = \frac{N_p}{4\pi^2 \epsilon_x \epsilon_y}$$

- Chromaticity sextupoles
- Magnet imperfections and misalignments
- Insertion devices (wigglers, undulators)
- Injection elements
- Ground motion
- Magnet fringe fields
- Space-charge effect (in the vertical plane for damping rings)
- Electron cloud (Ion) effects

- Limitations affecting beam **brightness**
  - Reduced injection efficiency
  - Particle losses causing
    - Reduced lifetime
    - Reduced machine availability
  - Emittance blow-up which can lead to particle loss
- **Cost** issues
  - Number of magnet correctors and families (power convertors)
  - Magnetic field and alignment tolerances

Contents of the 1st lecture

- Accelerator performance parameters and non-linear effects
- Linear and non-linear oscillators
  - Integral and frequency of motion
  - The pendulum
  - Damped harmonic oscillator
- Phase space dynamics
  - Fixed point analysis
- Non-autonomous systems
  - Driven harmonic oscillator, resonance conditions
- Linear equations with periodic coefficients – Hill's equations
  - Floquet solutions and normalized coordinates
- Perturbation theory
  - Non-linear oscillator
  - Perturbation by periodic function – single dipole perturbation
  - Application to single multipole – resonance conditions
  - Examples: single quadrupole, sextupole, octupole perturbation
  - General multi-pole perturbation– example: linear coupling
  - Resonance conditions and working point choice
- Summary
- Appendix I: Multipole expansion

Reminder: Harmonic oscillator

- Described by the differential equation:

$$\frac{d^2u(t)}{dt^2} + \omega_0^2 u(t) = 0$$

- The **solution** obtained by the substitution $u(t) = e^{\lambda t}$

  and the solutions of the characteristic polynomial are

  $\lambda_\pm = \pm i\omega_0$ which yields the general solution

$$u(t) = ce^{i\omega_0 t} + c^* e^{-i\omega_0 t} = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t) = A \sin(\omega_0 t + \phi)$$

Reminder: Harmonic oscillator

- Described by the differential equation:

$$\frac{d^2u(t)}{dt^2} + \omega_0^2 u(t) = 0$$

- The **solution** obtained by the substitution $u(t) = e^{\lambda t}$

  and the solutions of the characteristic polynomial are

  $\lambda_\pm = \pm i\omega_0$ which yields the general solution

$$u(t) = ce^{i\omega_0 t} + c^* e^{-i\omega_0 t} = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t) = A \sin(\omega_0 t + \phi)$$

- The **amplitude** and **phase** depend on the initial conditions

$$A = \frac{\left(\frac{du}{dt}(0)^2 + \omega_0^2 u(0)^2\right)^{1/2}}{\omega_0}, \quad \tan(\phi) = \frac{\frac{du}{dt}(0)}{\omega_0 u(0)}$$

- A **negative** sign in the differential equation provides a solution described by an **hyperbolic sine** function
- Note also that for **no restoring force** $\omega_0 = 0$, the motion is unbounded

Integral of motion

- Rewrite the differential equation of the harmonic oscillator as a **pair of coupled 1st order equations**

$$\frac{du(t)}{dt} = p_u(t)$$

$$\frac{dp_u(t)}{dt} = -\omega_0^2 u(t)$$

which can be combined to provide

$$\frac{dp_u}{dt}p_u + \omega_0^2 u \frac{du}{dt} = \frac{1}{2}\frac{d}{dt}\left(p_u^2 + \omega_0^2 u^2\right) = 0$$

or

$$\frac{1}{2}\left(p_u^2 + \omega_0^2 u^2\right) = I_1$$

with I_1 an **integral of motion** identified as the mechanical energy of the system

Integral of motion

- Rewrite the differential equation of the harmonic oscillator as a **pair of coupled 1st order equations**

$$\frac{du(t)}{dt} = p_u(t)$$
$$\frac{dp_u(t)}{dt} = -\omega_0^2 u(t)$$

which can be combined to provide

$$\frac{dp_u}{dt}p_u + \omega_0^2 u\frac{du}{dt} = \frac{1}{2}\frac{d}{dt}\left(p_u^2 + \omega_0^2 u^2\right) = 0$$

or

$$\frac{1}{2}\left(p_u^2 + \omega_0^2 u^2\right) = I_1$$

with I_1 an **integral of motion** identified as the mechanical energy of the system

- Solving the previous equation for p_u, the system can be reduced to a unique **1st order equation**

$$\frac{du}{dt} = \sqrt{2I_1 - \omega_0^2 u^2}$$

- The last equation can be be solved as an explicit integral or **"quadrature"**

$$\int dt = \int \frac{du}{\sqrt{2I_1 - \omega_0^2 u^2}}$$, yielding $$t + I_2 = \frac{1}{\omega_0} \arcsin\left(\frac{u\omega_0}{\sqrt{2I_1}}\right)$$

or the well-known solution $$u(t) = \frac{\sqrt{2I_1}}{\omega_0} \sin(\omega_0 t + \omega_0 I_2)$$

Integration by quadrature

- The last equation can be be solved as an explicit integral or **“quadrature”**

$$\int dt = \int \frac{du}{\sqrt{2I_1 - \omega_0^2 u^2}}$$, yielding $$t + I_2 = \frac{1}{\omega_0} \arcsin\left(\frac{u\omega_0}{\sqrt{2I_1}}\right)$$

or the well-known solution $$u(t) = \frac{\sqrt{2I_1}}{\omega_0} \sin(\omega_0 t + \omega_0 I_2)$$

- Although the previous route may seem complicated, it becomes more **natural** when **non-linear terms** appear, where a substitution of the type $u(t) = e^{\lambda t}$ is not applicable
- The ability to **integrate** a differential equation is not just a nice mathematical feature, but deeply characterizes the **dynamical behavior** of the system described by the equation

Frequency of motion

- The **period** of the harmonic oscillator is calculated through the previous integral after integration between two **extrema**, i.e. when the velocity

$$\frac{du}{dt} = \sqrt{2I_1 - \omega_0^2 u^2} \text{ vanishes, at } u_{\rm ext} = \pm \frac{\sqrt{2I_1}}{\omega_0}$$

Frequency of motion

- The **period** of the harmonic oscillator is calculated through the previous integral after integration between two **extrema**, i.e. when the velocity

$$\frac{du}{dt} = \sqrt{2I_1 - \omega_0^2 u^2} \text{ vanishes, at } u_{\rm ext} = \pm\frac{\sqrt{2I_1}}{\omega_0}$$

- The integration yields

$$T = 2\int_{-\frac{\sqrt{2I_1}}{\omega_0}}^{\frac{\sqrt{2I_1}}{\omega_0}} \frac{du}{\sqrt{2I_1 - \omega_0^2 u^2}} = \frac{2\pi}{\omega_0}$$

- The **frequency** (or the period) of linear systems is **independent** of the integral of motion (energy)

Frequency of motion

- The previous remark is **not true for non-linear systems**, e.g. for an oscillator with a **non-linear restoring force** $\frac{d^2u}{dt^2} + k\ u(t)^3 = 0$
- The **integral** of motion is $I_1 = \frac{1}{2}p_u^2 + \frac{1}{4}k\ u^4$

- The previous remark is **not true for non-linear systems**, e.g. for an oscillator with a **non-linear restoring force** $\frac{d^2u}{dt^2} + k\, u(t)^3 = 0$
- The **integral** of motion is $I_1 = \frac{1}{2}p_u^2 + \frac{1}{4}k\, u^4$
- Solving for vanishing velocity, we get $u_{\rm ext} = \pm\left(\frac{2I_1}{k}\right)^{1/4}$
- The integration yields

$$T = 2\int_{-(4I_1/k)^{1/4}}^{(4I_1/k)^{1/4}} \frac{du}{\sqrt{2I_1 - \frac{1}{2}k\, u^4}} = \sqrt{\frac{1}{2\pi}}\Gamma^2(\frac{1}{4})\,(I_1\, k)^{-1/4}$$

i.e. the **period** (frequency) depends on the integral of motion (energy), i.e. the maximum “amplitude”

The pendulum

- An important non-linear equation which can be integrated is the one of the **pendulum**, for a string of length L and gravitational constant g

$$\frac{d^2\phi}{dt^2} + \frac{g}{L}\sin\phi = 0$$

- For small displacements it reduces to an harmonic oscillator with frequency $\omega_0 = \sqrt{\frac{g}{L}}$
- The integral of motion (**scaled energy**) is

$$\frac{1}{2}\left(\frac{d\phi}{dt}\right)^2 - \frac{g}{L}\cos\phi = I_1 = E'$$

and the quadrature is written as $t = \int \frac{d\phi}{\sqrt{2(I_1 + \frac{g}{L}\cos\phi)}}$
assuming that for $t = 0\,,\quad \phi = 0$

Solution for the pendulum

- The integral $t = \int \frac{d\phi}{\sqrt{2(I_1 + \frac{g}{L}\cos\phi)}}$ can be solved, using the substitution $\cos\phi = 1 - 2k^2\sin^2\theta$ with $k = \sqrt{1/2(1 + I_1 L/g)}$.
- The integral then becomes

$$t = \sqrt{\frac{L}{g}} \int_0^\theta \frac{d\theta}{\sqrt{1 - k^2\sin^2\theta}}$$

- It is solved using **Jacobi elliptic functions**, with the final result:

$$\phi(t) = 2\arcsin\left[k\,\mathrm{sn}\left(t\sqrt{\frac{g}{L}}, k\right)\right]$$

Period of the pendulum

- For recovering the **period**, the integration is performed between the two extrema, i.e. $\phi = 0$ and $\phi = \arccos(-I_1 L/g)$, corresponding to $\theta = 0$ and $\theta = \pi/2$

- The **period** is

$$T = 4\sqrt{\frac{L}{g}}\int_0^{\pi/2}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = 4\sqrt{\frac{L}{g}}\mathcal{K}(k)$$

  i.e. the **complete elliptic integral** multiplied by four times the period of the harmonic oscillator

- By expanding

$$\mathcal{K}(k) = \frac{\pi}{2}\sum_{n=0}^{\infty}\left(\frac{(2n)!}{2^{2n}(n!)^2}\right)^2 k^{2n} = \frac{\pi}{2}\left(1+\frac{1}{4}k^2+\cdots\right)$$

  with $k = \sqrt{1/2(1+I_1 L/g)}$, the **“amplitude”**

Contents of the 1st lecture

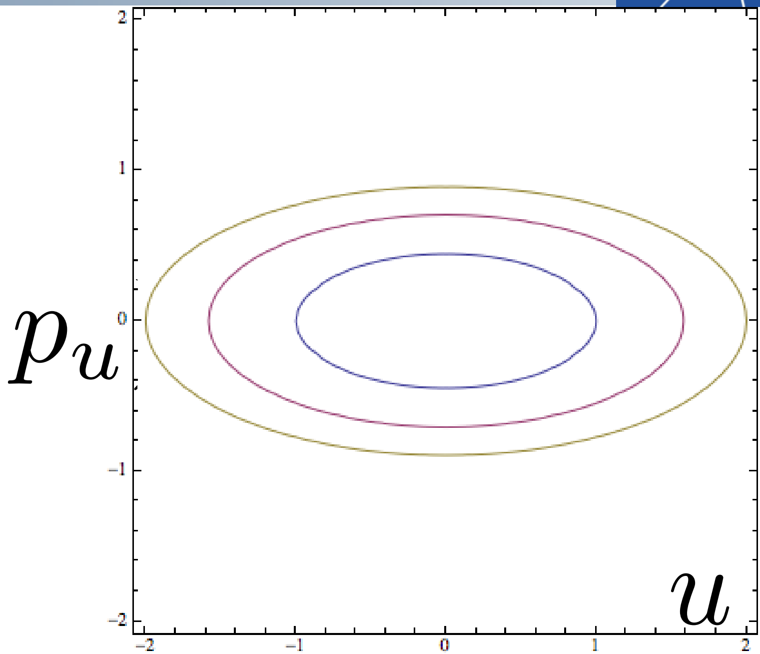
- Accelerator performance parameters and non-linear effects
- Linear and non-linear oscillators
  - Integral and frequency of motion
  - The pendulum
- Phase space dynamics
  - Fixed point analysis
- Non-autonomous systems
  - Driven harmonic oscillator, resonance conditions
- Linear equations with periodic coefficients – Hill's equations
  - Floquet solutions and normalized coordinates
- Perturbation theory
  - Non-linear oscillator
  - Perturbation by periodic function – single dipole perturbation
  - Application to single multipole – resonance conditions
  - Examples: single quadrupole, sextupole, octupole perturbation
  - General multi-pole perturbation– example: linear coupling
  - Resonance conditions and working point choice
- Summary
- Appendix I: Damped harmonic oscillator

