Summary

integrated dipole field over a turn	$\int B dl \approx N L_{\text{Bend}} B = 2\pi \frac{P_0}{q}$
transfer matrix of a FODO cell	$M_{\text{FODO}} = \begin{pmatrix} 1 + \frac{L}{2f} & L + \frac{L^2}{4f} \\ -\frac{2L}{f^2} & 1 - \frac{L}{2f} - \frac{L^2}{4f^2} \end{pmatrix}$
stability in a FODO cell	$f > L/4$
phase advance in a FODO cell	$\mu = \arccos\left(\frac{1}{2}\text{trace}\,(M)\right)$
matching sections provide	$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_s = M_{3\times 3} \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_0$

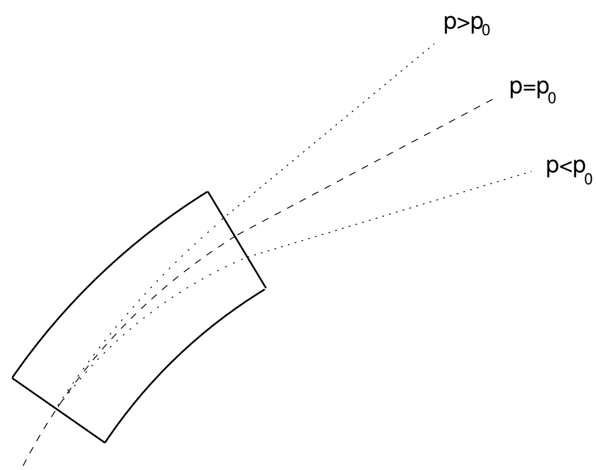
Part 4.

Dispersion

Introducing dispersion: $D(s)$

So far we have studied monochromatic beams of particles, but this is slightly unrealistic: We always have some (small?) momentum spread among all particles:
$\Delta P = P - P_0 \neq 0$.

Consider three particles with P respectively: less than, greater than, and equal to P_0 , traveling through a dipole. Remembering $B\rho = \frac{P}{q}$:



The system introduces a correlation of momentum with transverse position. This correlation is known as **dispersion** (an intrinsic property of the dipole magnets).

The Inhomogeneous Hill's equation

Let's go back to the magnetic rigidity. If $P \neq P_0$ (define $\delta = \frac{P-P_0}{P_0} = \frac{\Delta P}{P_0}$) we can work out how the bending radius ρ depends on the particle momentum, w.r.t. ρ_0:

$$\Rightarrow B\rho = \frac{P}{q} \quad = \frac{P_0(1+\delta)}{q} = B\rho_0(1+\delta) \qquad \Rightarrow \quad \rho = \rho_0(1+\delta).$$

When we derived the equation of motion at some point we had (slide 18):

$$\underbrace{x''}_{\text{term 1}} - \underbrace{\frac{1}{\rho+x}}_{\text{term 2}} = -\frac{B_y}{P/q} \quad \text{that later became: } x'' + \left(\frac{1}{\rho^2}+k\right)x = 0$$

On the way we had "Taylor expanded" term 2: $\quad \dfrac{1}{\rho+x} \approx \dfrac{1}{\rho}\left(1-\dfrac{x}{\rho}\right).$

Now we need to redo it for ρ as $\rho_0(1+\delta)$: $\quad \dfrac{1}{\rho+x} = \dfrac{1}{\rho_0(1+\delta)+x} \approx \dfrac{1}{\rho_0}\left(1-\dfrac{x}{\rho_0}-\delta\right)$

and the equation of motion becomes:

$$x'' + \left(\frac{1}{\rho_0^2}+k\right)x - \frac{\delta}{\rho_0} = 0.$$

If we drop the suffix 0 and explicit δ, this is "the inhomogeneous Hill's equation":

$$\boxed{x'' + \left(\frac{1}{\rho^2}+k\right)x = \frac{1}{\rho}\frac{\Delta P}{P_0}}$$

Solution of the inhomogeneous Hill's equation

A particle with $\Delta P = P - P_0 \neq 0$ satisfies the inhomogeneous Hill equation for the horizontal motion:

$$x''(s) + K(s)\,x(s) = \frac{1}{\rho}\frac{\Delta P}{P_0}$$

the total deviation of the particle from the reference orbit can be written as

$$x(s) = x_\beta(s) + x_D(s)$$

where:

- $x_\beta(s)$ describes the betatron oscillation around the new closed orbit, and it's the solution of the homogeneous equation $x''_\beta(s) + K(s)\,x_\beta(s) = 0$
- $x_D(s)$ describes the deviation of the closed orbit for an off-momentum particle with $P = P_0 + \Delta P$. It is rewritten as $x_D(s) = D(s)\,\frac{\Delta P}{P_0}$, where $D(s)$ is the solution of the equation

$$D''(s) + K(s)\,D(s) = \frac{1}{\rho}$$

$D(s)$ is the *dispersion function*.

Dispersion function and orbit

The dispersion function $D(s)$ is the solution of the inhomogeneous Hill's equation:

$$D''(s) + K(s)\,D(s) = \frac{1}{\rho}$$

$D(s)$:

- is that special orbit that an ideal particle would have for $\Delta P/P_0 = 1$
- It can be proved that the solution is:

$$D(s) = S(s)\int_0^s \frac{1}{\rho(t)} C(t)\,\mathrm{d}t - C(s)\int_0^s \frac{1}{\rho(t)} S(t)\,\mathrm{d}t$$

Once one knows $D(s)$, the orbit $x(s) = x_\beta(s) + x_D(s)$, with $x_D(s) = D(s)\,\frac{\Delta P}{P_0}$, can be rewritten as

$$\begin{aligned} x(s) &= x_\beta(s) + x_D(s) \\ &= C(s)\,x_0 + S(s)\,x'_0 + D(s)\,\frac{\Delta P}{P_0} \end{aligned}$$

Dispersion function and orbit

The equation of motion:

$$\begin{aligned} x(s) &= x_\beta(s) + x_D(s) \\ &= C(s)\,x_0 + S(s)\,x_0' + D(s)\,\frac{\Delta P}{P_0} \end{aligned}$$

can be written in matrix form:

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_0 + \frac{\Delta P}{P_0} \begin{pmatrix} D \\ D' \end{pmatrix}_0$$

Or, in a more compact way:

$$\begin{pmatrix} x \\ x' \\ \Delta P/P_0 \end{pmatrix}_s = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \\ \Delta P/P_0 \end{pmatrix}_0$$

Dispersion function and orbit

We need to study the motion for particles with $\Delta P = P - P_0 \neq 0$:

$$x''(s) + K(s)\,x(s) = \frac{1}{\rho}\frac{\Delta P}{P_0}$$

The general solution of this equation is:

$$x(s) = x_\beta(s) + x_D(s) \qquad \begin{cases} x_\beta''(s) + K(s)\,x_\beta(s) = 0 \\ D''(s) + K(s)\,D(s) = \dfrac{1}{\rho} \end{cases}$$

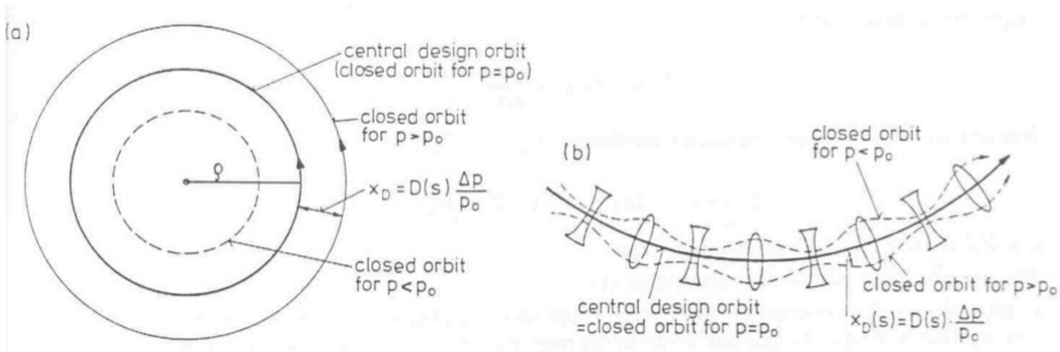
with $x_D(s) = D(s)\,\frac{\Delta P}{P_0}$.