Phase space dynamics

- Valuable description when examining trajectories in **phase space** (u, p_u)
- Existence of integral of motion imposes geometrical constraints on phase flow
- For the simple **harmonic oscillator**

$$H = \frac{1}{2}\left(p_u^2 + \omega_0^2 u^2\right)$$

phase space curves are **ellipses** around the equilibrium point parameterized by the integral of motion Hamiltonian (energy)



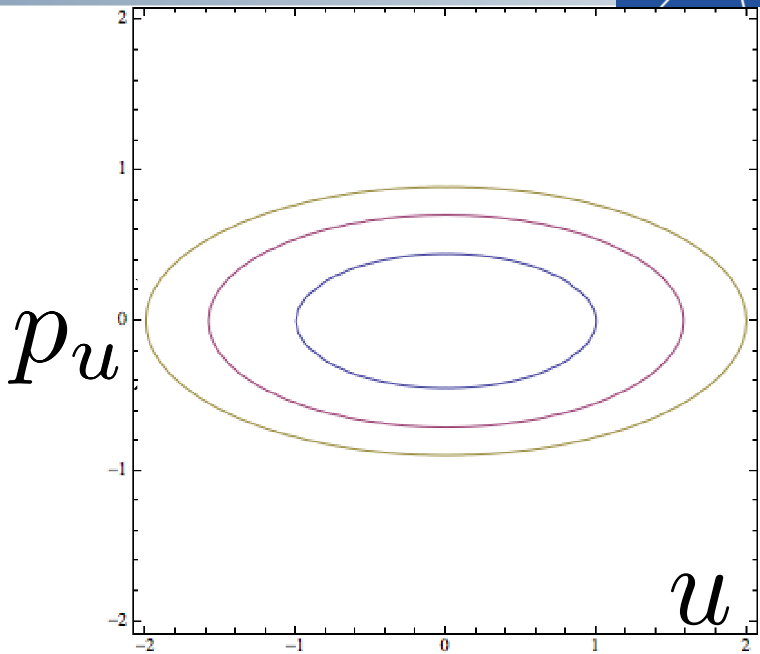
Phase space dynamics

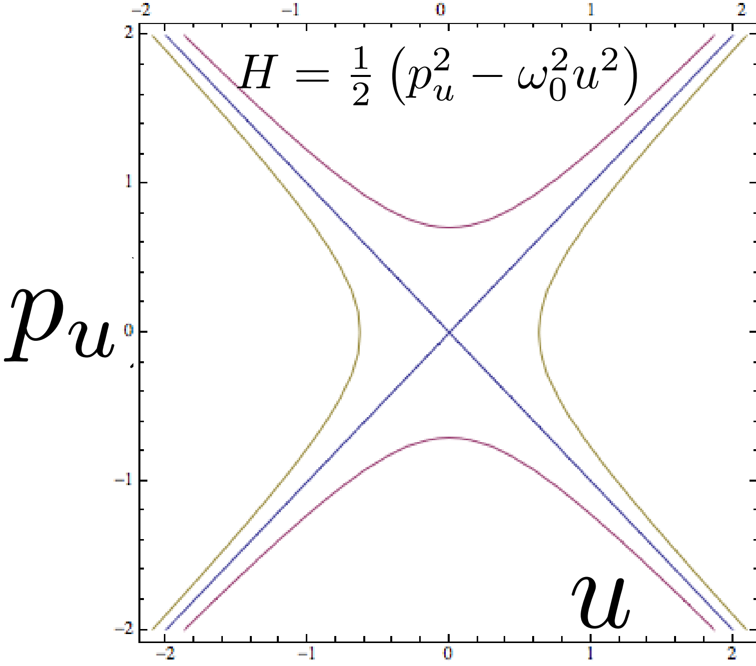
- Valuable description when examining trajectories in **phase space** (u, p_u)
- Existence of integral of motion imposes geometrical constraints on phase flow
- For the simple **harmonic oscillator**

$$H = \frac{1}{2}\left(p_u^2 + \omega_0^2 u^2\right)$$

phase space curves are **ellipses** around the equilibrium point parameterized by the integral of motion Hamiltonian (energy)

- By simply **changing** the **sign** of the potential in the harmonic oscillator, the phase trajectories become **hyperbolas**, symmetric around the equilibrium point where two straight lines cross, moving towards and away from it





Non-linear oscillators

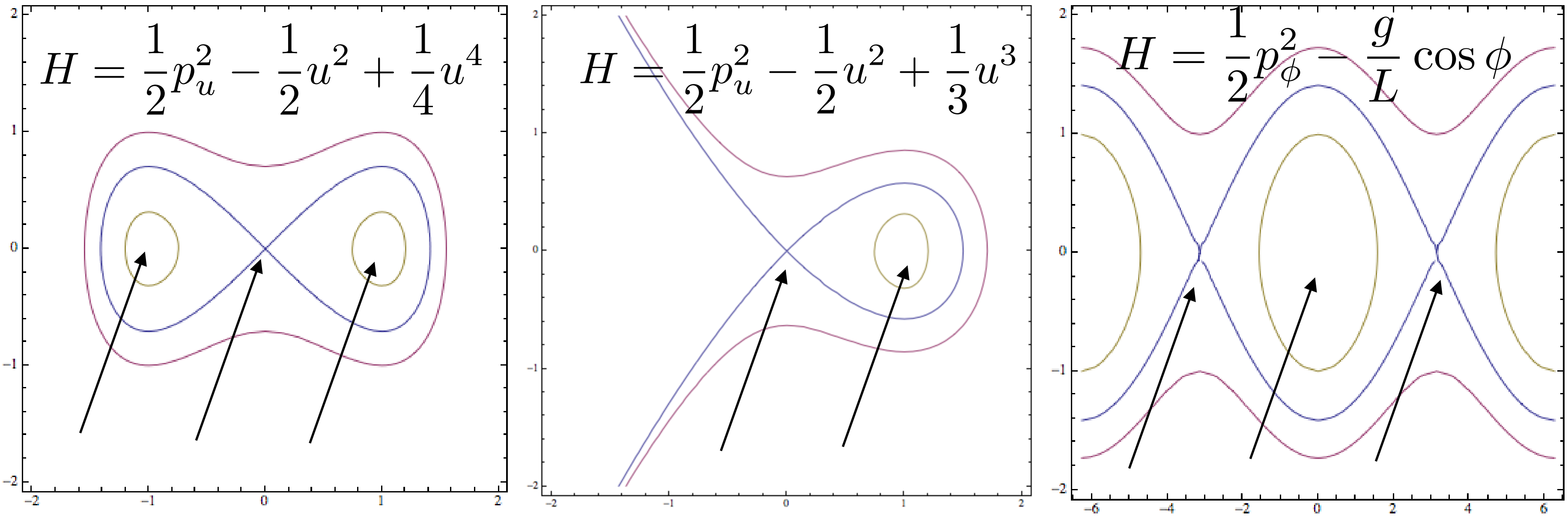
- Conservative non-linear oscillators have Hamiltonian

$$H = E = \frac{1}{2}p_u^2 + V(u)$$

with the potential being a general (polynomial) function of positions

- **Equilibrium points** are associated with extrema of the potential

Non-linear oscillators



- Conservative non-linear oscillators have Hamiltonian

$$H = E = \frac{1}{2}p_u^2 + V(u)$$

  with the potential being a general (polynomial) function of positions
- **Equilibrium points** are associated with extrema of the potential
- Considering three non-linear oscillators
  - **Quartic** potential (left): two minima and one maximum
  - **Cubic** potential (center): one minimum and one maximum
  - **Pendulum** (right): periodic minima and maxima

Fixed point analysis

- Consider a general second order system
$$\frac{du}{dt} = f_1(u, p_u)$$
$$\frac{dp_u}{dt} = f_2(u, p_u)$$
- Equilibrium or "**fixed**" points $f_1(u_0, p_{u0}) = f_2(u_0, p_{u0}) = 0$ are determinant for topology of trajectories at their vicinity

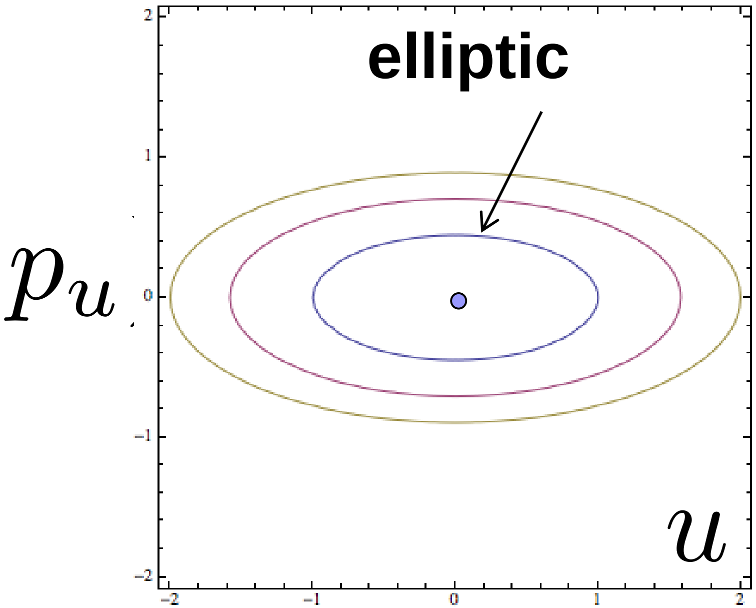
Fixed point analysis

- Consider a general second order system
$$\frac{du}{dt} = f_1(u, p_u)$$
$$\frac{dp_u}{dt} = f_2(u, p_u)$$
- Equilibrium or “**fixed**” points $f_1(u_0, p_{u0}) = f_2(u_0, p_{u0}) = 0$ are determinant for topology of trajectories at their vicinity
- The **linearized equations** of motion at their vicinity are

$$\frac{d}{dt}\begin{bmatrix}\delta u\\ \delta p_u\end{bmatrix} = \mathcal{M}_J \begin{bmatrix}\delta u\\ \delta p_u\end{bmatrix} = \underbrace{\begin{bmatrix}\frac{\partial f_1(u_0,p_{u0})}{\partial u} & \frac{\partial f_1(u_0,p_{u0})}{\partial p_u}\\ \frac{\partial f_2(u_0,p_{u0})}{\partial u} & \frac{\partial f_2(u_0,p_{u0})}{\partial p_u}\end{bmatrix}}_{\textbf{Jacobian}\ \text{matrix}} \begin{bmatrix}\delta u\\ \delta p_u\end{bmatrix}$$

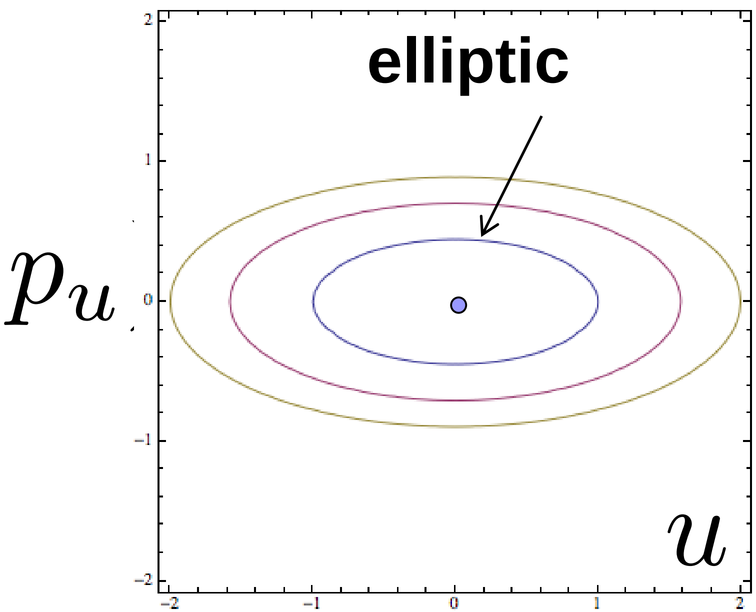
- Fixed point nature is revealed by **eigenvalues** of $\mathcal{M}_J$, i.e. solutions of the characteristic polynomial $\det|\mathcal{M}_J - \lambda \mathbf{I}| = 0$

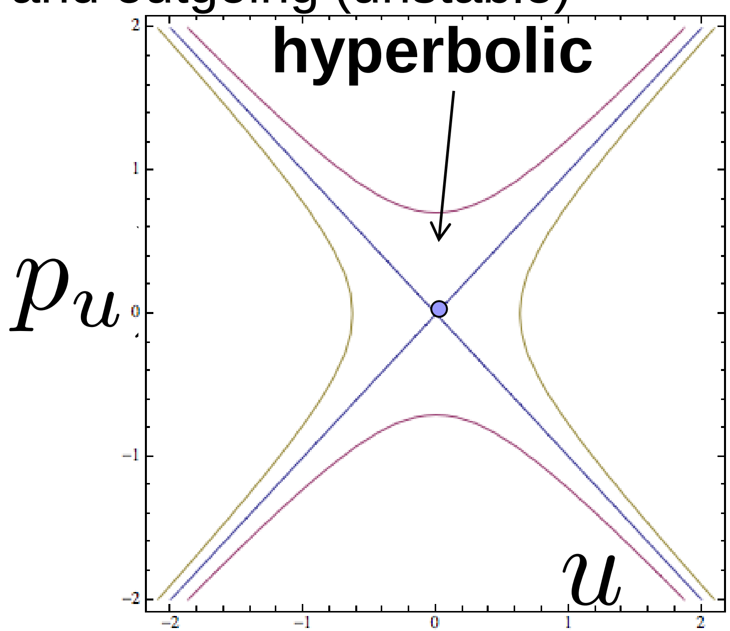
- For **conservative systems** of 1 degree of freedom, the second order characteristic polynomial for any fixed point has two possible solutions:
  - Two **complex eigenvalues** with opposite sign, corresponding to **elliptic** fixed points. Phase space flow is described by **ellipses**, with particles evolving clockwise or anti-clockwise



Fixed point for conservative systems

- For conservative systems of 1 degree of freedom, the second order characteristic polynomial for any fixed point has two possible solutions:
  - Two **complex eigenvalues** with opposite sign, corresponding to **elliptic** fixed points. Phase space flow is described by **ellipses**, with particles evolving clockwise or anti-clockwise
  - Two **real eigenvalues** with opposite sign, corresponding to **hyperbolic** (or saddle) fixed points. Flow described by two lines (or manifolds), incoming (stable) and outgoing (unstable)



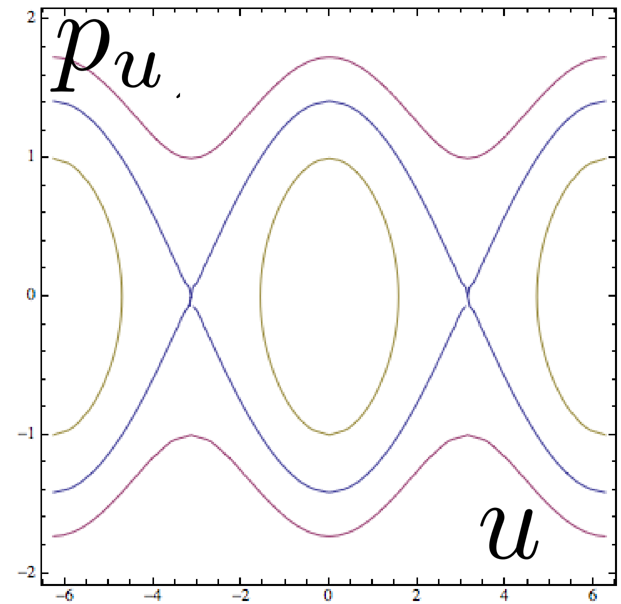


Pendulum fixed point analysis

- The "fixed" points for a pendulum can be found at

$$(\phi_n, p_\phi) = (\pm n\pi, 0) \;,\; n = 0, 1, 2 \ldots$$

- The Jacobian matrix is $\begin{bmatrix} 0 & 1 \\ -\frac{g}{L}\cos\phi_n & 0 \end{bmatrix}$
- The eigenvalues are $\lambda_{1,2} = \pm i\sqrt{\frac{g}{L}\cos\phi_n}$



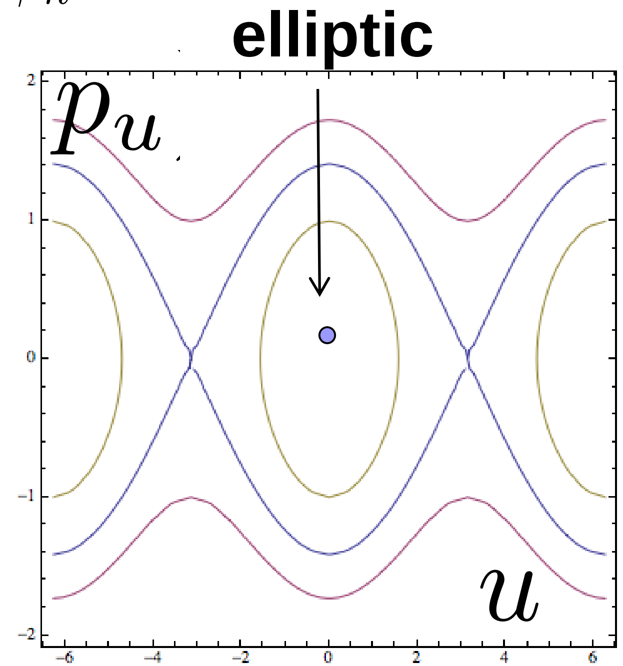
Pendulum fixed point analysis

- The "fixed" points for a pendulum can be found at

$$(\phi_n, p_\phi) = (\pm n\pi, 0)\ ,\ n = 0, 1, 2\ldots$$

- The Jacobian matrix is $\begin{bmatrix} 0 & 1 \\ -\frac{g}{L}\cos\phi_n & 0 \end{bmatrix}$
- The eigenvalues are $\lambda_{1,2} = \pm i\sqrt{\frac{g}{L}\cos\phi_n}$
- Two cases can be distinguished:
  - $\phi_n = 2n\pi$, for which $\lambda_{1,2} = \pm i\sqrt{\frac{g}{L}}$

    corresponding to **elliptic** fixed points

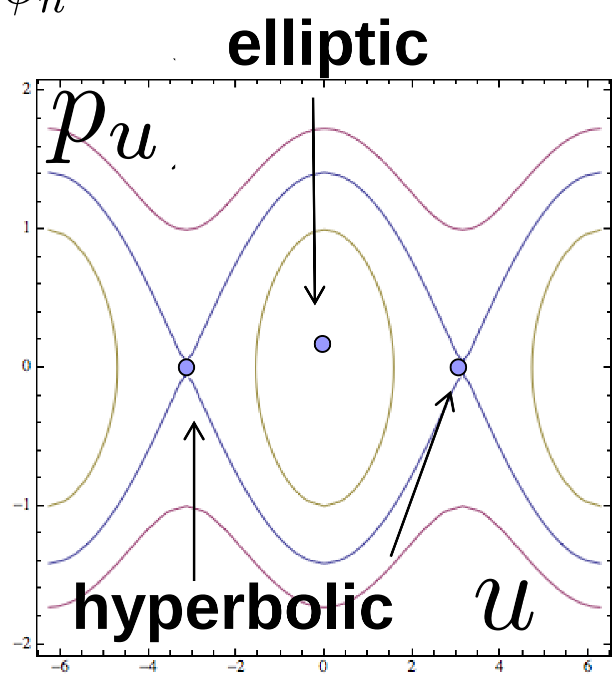


Pendulum fixed point analysis

- The "fixed" points for a pendulum can be found at

$$(\phi_n, p_\phi) = (\pm n\pi, 0)\ ,\ n = 0, 1, 2\dots$$

- The Jacobian matrix is $\begin{bmatrix} 0 & 1 \\ -\frac{g}{L}\cos\phi_n & 0 \end{bmatrix}$
- The eigenvalues are $\lambda_{1,2} = \pm i\sqrt{\frac{g}{L}\cos\phi_n}$
- Two cases can be distinguished:
  - $\phi_n = 2n\pi$, for which $\lambda_{1,2} = \pm i\sqrt{\frac{g}{L}}$ corresponding to **elliptic** fixed points
  - $\phi_n = (2n+1)\pi$, for which $\lambda_{1,2} = \pm\sqrt{\frac{g}{L}}$ corresponding to **hyperbolic** fixed points
  - The **separatrix** are the stable and unstable manifolds through the hyperbolic points, separating bounded **librations** and unbounded **rotations**



Contents of the 1st lecture

- Accelerator performance parameters and non-linear effects
- Linear and non-linear oscillators
  - Integral and frequency of motion
  - The pendulum
  - Damped harmonic oscillator
- Phase space dynamics
  - Fixed point analysis
- Non-autonomous systems
  - Driven (damped) harmonic oscillator, resonance conditions
- Linear equations with periodic coefficients – Hill's equations
  - Floquet solutions and normalized coordinates
- Perturbation theory
  - Non-linear oscillator
  - Perturbation by periodic function – single dipole perturbation
  - Application to single multipole – resonance conditions
  - Examples: single quadrupole, sextupole, octupole perturbation
  - General multi-pole perturbation– example: linear coupling
  - Resonance conditions and working point choice
- Summary
- Appendix I: Multipole expansion

Non-autonomous systems

- Consider a linear system with explicit dependence in time

$$\frac{d^2u}{dt^2} + \omega_0^2 u = F(t)$$

- **Time** now is **not** an **independent** variable but can be considered as an **extra dimension** leading to a completely new type of behavior

Non-autonomous systems

- Consider a linear system with explicit dependence in time

$$\frac{d^2u}{dt^2} + \omega_0^2 u = F(t)$$

- **Time** now is **not** an **independent** variable but can be considered as an **extra dimension** leading to a completely new type of behavior
- Consider two independent solutions of the homogeneous equation $u_1(t)$ and $u_2(t)$
- The general solution is a sum of the **homogeneous** solutions $u_h(t) = c_1 u_1(t) + c_2 u_2(t)$ and a **particular** solution, $u_p(t) = c_3 u_1(t) + c_4 u_2(t)$, where the coefficients are computed as

$$c_3 = \int \frac{u_2(t)F(t)}{W(t)} dt \quad , c_4 = \int \frac{u_1(t)F(t)}{W(t)} dt$$

with the **Wronskian** of the system

$$W(t) = u_1(t)\frac{du_2(t)}{dt} - u_2(t)\frac{du_1(t)}{dt}$$

Driven harmonic oscillator

- Consider periodic force pumping energy into the system

$$\frac{d^2u(t)}{dt^2} + \omega_0^2 u(t) = \frac{F}{m}\cos(\omega t)$$

- General solution is a combination of the **homogeneous** and a **particular** solution found as

$$u(t) = u_0 \sin(\omega_0 t + \phi_0) + \frac{F}{m(\omega_0^2 - \omega^2)}\cos(\omega t)$$

Driven harmonic oscillator

- Consider periodic force pumping energy into the system

$$\frac{d^2u(t)}{dt^2} + \omega_0^2 u(t) = \frac{F}{m}\cos(\omega t)$$

- General solution is a combination of the **homogeneous** and a **particular** solution found as

$$u(t) = u_0 \sin(\omega_0 t + \phi_0) + \frac{F}{m(\omega_0^2 - \omega^2)}\cos(\omega t)$$

- Obviously a **resonance** condition appears when driving frequency hits the oscillator eigen-frequency.

Driven harmonic oscillator

- Consider periodic force pumping energy into the system

$$\frac{d^2u(t)}{dt^2} + \omega_0^2 u(t) = \frac{F}{m}\cos(\omega t)$$

- General solution is a combination of the **homogeneous** and a **particular** solution found as

$$u(t) = u_0 \sin(\omega_0 t + \phi_0) + \frac{F}{m(\omega_0^2 - \omega^2)}\cos(\omega t)$$

- Obviously a **resonance** condition appears when driving frequency hits the oscillator eigen-frequency.
- In the limit of $\omega \to \omega_0$ the solution becomes

$$u(t) = \hat{u}_0 \sin(\omega_0 t + \hat{\phi}_0) + \frac{F}{2m\omega_0} t \sin(\omega_0 t)$$

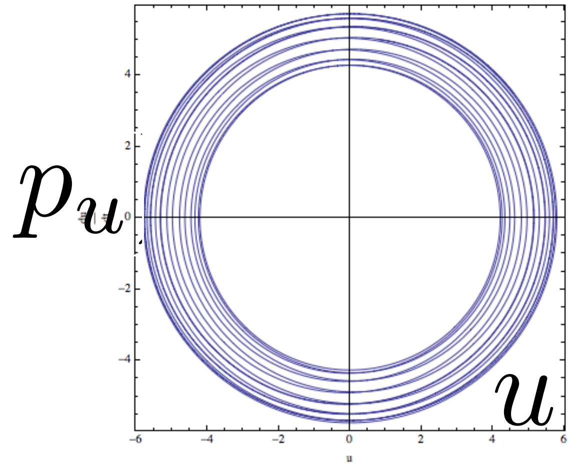
- The 2nd **secular term** implies unbounded growth of amplitude at resonance

Phase space for time-dependent systems

- Consider now a simple harmonic oscillator where the **frequency** is **time-dependent**

$$H = \frac{1}{2}\left(p_u^2 + \omega_0^2(t)u^2\right)$$

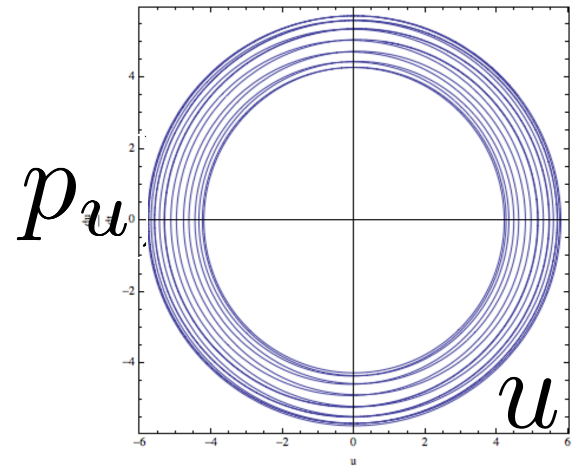
- Plotting the evolution in phase space, provides trajectories that **intersect** each other
- The phase space has **time** as **extra dimension**



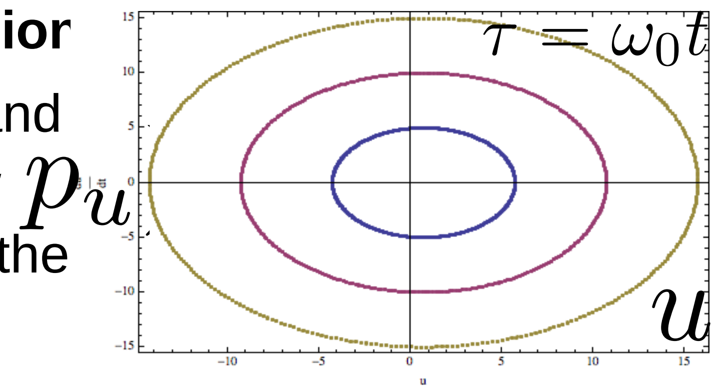
Phase space for time-dependent systems

- Consider now a simple harmonic oscillator where the **frequency** is **time-dependent**

$$H = \frac{1}{2}\left(p_u^2 + \omega_0^2(t)u^2\right)$$



- Plotting the evolution in phase space, provides trajectories that **intersect** each other
- The phase space has **time** as **extra dimension**
- By **rescaling** the **time** to become $\tau = \omega_0 t$ and considering every integer interval of the **new "time"** variable, the **phase space** looks like the one of the **harmonic oscillator**

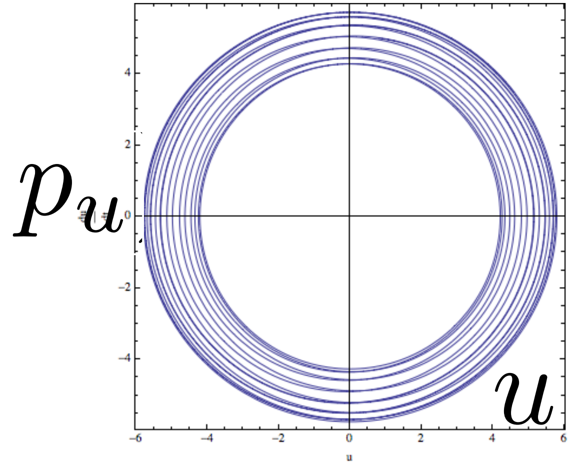


- This is the simplest version of a **Poincaré surface of section**, which is useful for studying geometrically phase space of multi-dimensional systems

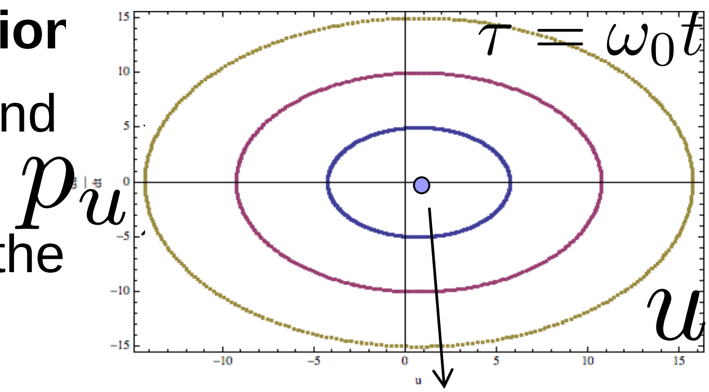
Phase space for time-dependent systems

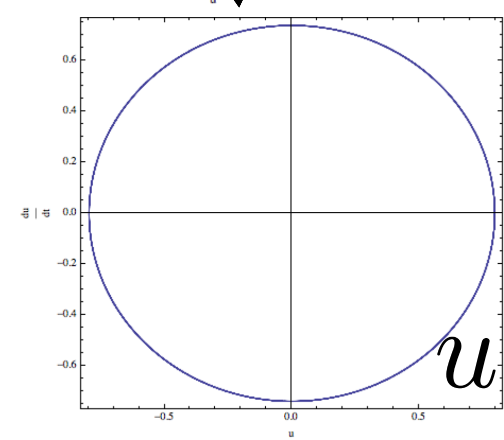
- Consider now a simple harmonic oscillator where the **frequency** is **time-dependent**

$$H = \frac{1}{2}\left(p_u^2 + \omega_0^2(t)u^2\right)$$



- Plotting the evolution in phase space, provides trajectories that **intersect** each other
- The phase space has **time** as **extra dimension**
- By **rescaling** the **time** to become $\tau = \omega_0 t$ and considering every integer interval of the **new** "**time**" variable, the **phase space** looks like the one of the **harmonic oscillator**





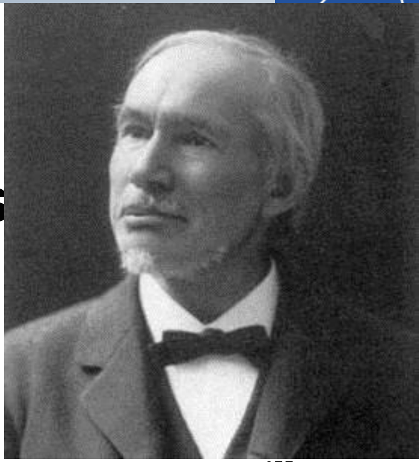
- This is the simplest version of a **Poincaré surface of section**, which is useful for studying geometrically phase space of multi-dimensional systems
- The **fixed point** in the surface of section is now a periodic orbit

Contents of the 1st lecture

- Accelerator performance parameters and non-linear effects
- Linear and non-linear oscillators
  - Integral and frequency of motion
  - The pendulum
  - Damped harmonic oscillator
- Phase space dynamics
  - Fixed point analysis
- Non-autonomous systems
  - Driven harmonic oscillator, resonance conditions
- Linear equations with periodic coefficients – Hill's equations
  - Floquet solutions and normalized coordinates
- Perturbation theory
  - Non-linear oscillator
  - Perturbation by periodic function – single dipole perturbation
  - Application to single multipole – resonance conditions
  - Examples: single quadrupole, sextupole, octupole perturbation
  - General multi-pole perturbation– example: linear coupling
  - Resonance conditions and working point choice
- Summary
- Appendix: Damped harmonic oscillator

Linear equation with periodic coefficients

- A very important class of equations especially for beam dynamics (but also solid state physics) are **linear equations** with **periodic coefficients**

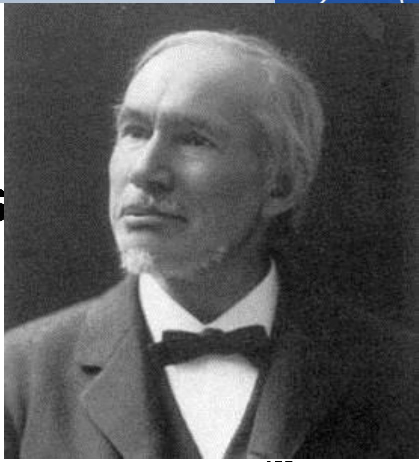


George Hill

$$\frac{d^2u}{dt^2} + K(t)u = 0$$

with $K(t) = K(t+T)$ a periodic function of time

Linear equation with periodic coefficients


George Hill

- A very important class of equations especially for beam dynamics (but also solid state physics) are **linear equations** with **periodic coefficients**

$$\frac{d^2u}{dt^2} + K(t)u = 0$$

with $K(t) = K(t+T)$ a periodic function of time

- These are called **Hill's equations** and can be thought as equations of harmonic oscillator with time dependent (periodic) frequency
- There are two solutions that can be written as $u(t) = \Re\left\{w(t)e^{i\psi(t)}\right\}$ with $w(t) = w(t+T)$ periodic but also $e^{i\psi(t+T)-i\psi(t)} = e^{i\sigma}$ with a constant which implies that $\frac{d\psi}{dt}(t+T) = \frac{d\psi}{dt}(t)$ is periodic
- The solutions are derived based on **Floquet** theory

Amplitude, phase and invariant

- Differentiating the solutions twice and substituting to Hill's equation, the following two equations are obtained

$$\begin{aligned} \frac{d^2w}{dt^2} - w\left(\frac{d\psi}{dt}\right)^2 + K(t)w &= 0 \\ 2\frac{dw}{dt}\frac{d\psi}{dt} + w\frac{d^2\psi}{dt^2} &= 0 \end{aligned}$$

- The 2nd one can be integrated to give $\frac{d\psi}{dt} = \frac{1}{w^2}$, i.e. the relation between the **"phase"** and the **amplitude**

Amplitude, phase and invariant

- Differentiating the solutions twice and substituting to Hill's equation, the following two equations are obtained

$$\begin{aligned} \frac{d^2w}{dt^2} - w(\frac{d\psi}{dt})^2 + K(t)w &= 0 \\ 2\frac{dw}{dt}\frac{d\psi}{dt} + w\frac{d^2\psi}{dt^2} &= 0 \end{aligned}$$

- The 2nd one can be integrated to give $\frac{d\psi}{dt} = \frac{1}{w^2}$, i.e. the relation between the **"phase"** and the **amplitude**
- Substituting this to the 1st equation, the amplitude equation is derived (or the **beta function** in accelerator jargon)

$$\frac{d^2w}{dt^2} + K(t)w - \frac{1}{w^3} = 0$$

- By evaluating the quadratic sum of the solution and its derivative an invariant can be constructed, with the form

$$I(u, \frac{du}{dt}, t) = \left[\frac{u^2}{w^2} + \left(w\frac{du}{dt} - \frac{dw}{dt}u\right)^2\right]$$

Normalized coordinates

- Recall the Floquet solutions for betatron motion
$$u(s) = \sqrt{\epsilon\beta(s)}\cos(\psi(s)+\psi_0)$$
$$u'(s) = -\sqrt{\frac{\epsilon}{\beta(s)}}\left(\sin(\psi(s)+\psi_0)+\alpha(s)\cos(\psi(s)+\psi_0)\right)$$
- Introduce **new variables**

$$\mathcal{U} = \frac{u}{\sqrt{\beta}}\ ,\ \ \mathcal{U}' = \frac{d\mathcal{U}}{d\phi} = \frac{\alpha}{\sqrt{\beta}}u + \sqrt{\beta}u'\ ,\ \ \phi = \frac{\psi}{\nu} = \frac{1}{\nu}\int\frac{ds}{\beta(s)}$$

- In matrix form
$$\begin{pmatrix}\mathcal{U}\\ \mathcal{U}'\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{\beta}} & 0\\ \frac{\alpha}{\sqrt{\beta}} & \sqrt{\beta}\end{pmatrix}\begin{pmatrix}u\\ u'\end{pmatrix}$$

Normalized coordinates

- Recall the Floquet solutions for betatron motion

$$u(s) = \sqrt{\epsilon\beta(s)}\cos(\psi(s)+\psi_0)$$

$$u'(s) = -\sqrt{\frac{\epsilon}{\beta(s)}}\left(\sin(\psi(s)+\psi_0)+\alpha(s)\cos(\psi(s)+\psi_0)\right)$$

- Introduce **new variables**

$$\mathcal{U} = \frac{u}{\sqrt{\beta}}\ , \quad \mathcal{U}' = \frac{d\mathcal{U}}{d\phi} = \frac{\alpha}{\sqrt{\beta}}u + \sqrt{\beta}u'\ , \quad \phi = \frac{\psi}{\nu} = \frac{1}{\nu}\int\frac{ds}{\beta(s)}$$

- In matrix form

$$\begin{pmatrix}\mathcal{U}\\ \mathcal{U}'\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{\beta}} & 0\\ \frac{\alpha}{\sqrt{\beta}} & \sqrt{\beta}\end{pmatrix}\begin{pmatrix}u\\ u'\end{pmatrix}$$

- Hill's equation becomes

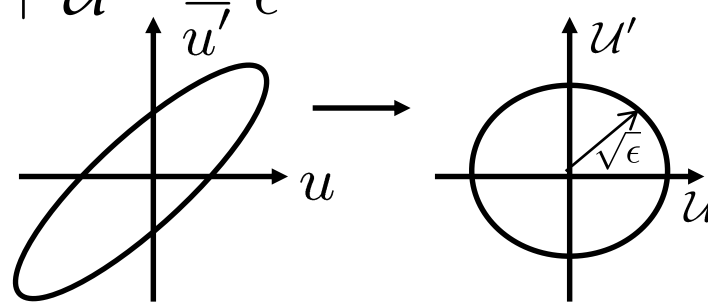
$$\frac{1}{\nu^2\beta^{3/2}}\left(\frac{d^2\mathcal{U}}{d\phi^2}+\nu^2\mathcal{U}\right) = 0$$

- System becomes **harmonic oscillator** with frequency

$$\begin{pmatrix}\mathcal{U}\\ \mathcal{U}'\end{pmatrix} = \sqrt{\epsilon}\begin{pmatrix}\cos(\nu\phi)\\ -\sin(\nu\phi)\end{pmatrix}$$

$$\mathcal{U}^2 + \mathcal{U}'^2 = \epsilon$$

- **Floquet transformation** transforms phase space in circles



Perturbation of Hill's equations

- Hill's equations in normalized coordinates with harmonic perturbation, using $\mathcal{U} = \mathcal{U}_x$ or $\mathcal{U}_y$ and $\phi = \phi_x$ or ϕ_y

$$\frac{d^2\mathcal{U}}{d\phi^2} + \nu^2\mathcal{U} = \nu^2\beta^{3/2}F(\mathcal{U}_x(\phi_x),\mathcal{U}_y(\phi_y))$$

where the *F* is the Lorentz force from perturbing fields

- **Linear magnet imperfections**: deviation from the design dipole and quadrupole fields due to powering and alignment errors
- **Time varying fields**: feedback systems (damper) and wake fields due to collective effects (wall currents)
- **Non-linear magnets**: sextupole magnets for chromaticity correction and octupole magnets for Landau damping
- **Beam-beam interactions**: strongly non-linear field
- **Space charge effects**: very important for high intensity beams
- **non-linear magnetic field imperfections**: particularly difficult to control for super conducting magnets where the field quality is entirely determined by the coil winding accuracy

Magnetic multipole expansion

- From Gauss law of magnetostatics, a vector potential exist

$$\nabla \cdot \mathbf{B} = 0 \quad \rightarrow \quad \exists \mathbf{A} : \quad \mathbf{B} = \nabla \times \mathbf{A}$$

- Assuming transverse 2D field, vector potential has only one component A_s. The Ampere's law in vacuum (inside the beam pipe) $\nabla \times \mathbf{B} = 0 \quad \rightarrow \quad \exists V : \quad \mathbf{B} = -\nabla V$
- Using the previous equations, the relations between field components and potentials are

$$B_x = -\frac{\partial V}{\partial x} = \frac{\partial A_s}{\partial y} , \quad B_y = -\frac{\partial V}{\partial y} = -\frac{\partial A_s}{\partial x}$$

Magnetic multipole expansion

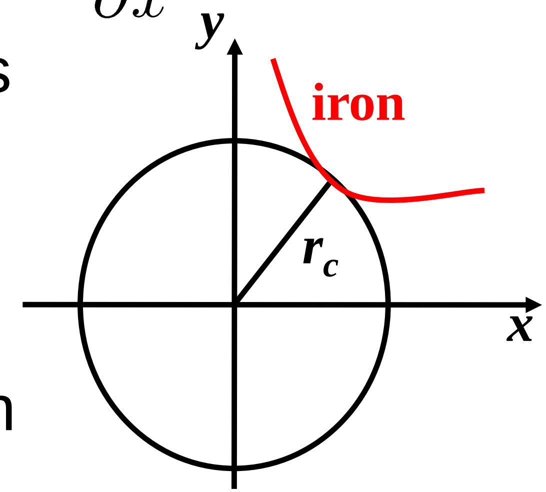
- From Gauss law of magnetostatics, a vector potential exist

$$\nabla \cdot \mathbf{B} = 0 \quad \rightarrow \quad \exists \mathbf{A} : \quad \mathbf{B} = \nabla \times \mathbf{A}$$

- Assuming transverse 2D field, vector potential has only one component A_s. The Ampere's law in vacuum (inside the beam pipe) $\nabla \times \mathbf{B} = 0 \quad \rightarrow \quad \exists V : \quad \mathbf{B} = -\nabla V$
- Using the previous equations, the relations between field components and potentials are

$$B_x = -\frac{\partial V}{\partial x} = \frac{\partial A_s}{\partial y}, \qquad B_y = -\frac{\partial V}{\partial y} = -\frac{\partial A_s}{\partial x}$$

i.e. **Riemann conditions** of analytic functions



Exists **complex potential** of $z = x + iy$ with **power series expansion** convergent in a circle with radius $|z| = r_c$ (distance from iron

$$\mathcal{A}(x+iy) = A_s(x,y) + iV(x,y) = \sum_{n=1}^{\infty} \kappa_n z^n = \sum_{n=1}^{\infty} (\lambda_n + i\mu_n)(x+iy)^n$$

Multipole expansion II

- From the complex potential we can derive the fields

$$B_y + iB_x = -\frac{\partial}{\partial x}(A_s(x,y) + iV(x,y)) = -\sum_{n=1}^{\infty} n(\lambda_n + i\mu_n)(x+iy)^{n-1}$$

- Setting $b_n = -n\lambda_n \; , \quad a_n = n\mu_n$

$$B_y + iB_x = \sum_{n=1}^{\infty}(b_n - ia_n)(x+iy)^{n-1}$$

Multipole expansion II

- From the complex potential we can derive the fields

$$B_y + iB_x = -\frac{\partial}{\partial x}(A_s(x,y) + iV(x,y)) = -\sum_{n=1}^{\infty} n(\lambda_n + i\mu_n)(x+iy)^{n-1}$$

- Setting $b_n = -n\lambda_n \;, \quad a_n = n\mu_n$

$$B_y + iB_x = \sum_{n=1}^{\infty}(b_n - ia_n)(x+iy)^{n-1}$$

- Define normalized **multipole** coefficients

$$b'_n = \frac{b_n}{10^{-4}B_0} r_0^{n-1} \;, \; a'_n = \frac{a_n}{10^{-4}B_0} r_0^{n-1}$$

on a reference radius r_0, 10^{-4} of the main field to get

$$B_y + iB_x = 10^{-4}B_0 \sum_{n=1}^{\infty}(b'_n - ia'_n)(\frac{x+iy}{r_0})^{n-1}$$

- **Note**: $n' = n-1$ is the US convention

Contents of the 1st lecture

- Accelerator performance parameters and non-linear effects
- Linear and non-linear oscillators
  - Integral and frequency of motion
  - The pendulum
  - Damped harmonic oscillator
- Phase space dynamics
  - Fixed point analysis
- Non-autonomous systems
  - Driven harmonic oscillator, resonance conditions
- Linear equations with periodic coefficients – Hill's equations
  - Floquet solutions and normalized coordinates
- Perturbation theory
  - Non-linear oscillator
  - Perturbation by periodic function – single dipole perturbation
  - Application to single multipole – resonance conditions
  - Examples: single quadrupole, sextupole, octupole perturbation
  - General multi-pole perturbation– example: linear coupling
  - Resonance conditions and working point choice
- Summary
- Appendix: Damped harmonic oscillator

Perturbation theory

- **Completely integrable systems** are **exceptional**
- For understanding dynamics of general non-linear systems composed of a part whose solution $u_0(t)$ is known and a part parameterized by a small constant ϵ, **perturbation theory** is employed

Perturbation theory

- **Completely integrable systems** are **exceptional**
- For understanding dynamics of general non-linear systems composed of a part whose solution $u_0(t)$ is known and a part parameterized by a small constant ϵ, **perturbation theory** is employed
- The general idea is to **expand** the solution in a power series $u(t) = u_0(t) + \epsilon u_1(t) + \epsilon^2 u_2(t) + \ldots$ and **compute recursively** the corrections $u_1(t), u_2(t), \ldots$ hoping that a few terms will be sufficient to find an accurate representation of the general solution

- **Completely integrable systems** are **exceptional**
- For understanding dynamics of general non-linear systems composed of a part whose solution $u_0(t)$ is known and a part parameterized by a small constant ϵ , **perturbation theory** is employed
- The general idea is to **expand** the solution in a power series $u(t) = u_0(t) + \epsilon u_1(t) + \epsilon^2 u_2(t) + \ldots$ and **compute recursively** the corrections $u_1(t), u_2(t), \ldots$ hoping that a few terms will be sufficient to find an accurate representation of the general solution
- This **may not be true** for all times, i.e. facing **series convergence** problems
- In addition, any **series expansion** breaks in the **vicinity** of a **resonance**

Perturbation of non-linear oscillator

- Consider a non-linear harmonic oscillator, $\frac{d^2u}{dt^2} + \omega_0^2 u - \frac{1}{6}\epsilon\omega_0^2 u^3 = 0$
- This is just the **pendulum expanded to 3rd order** in u
- Note that ϵ is a dimensionless measure of smallness, which may represent a scaling factor of u (e.g. $\epsilon = 1$ without loss of generality)

Perturbation of non-linear oscillator

- Consider a non-linear harmonic oscillator, $\frac{d^2u}{dt^2} + \omega_0^2 u - \frac{1}{6}\epsilon\omega_0^2 u^3 = 0$
- This is just the **pendulum expanded to 3rd order in** u
- Note that ϵ is a dimensionless measure of smallness, which may represent a scaling factor of u (e.g. $\epsilon = 1$ without loss of generality)
- Expanding $u(t) = u_0(t) + \epsilon u_1(t) + \epsilon^2 u_2(t) + \ldots$ and separating the equations with equal power in ϵ:
  - Order 0: $\frac{d^2u_0}{dt^2} + \omega_0^2 u_0 = 0 \Rightarrow u_0(t) = A\cos(\omega_0 t)$
  - Order 1: $\frac{d^2u_1}{dt^2} + \omega_0^2 u_1 = \frac{\omega_0^2 u_0^3}{6} = \frac{\omega_0^2 A^3}{6}\cos^3(\omega_0 t) = \frac{\omega_0^2 A^3}{24}(\cos(3\omega_0 t) + 3\cos(\omega_0 t))$

Perturbation of non-linear oscillator

- Consider a non-linear harmonic oscillator, $\frac{d^2u}{dt^2} + \omega_0^2 u - \frac{1}{6}\epsilon\omega_0^2 u^3 = 0$
- This is just the **pendulum expanded to 3rd order in** u
- Note that ϵ is a dimensionless measure of smallness, which may represent a scaling factor of u (e.g. $\epsilon = 1$ without loss of generality)
- Expanding $u(t) = u_0(t) + \epsilon u_1(t) + \epsilon^2 u_2(t) + \ldots$ and separating the equations with equal power in ϵ:
  - Order 0: $\frac{d^2u_0}{dt^2} + \omega_0^2 u_0 = 0 \Rightarrow u_0(t) = A\cos(\omega_0 t)$
  - Order 1: $\frac{d^2u_1}{dt^2} + \omega_0^2 u_1 = \frac{\omega_0^2 u_0^3}{6} = \frac{\omega_0^2 A^3}{6}\cos^3(\omega_0 t) = \frac{\omega_0^2 A^3}{24}(\cos(3\omega_0 t) + 3\cos(\omega_0 t))$
- The 2nd equation has a particular solution with two terms. A well behaved one $u_{1a}(t) = -\frac{A^3}{192}\cos(3\omega t)$ and $u_{1b}(t) = \frac{A^3}{64}(\omega_0 t\cos(\omega_0 t) + 2\cos(\omega_0 t))$ the first part of which grows linearly with time (**secular** term)
- But this cannot be true, the pendulum **does not present** such **behavior**. **What did it go wrong?**

- It was already shown that the pendulum has an **amplitude dependent frequency**, so the frequency has to be developed as well (**Poincaré-Linstead** method):

$$\omega = \omega_0 + \epsilon\omega_1 + \epsilon^2\omega_2 + \ldots$$

- It was already shown that the pendulum has an **amplitude dependent frequency**, so the frequency has to be developed as well (**Poincaré-Linstead** method):

$$\omega = \omega_0 + \epsilon\omega_1 + \epsilon^2\omega_2 + \ldots$$

- Assume that the solution is a periodic function of $\tau = \omega t$ which becomes the new independent variable. The equation at zero order gives the solution $u_0(\tau) = A\cos(\tau)$ and at leading perturbation order becomes

$$\omega_0 \frac{d^2u_1}{d\tau^2} + \omega_0 u_1 = -2\omega_1 \frac{d^2u_0}{d\tau^2} + \frac{\omega_0}{6}u_0^3 = \frac{\omega_0 A^3}{24}\cos(3\tau) + \left(\frac{\omega_0 A^3}{8} + 2A\omega_1\right)\cos(\tau)$$

- It was already shown that the pendulum has an **amplitude dependent frequency**, so the frequency has to be developed as well (**Poincaré-Linstead** method):

$$\omega = \omega_0 + \epsilon\omega_1 + \epsilon^2\omega_2 + \dots$$

- Assume that the solution is a periodic function of $\tau = \omega t$ which becomes the new independent variable. The equation at zero order gives the solution $u_0(\tau) = A\cos(\tau)$ and at leading perturbation order becomes

$$\omega_0 \frac{d^2u_1}{d\tau^2} + \omega_0 u_1 = -2\omega_1 \frac{d^2u_0}{d\tau^2} + \frac{\omega_0}{6}u_0^3 = \frac{\omega_0 A^3}{24}\cos(3\tau) + \left(\frac{\omega_0 A^3}{8} + 2A\omega_1\right)\cos(\tau)$$

- The last term has to be zero, if not it gives secular terms, thus $\omega_1 = -\frac{A^2\omega_0}{16}$ which **reveals** the **reduction** of the **frequency** with the **oscillation amplitude**
- Finally, the solution $u_1(t) = \frac{A^3}{192}\left(\cos(\omega_0 t) - \cos(3\omega_0 t)\right)$ is the leading order correction due to the non-linear term

Perturbation by periodic function

- In beam dynamics, perturbing fields are **periodic functions**
- The problem to solve is a generalization of the driven harmonic oscillator, $\frac{d^2u}{dt^2} + \omega_0^2 u(t) = g(t)$ with a general periodic function $g(t)$, with frequency ω
- The right side can be **Fourier analyzed**: $g(t) = \sum_{m=-\infty}^{m=+\infty} a_m e^{im\omega t}$

Perturbation by periodic function

- In beam dynamics, perturbing fields are **periodic functions**
- The problem to solve is a generalization of the driven harmonic oscillator, $\frac{d^2u}{dt^2} + \omega_0^2 u(t) = g(t)$ with a general periodic function $g(t)$, with frequency ω
- The right side can be **Fourier analyzed**: $g(t) = \sum_{m=-\infty}^{m=+\infty} a_m e^{im\omega t}$
- The **homogeneous** solution is $u_h(t) = u_0(t)\sin(\omega_0 t + \phi_0)$
- The **particular** solution can be found by considering that $u(t)$ has the same form as $g(t)$: $u_p(t) = \sum_{m=-\infty}^{m=+\infty} u_{pm} e^{im\omega t}$
- By substituting the following relation is derived for the Fourier coefficients of the particular solution $u_{pm} = \frac{a_m}{\omega_0^2 - m^2\omega^2}$
- There is a **resonance condition** for infinite number of frequencies satisfying $\omega_0^2 = m^2\omega^2$

Perturbation by single dipole

- Hill's equations in normalized coordinates with **single dipole perturbation**:

$$\frac{d^2\mathcal{U}}{d\phi^2} + \nu_0^2\mathcal{U} = \nu_0^2\beta^{3/2}b_1(\phi) = \overline{b_1}(\phi)$$

- The dipole perturbation is **periodic**, so it can be expanded in a **Fourier series**

$$\overline{b_1}(\phi) = \sum_{m=-\infty}^{\infty} \overline{b_{1m}} e^{im\phi}$$

Perturbation by single dipole

- Hill's equations in normalized coordinates with **single dipole perturbation**:

$$\frac{d^2\mathcal{U}}{d\phi^2} + \nu_0^2\mathcal{U} = \nu_0^2\beta^{3/2}b_1(\phi) = \overline{b_1}(\phi)$$

- The dipole perturbation is **periodic**, so it can be expanded in a **Fourier series**

$$\overline{b_1}(\phi) = \sum_{m=-\infty}^{\infty} \overline{b_{1m}} e^{im\phi}$$

- Note, as before that a periodic kick introduces **infinite number** of **integer driving frequencies**
- The resonance condition occurs when $\nu_0 = m$ i.e. **integer tunes** should be avoided (remember orbit distortion due to single dipole kick)

- For a generalized multi-pole perturbation, Hill's equation is:

$$\frac{d^2\mathcal{U}}{d\phi^2} + \nu_0^2\mathcal{U} = \nu_0^2\beta^{\frac{n}{2}+1}b_n(\phi)\mathcal{U}^{n-1} = \overline{b_n}(\phi)\mathcal{U}^{n-1}$$

- As before, the multipole coefficient can be expanded in Fourier series

$$\overline{b_n}(\phi) = \sum_{m=-\infty}^{\infty}\overline{b_{nm}}e^{im\phi}$$

Perturbation by single multi-pole

- For a generalized multi-pole perturbation, Hill's equation is:

$$\frac{d^2\mathcal{U}}{d\phi^2} + \nu_0^2\mathcal{U} = \nu_0^2\beta^{\frac{n}{2}+1}b_n(\phi)\mathcal{U}^{n-1} = \overline{b_n}(\phi)\mathcal{U}^{n-1}$$

- As before, the multipole coefficient can be expanded in Fourier series $\overline{b_n}(\phi) = \sum_{m=-\infty}^{\infty} \overline{b_{nm}} e^{im\phi}$
- Following the perturbation steps, the zero-order solution is given by the homogeneous equation $\mathcal{U}_0 = W_1 e^{i\nu_0\phi} + W_{-1}e^{-i\nu_0\phi}$
- Then the position can be expressed as

$$\mathcal{U}_0^{n-1} = \sum_{k=0}^{n-1} \binom{n-1}{k} W_1^{n-1-k} W_{-1}^{k} e^{i(n-1-2k)\nu_0\phi}$$

Perturbation by single multi-pole

- For a generalized multi-pole perturbation, Hill's equation is:

$$\frac{d^2\mathcal{U}}{d\phi^2} + \nu_0^2\mathcal{U} = \nu_0^2\beta^{\frac{n}{2}+1}b_n(\phi)\mathcal{U}^{n-1} = \overline{b_n}(\phi)\mathcal{U}^{n-1}$$

- As before, the multipole coefficient can be expanded in Fourier series $\overline{b_n}(\phi) = \sum_{m=-\infty}^{\infty} \overline{b_{nm}} e^{im\phi}$
- Following the perturbation steps, the zero-order solution is given by the homogeneous equation $\mathcal{U}_0 = W_1 e^{i\nu_0\phi} + W_{-1}e^{-i\nu_0\phi}$
- Then the position can be expressed as

$$\mathcal{U}_0^{n-1} = \sum_{k=0}^{n-1} \underbrace{\binom{n-1}{k} W_1^{n-1-k} W_{-1}^k}_{\overline{W}_q} e^{i\overbrace{(n-1-2k)}^{q = -n+1, -n+3, \ldots, n-1}\nu_0\phi} = \sum_{q=-n+1}^{n-1} \overline{W}_q e^{iq\nu_0\phi}$$

with $\overline{W}_{n-2} = \overline{W}_{n-4} = \overline{W}_{n-6} = \cdots = \overline{W}_{-n+2} = 0$

Perturbation by single multi-pole

- For a generalized multi-pole perturbation, Hill's equation is:

$$\frac{d^2\mathcal{U}}{d\phi^2} + \nu_0^2\mathcal{U} = \nu_0^2\beta^{\frac{n}{2}+1}b_n(\phi)\mathcal{U}^{n-1} = \overline{b_n}(\phi)\mathcal{U}^{n-1}$$

- As before, the multipole coefficient can be expanded in Fourier series $\overline{b_n}(\phi) = \sum_{m=-\infty}^{\infty} \overline{b_{nm}} e^{im\phi}$
- Following the perturbation steps, the zero-order solution is given by the homogeneous equation $\mathcal{U}_0 = W_1 e^{i\nu_0\phi} + W_{-1}e^{-i\nu_0\phi}$
- Then the position can be expressed as

$$q = -n+1, -n+3, \ldots, n-1$$

$$\mathcal{U}_0^{n-1} = \sum_{k=0}^{n-1} \overbrace{\binom{n-1}{k} W_1^{n-1-k} W_{-1}^{k}}^{\overline{W}_q} e^{i\overbrace{(n-1-2k)}^{q}\nu_0\phi} = \sum_{q=-n+1}^{n-1} \overline{W}_q e^{iq\nu_0\phi}$$

with $\quad \overline{W}_{n-2} = \overline{W}_{n-4} = \overline{W}_{n-6} = \cdots = \overline{W}_{-n+2} = 0$

- The first order solution is written as

$$\frac{d^2\mathcal{U}_1}{d\phi^2} + \nu_0^2\mathcal{U}_1 = \overline{b_n}(\phi)\mathcal{U}_0^{n-1} = \sum_{q=-n+1}^{n-1} \sum_{m=-\infty}^{m=\infty} \bar{b}_{nm}\overline{W}_q e^{i(m+q\nu_0)\phi}$$

- Following the discussion on the periodic perturbation, the solution can be found by setting the **leading order solution** to be **periodic** with the **same frequency** as the right hand side

$$\mathcal{U}_1 = \sum_{q=-n+1}^{n-1} \sum_{m=-\infty}^{m=\infty} \mathcal{U}_{1mq} e^{i(m+q\nu_0)\phi}$$

- Following the discussion on the periodic perturbation, the solution can be found by setting the **leading order solution** to be **periodic** with the **same frequency** as the right hand side

$$\mathcal{U}_1 = \sum_{q=-n+1}^{n-1} \sum_{m=-\infty}^{m=\infty} \mathcal{U}_{1mq} e^{i(m+q\nu_0)\phi}$$

- Equating terms of **equal exponential powers**, the Fourier amplitudes are found to satisfy the relationship

$$\mathcal{U}_{1mq} = \frac{\tilde{b}_{nm} \widetilde{W}_q}{\nu_0^2 - (m + q\nu_0)^2}$$

Resonances for single multi-pole

- Following the discussion on the periodic perturbation, the solution can be found by setting the **leading order solution** to be **periodic** with the **same frequency** as the right hand side

$$\mathcal{U}_1 = \sum_{q=-n+1}^{n-1} \sum_{m=-\infty}^{m=\infty} \mathcal{U}_{1mq} e^{i(m+q\nu_0)\phi}$$

- Equating terms of **equal exponential powers**, the Fourier amplitudes are found to satisfy the relationship

$$\mathcal{U}_{1mq} = \frac{\bar{b}_{nm} \bar{W}_q}{\nu_0^2 - (m+q\nu_0)^2}$$

$$m \pm |q| \nu_0 = \nu_0$$

- This provides the **resonance condition**

or $\nu_0 = \frac{m}{1 \pm |q|}$ which means that there are resonant

frequencies for and **"infinite" number of rationals**

- Note that for even multi-poles and $q = 1$ or $m = 0$, there is a Fourier coefficient $\bar{b}_{n0}$, which is independent of ϕ and represents the average value of the periodic perturbation
- The perturbing term in the r.h.s. is

$$\bar{b}_{n0}\overline{W}_1 e^{i\nu_0\phi} = \nu_0^2 \beta^{\frac{n}{2}+1} b_{n0} \binom{n-1}{\frac{n}{2}-1} W_1^{n-1} W_{-1}^{\frac{n}{2}-1} e^{i\nu_0\phi}$$

which can be obtained for $k = \dfrac{n}{2} - 1$ (it is indeed an integer only for even multi-poles)

Tune-shift for single multi-pole

- Note that for even multi-poles and $q = 1$ or $m = 0$, there is a Fourier coefficient $\bar{b}_{n0}$, which is independent of ϕ and represents the average value of the periodic perturbation
- The perturbing term in the r.h.s. is

$$\bar{b}_{n0}\overline{W}_1 e^{i\nu_0\phi} = \nu_0^2 \beta^{\frac{n}{2}+1} b_{n0} \binom{n-1}{\frac{n}{2}-1} W_1^{n-1} W_{-1}^{\frac{n}{2}-1} e^{i\nu_0\phi}$$

which can be obtained for $k = \frac{n}{2} - 1$ (it is indeed an integer only for even multi-poles)

- Following the approach of the perturbed non-linear harmonic oscillator, this term will be secular unless a perturbation in the frequency is considered, thereby resulting to a **tune-shift** equal to

$$\delta\nu = -\frac{\nu_0 \beta^{\frac{n}{2}+1} b_{n0}}{2} \binom{n-1}{\frac{n}{2}-1} \widetilde{W}^{n-2} \quad \text{with} \quad \widetilde{W}^2 = W_1 W_{-1}$$

- This tune-shift is **amplitude dependent** for $n > 2$

Example: single quadrupole perturbation

- Consider single quadrupole kick in one normalized plane:

$$\frac{d^2\mathcal{U}}{d\phi^2} + \nu_0^2\mathcal{U} = \nu_0^2\beta^2 b_2(\phi)\mathcal{U} = \overline{b_2}(\phi)\mathcal{U}$$

- The quadrupole perturbation can be expanded in a Fourier series

$$\overline{b_2}(\phi) = \sum_{m=-\infty}^{\infty} \overline{b_{2m}} e^{im\phi}$$

Example: single quadrupole perturbation

- Consider single quadrupole kick in one normalized plane:

$$\frac{d^2\mathcal{U}}{d\phi^2} + \nu_0^2\mathcal{U} = \nu_0^2\beta^2 b_2(\phi)\mathcal{U} = \overline{b_2}(\phi)\mathcal{U}$$

- The quadrupole perturbation can be expanded in a Fourier series

$$\overline{b_2}(\phi) = \sum_{m=-\infty}^{\infty} \overline{b_{2m}} e^{im\phi}$$

- Following the perturbation approach, the 1st order equation becomes

$$\frac{d^2\mathcal{U}_1}{d\phi^2} + \nu_0^2\mathcal{U}_1 = \sum_{q=-1}^{1}\sum_{m=-\infty}^{\infty} \overline{W}_q \overline{b_{2m}} e^{i(m+q\nu_0)\phi}$$

with $\overline{W}_0 = 0$

- For $q = -1$, the resonance conditions are $m - \nu_0 = \nu_0 \rightarrow \nu_0 = \frac{m}{2}$

i.e. **integer** and **half-integer tunes** should be avoided

- Consider single quadrupole kick in one normalized plane:

$$\frac{d^2\mathcal{U}}{d\phi^2} + \nu_0^2\mathcal{U} = \nu_0^2\beta^2 b_2(\phi)\mathcal{U} = \overline{b_2}(\phi)\mathcal{U}$$

- The quadrupole perturbation can be expanded in a Fourier series

$$\overline{b_2}(\phi) = \sum_{m=-\infty}^{\infty} \overline{b_{2m}} e^{im\phi}$$

- Following the perturbation approach, the 1st order equation becomes $\frac{d^2\mathcal{U}_1}{d\phi^2} + \nu_0^2\mathcal{U}_1 = \sum_{q=-1}^{1}\sum_{m=-\infty}^{\infty} \overline{W}_q\overline{b_{2m}} e^{i(m+q\nu_0)\phi}$ with $\overline{W}_0 = 0$
- For $q = -1$, the resonance conditions are $m - \nu_0 = \nu_0 \rightarrow \nu_0 = \frac{m}{2}$

i.e. **integer** and **half-integer tunes** should be avoided

- For $q = 1$, the condition $m + \nu_0 = \nu_0 \rightarrow m = 0$ corresponds to a non-vanishing average value $\overline{b_{20}}$, which can be absorbed in the left-hand side providing a **tune-shift**:

$\nu^2 = \nu_0^2 - \bar{b}_{20}$ or $\delta\nu \approx -\frac{\bar{b}_{20}}{2\nu_0} = -\frac{\nu_0\beta^2 b_{20}}{2}$

Single Sextupole Perturbation

- Consider a localized sextupole perturbation in the horizontal plane

$$\frac{d^2\mathcal{U}}{d\phi^2} + \nu_0^2\mathcal{U} = \nu_0^2\beta^{\frac{5}{2}}b_3(\phi)\mathcal{U}^2 = \overline{b_3}(\phi)\mathcal{U}^2$$

- After replacing the perturbation by its Fourier transform and inserting the unperturbed solution to the right hand side

$$\frac{d^2\mathcal{U}_1}{d\phi^2} + \nu_0^2\mathcal{U}_1 = \sum_{q=-2}^{2}\sum_{m=-\infty}^{\infty}\overline{W}_q\overline{b_{3m}}e^{i(m+q\nu_0)\phi}$$ with $\overline{W}_{-1} = \overline{W}_1 = 0$

Single Sextupole Perturbation

- Consider a localized sextupole perturbation in the horizontal plane

$$\frac{d^2\mathcal{U}}{d\phi^2} + \nu_0^2\mathcal{U} = \nu_0^2\beta^{\frac{5}{2}}b_3(\phi)\mathcal{U}^2 = \overline{b_3}(\phi)\mathcal{U}^2$$

- After replacing the perturbation by its Fourier transform and inserting the unperturbed solution to the right hand side

$$\frac{d^2\mathcal{U}_1}{d\phi^2} + \nu_0^2\mathcal{U}_1 = \sum_{q=-2}^{2}\sum_{m=-\infty}^{\infty}\overline{W}_q\overline{b_{3m}}e^{i(m+q\nu_0)\phi}$$ with $\overline{W}_{-1} = \overline{W}_1 = 0$

3rd integer $\rightarrow 3\nu_0 = m \text{ for } q = -2$

- Resonance conditions: **integer** $\rightarrow \nu_0 = m \text{ for } q = 0, 2$

Single Sextupole Perturbation

- Consider a localized sextupole perturbation in the horizontal plane

$$\frac{d^2\mathcal{U}}{d\phi^2} + \nu_0^2\mathcal{U} = \nu_0^2\beta^{\frac{5}{2}} b_3(\phi)\mathcal{U}^2 = \overline{b_3}(\phi)\mathcal{U}^2$$

- After replacing the perturbation by its Fourier transform and inserting the unperturbed solution to the right hand side

$$\frac{d^2\mathcal{U}_1}{d\phi^2} + \nu_0^2\mathcal{U}_1 = \sum_{q=-2}^{2}\sum_{m=-\infty}^{\infty} \overline{W}_q \overline{b_{3m}} e^{i(m+q\nu_0)\phi}$$ with $\overline{W}_{-1} = \overline{W}_1 = 0$

3rd integer $\rightarrow 3\nu_0 = m \text{ for } q = -2$

- Resonance conditions: **integer** $\rightarrow \nu_0 = m \text{ for } q = 0, 2$
- Note that there is **not a tune-spread associated**. This is only true for small perturbations (**first order** perturbation treatment)
- Although perturbation treatment can provide approximations for evolution of motion, there is **no exact solution**

General multi-pole perturbation

- Equations of motion including any multi-pole error term, in both planes

$$\frac{d^2\mathcal{U}_x}{d\phi_x^2} + \nu_{0x}^2\mathcal{U}_x = \overline{b_{n,r}}(\phi_x)\mathcal{U}_x^{n-1}\mathcal{U}_y^{r-1}$$

- Expanding **perturbation coefficient** in **Fourier series** and inserting the **solution** of the **unperturbed system** on the rhs gives the following series:

$$\overline{b_{nr}}(\phi_x) = \sum_{m=-\infty}^{\infty} \overline{b_{nrm}} e^{im\phi_x}$$

$$\mathcal{U}_x^{n-1} \approx \mathcal{U}_{0x}^{n-1} = \sum_{q_x=-n+1}^{n-1} \overline{W}_{q_x} e^{iq_x\nu_0\phi_x}$$

$$\mathcal{U}_y^{r-1} \approx \mathcal{U}_{0y}^{r-1} = \sum_{q_y=-r+1}^{r-1} \overline{W}_{q_y} e^{iq_y\nu_{0y}\phi_x}$$

General multi-pole perturbation

- Equations of motion including any multi-pole error term, in both planes

$$\frac{d^2\mathcal{U}_x}{d\phi_x^2} + \nu_{0x}^2\mathcal{U}_x = \overline{b_{n,r}}(\phi_x)\mathcal{U}_x^{n-1}\mathcal{U}_y^{r-1}$$

- Expanding **perturbation coefficient** in **Fourier series** and inserting the **solution** of the **unperturbed system** on the rhs gives the following series:

$$\overline{b_{nr}}(\phi_x) = \sum_{m=-\infty}^{\infty} \overline{b_{nrm}} e^{im\phi_x}$$

$$\mathcal{U}_x^{n-1} \approx \mathcal{U}_{0x}^{n-1} = \sum_{q_x=-n+1}^{n-1} \overline{W}_{q_x} e^{iq_x\nu_0\phi_x}$$

$$\mathcal{U}_y^{r-1} \approx \mathcal{U}_{0y}^{r-1} = \sum_{q_y=-r+1}^{r-1} \overline{W}_{q_y} e^{iq_y\nu_{0y}\phi_x}$$

- The equation of motion becomes

$$\frac{d^2\mathcal{U}_x}{d\phi_x^2} + \nu_{0x}^2\mathcal{U}_x = \sum_{m,q_x,q_y} \overline{b_{nrm}} W_{q_x}^x W_{q_y}^y e^{i(m+q_x\nu_{0x}+q_y\nu_{0y})\phi_x}$$

- In principle, same perturbation steps can be followed for getting an approximate solution in both planes

Example: Linear Coupling

- For a localized skew quadrupole we have

$$\frac{d^2\mathcal{U}_x}{d\phi_x^2} + \nu_{0x}^2\mathcal{U}_x = \overline{b_{1,2}}(\phi_x)\mathcal{U}_y$$

- Expanding perturbation coefficient in Fourier series and inserting the solution of the unperturbed system gives the following equation:

$$\frac{d^2\mathcal{U}_x}{d\phi_x^2} + \nu_{0x}^2\mathcal{U}_x = \sum_{m=-\infty}^{\infty}\sum_{q_y=-1}^{q_y=1}\overline{b_{12m}}W_{q_y}^y e^{i(m+q_y\nu_{0y})\phi_x}$$ with $W_0^y = 0$

Example: Linear Coupling

- For a localized skew quadrupole we have

$$\frac{d^2\mathcal{U}_x}{d\phi_x^2} + \nu_{0x}^2\mathcal{U}_x = \overline{b_{1,2}}(\phi_x)\mathcal{U}_y$$

- Expanding perturbation coefficient in Fourier series and inserting the solution of the unperturbed system gives the following equation:

$$\frac{d^2\mathcal{U}_x}{d\phi_x^2} + \nu_{0x}^2\mathcal{U}_x = \sum_{m=-\infty}^{\infty}\sum_{q_y=-1}^{q_y=1}\overline{b_{12m}}W_{q_y}^y e^{i(m+q_y\nu_{0y})\phi_x}$$ with $W_0^y = 0$

- The coupling resonance are found for $q_y = \pm 1$

Linear sum resonance

$$m = \nu_{0x} + \nu_{0y}$$

Linear difference resonance

$$m = \nu_{0x} - \nu_{0y}$$

General resonance conditions

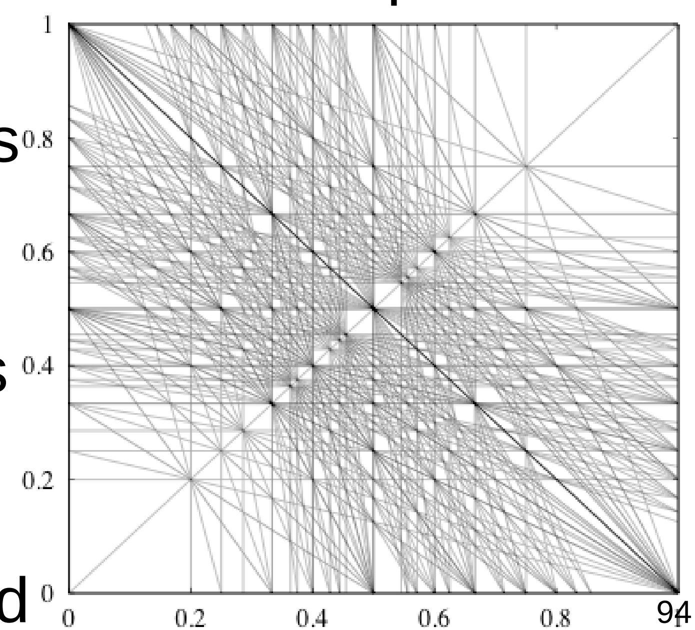
- The general resonance conditions is $m + q_x\nu_{0x} + q_y\nu_{0y} = \nu_{0x}$ or $m + q'_x\nu_{0x} + q_y\nu_{0y} = 0$, with order $|q_x| + |q_y| + 1$
- The same condition can be obtained in the vertical plane

General resonance conditions

- The general resonance conditions is $m + q_x\nu_{0x} + q_y\nu_{0y} = \nu_{0x}$ or $m + q_x'\nu_{0x} + q_y\nu_{0y} = 0$, with order $|q_x| + |q_y| + 1$
- The same condition can be obtained in the vertical plane
- For all the polynomial field terms of a $2n$-pole, the **main** excited resonances satisfy the condition $q_x' + q_y = n$ but there are also **sub-resonances** for which $q_x' + q_y < n$
- For **normal** (erect) multi-poles, the main resonances are $(q_x', q_y) = (n, 0),\ (n-2, \pm 2), \ldots$ whereas for **skew** multi-poles $(q_x', q_y) = (n-1, \pm 1),\ (n-3, \pm 3), \ldots$

General resonance conditions

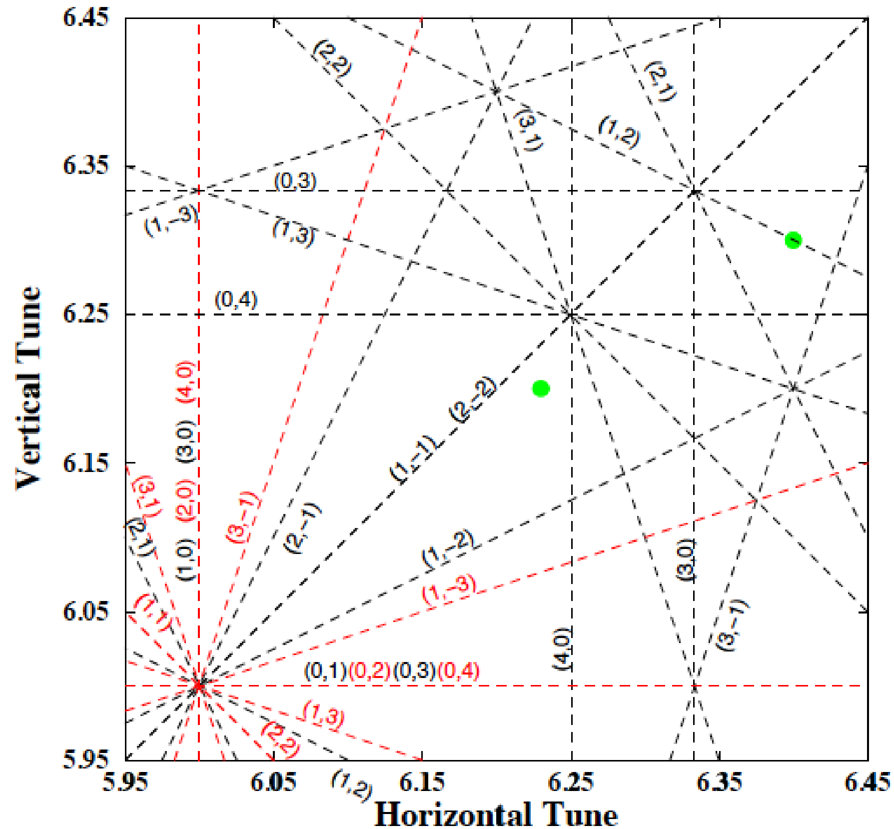
- The general resonance conditions is $m + q_x \nu_{0x} + q_y \nu_{0y} = \nu_{0x}$ or $m + q'_x \nu_{0x} + q_y \nu_{0y} = 0$, with order $|q_x| + |q_y| + 1$
- The same condition can be obtained in the vertical plane
- For all the polynomial field terms of a $2n$-pole, the **main** excited resonances satisfy the condition $q'_x + q_y = n$ but there are also **sub-resonances** for which $q'_x + q_y < n$
- For **normal** (erect) multi-poles, the main resonances are $(q'_x, q_y) = (n, 0), \ (n-2, \pm 2), \ldots$ whereas for **skew** multi-poles $(q'_x, q_y) = (n-1, \pm 1), \ (n-3, \pm 3), \ldots$
- If perturbation is large, **all** resonances can be potentially excited
- The **resonance conditions form lines** in frequency space and fill it up as the **order grows** (the rational numbers form a dense set inside the real numbers), but Fourier amplitudes should



- If lattice is made out of N **identical cells**, and the perturbation follows the same periodicity, resulting in a reduction of the resonance conditions to the ones satisfying $q_x\nu_{0x} + q_y\nu_{0y} = jN$
- These are called **systematic** resonances

Systematic and random resonances

- If lattice is made out of N **identical cells**, and the perturbation follows the same periodicity, resulting in a reduction of the resonance conditions to the ones satisfying $q_x \nu_{0x} + q_y \nu_{0y} = jN$
- These are called **systematic** resonance
- Practically, any (linear) lattice perturbation breaks super-periodicity and any **random** resonance can be excited
- Careful **choice** of the **working point** is necessary



Contents of the 1st lecture

- Accelerator performance parameters and non-linear effects
- Linear and non-linear oscillators
  - Integral and frequency of motion
  - The pendulum
  - Damped harmonic oscillator
- Phase space dynamics
  - Fixed point analysis
- Non-autonomous systems
  - Driven harmonic oscillator, resonance conditions
- Linear equations with periodic coefficients – Hill's equations
  - Floquet solutions and normalized coordinates
- Perturbation theory
  - Non-linear oscillator
  - Perturbation by periodic function – single dipole perturbation
  - Application to single multipole – resonance conditions
  - Examples: single quadrupole, sextupole, octupole perturbation
  - General multi-pole perturbation– example: linear coupling
  - Resonance conditions and working point choice
- Summary
- Appendix: Damped harmonic oscillator

Summary

- **Accelerator performance** depends heavily on the understanding and control of **non-linear effects**
- The ability to **integrate differential equations** has a deep impact to the **dynamics** of the system
- **Phase space** is the natural space to study this dynamics
- **Perturbation theory** helps integrate iteratively differential equations and reveals appearance of resonances
- **Periodic perturbations** drive **infinite number** of resonances
- There is an **amplitude dependent tune-shift** at 1^{st} order for **even multi-poles**
- Periodicity of the lattice very important for reducing number of lines excited at first order

Damped harmonic oscillator I

- Damped harmonic oscillator:

$$\frac{d^2u(t)}{dt^2} + \frac{\omega_0}{Q}\frac{du(t)}{dt} + \omega_0^2 u(t) = 0$$

  - $Q = \frac{1}{2\zeta}$ is the ratio between the stored and lost energy per cycle with ζ the damping ratio
  - ω_0 is the eigen-frequency of the harmonic oscillator

- General solution can be found by the same ansatz

$$u(t) = e^{\lambda t}$$

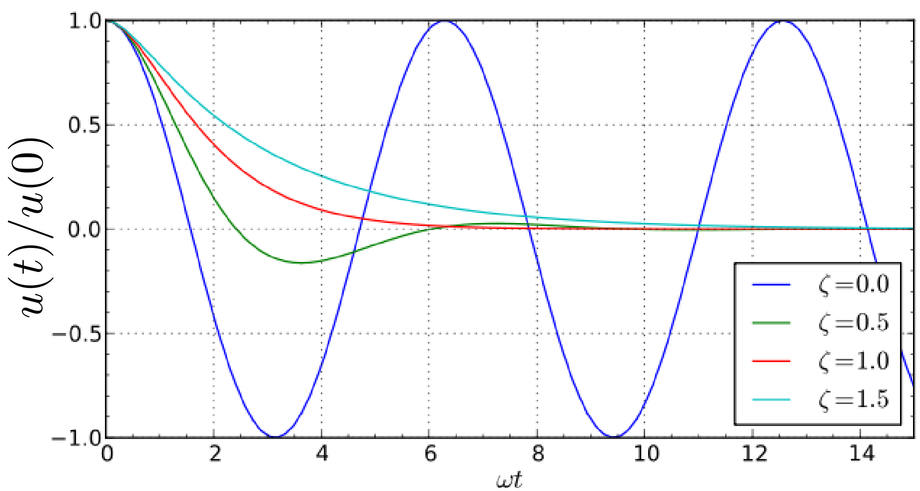
leading to an auxiliary 2nd order equation

$\lambda^2 + \frac{\omega_0}{Q}\lambda + \omega_0^2 = 0$ with solutions

$$\lambda_\pm = -\frac{\omega_0}{2Q}(-1 \pm \sqrt{1-4Q^2}) = -\omega_0\zeta(-1 \pm \sqrt{1-\frac{1}{\zeta^2}})$$

Damped harmonic oscillator II

- Three cases can be distinguished
  - *Overdamping* (λ real, i.e. $\zeta > 1$ or $Q < 1/2$): The system exponentially decays to equilibrium (slower for larger damping ratio values)
  - *Critical damping* ($\zeta = 1$): The system returns to equilibrium as quickly as possible without oscillating.
  - *Underdamping* (λ complex, i.e. $\zeta < 1$ or $Q > 1/2$): The system oscillates with the amplitude gradually decreasing to zero, with a slightly different frequency than $\omega_d = \omega_0\sqrt{1-\zeta^2}$



- Note that there is no integral of motion, in that case, as the energy is not conserved (dissipative system)

Damped oscillator with periodic driving

- Consider periodic force pumping energy into the system

$$\frac{d^2u(t)}{dt^2} + \frac{\omega_0}{Q}\frac{du(t)}{dt} + \omega_0^2 u(t) = \frac{F}{m}\cos(\omega t)$$

- The solution of the **homogeneous** system is

$$u_h(t) = u_0(t)e^{-\omega_0\zeta t}\sin(\omega_0\sqrt{1-\zeta^2}\,t + \phi_0)$$

Damped oscillator with periodic driving

- Consider periodic force pumping energy into the system

$$\frac{d^2u(t)}{dt^2} + \frac{\omega_0}{Q}\frac{du(t)}{dt} + \omega_0^2 u(t) = \frac{F}{m}\cos(\omega t)$$

- The solution of the **homogeneous** system is

$$u_h(t) = u_0(t) e^{-\omega_0 \zeta t} \sin(\omega_0 \sqrt{1-\zeta^2}\, t + \phi_0)$$

- The particular solution is

$$u_p(t) = \frac{F\cos(\omega t + \phi_0')}{m\,\omega_0^2 \sqrt{(1-\frac{\omega^2}{\omega_0^2})^2 + 4\zeta^2 \frac{\omega^2}{\omega_0^2}}}$$

- The **homogeneous solution vanishes** for $t \to \infty$, leaving only the particular one, for which there is an **amplitude maximum** for $\omega_0 = \omega$ but no divergence
- In that case, the energy pumped into the system compensates the friction, and a **steady state** is reached representing a **limit cycle**