Remarks

- $D(s)$ is that special orbit that a particle would have for $\Delta P/P_0 = 1$
- $x_D(s)$ describes the deviation of the new **closed orbit** for an off-momentum particle with a certain ΔP
- the orbit of a generic particle is the sum of the well known $x_\beta(s)$ and $x_D(s)$

Closed orbit of off-momentum particles

Orbit $x(s) = x_\beta(s) + D(s)\frac{\Delta P}{P_0}$.



Closed orbit for particles with momentum $P \neq P_0$ in
a weakly (a) and strongly (b) focusing circular accelerator.

- $x_D(s)$ describes the deviation from the reference orbit of an off-momentum particle with $P = P_0 + \Delta P$
- $x_\beta(s)$ describes the betatron oscillation around the orbit $x_D(s)$

Dispersion and orbit propagation

The dispersion orbit is solution of $D''(s) + K(s)D(s) = \frac{1}{\rho}$:

$$D(s) = S(s)\int_0^s \frac{1}{\rho(t)}C(t)\,\mathrm{d}t - C(s)\int_0^s \frac{1}{\rho(t)}S(t)\,\mathrm{d}t$$

Now the orbit:

$$x(s) = x_\beta(s) + x_D(s)$$
$$x(s) = C(s)x_0 + S(s)x'_0 + D(s)\frac{\Delta P}{P_0}$$

In matrix form
$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix}\begin{pmatrix} x \\ x' \end{pmatrix}_0 + \frac{\Delta P}{P_0}\begin{pmatrix} D \\ D' \end{pmatrix}_0$$

We can rewrite the solution in matrix form:

$$\begin{pmatrix} x \\ x' \\ \Delta P/P_0 \end{pmatrix}_s = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ x' \\ \Delta P/P_0 \end{pmatrix}_0$$

Exercise: show that $D(s)$ is a solution for the equation of motion, with the initial conditions $D_0 = D'_0 = 0$.

Examples of dispersion function

Let's study, for different magnetic elements, the solution of:

$$D(s) = S(s)\int_0^s \frac{1}{\rho(t)} C(t)\,\mathrm{d}t - C(s)\int_0^s \frac{1}{\rho(t)} S(t)\,\mathrm{d}t$$

at the exit of the element: that is, $D(s)$ with $s = L_{\text{magnet}}$

- Drift space:

$$M_{\text{Drift}} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$

$C(t) = 1,\ S(t) = L,\ \rho(t) = \infty \qquad \Rightarrow$ the integrals cancel

$$M_{\text{Drift}} = \begin{pmatrix} 1 & L & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Dispersion function in a sector dipole

- Sector dipole:
  $K = \frac{1}{\rho^2}$:

$$M_{\text{Dipole}} = \begin{pmatrix} \cos\left(\sqrt{K}L\right) & \frac{1}{\sqrt{K}}\sin\left(\sqrt{K}L\right) \\ -\sqrt{K}\sin\left(\sqrt{K}L\right) & \cos\left(\sqrt{K}L\right) \end{pmatrix} = \begin{pmatrix} \cos\frac{L}{\rho} & \rho\sin\frac{L}{\rho} \\ -\frac{1}{\rho}\sin\frac{L}{\rho} & \cos\frac{L}{\rho} \end{pmatrix}$$

which gives

$$D(L) = \rho\left(1 - \cos\frac{L}{\rho}\right)$$

$$D'(L) = \sin\frac{L}{\rho}$$

therefore

$$M_{\text{Dipole}} = \begin{pmatrix} \cos\frac{L}{\rho} & \rho\sin\frac{L}{\rho} & \rho\left(1-\cos\frac{L}{\rho}\right) \\ -\frac{1}{\rho}\sin\frac{L}{\rho} & \cos\frac{L}{\rho} & \sin\frac{L}{\rho} \\ 0 & 0 & 1 \end{pmatrix}$$

Notice: $\frac{L}{\rho} = \phi$ is the bending angle.

Dispersion function in a quadrupole

- Focusing quadrupole, $K > 0$:

$$M_{\mathrm{QF}} = \begin{pmatrix} \cos\left(\sqrt{K}L\right) & \frac{1}{\sqrt{K}}\sin\left(\sqrt{K}L\right) & 0 \\ -\sqrt{K}\sin\left(\sqrt{K}L\right) & \cos\left(\sqrt{K}L\right) & 0 \\ 0 & 0 & 1 \end{pmatrix};$$

- Defocusing quadrupole, $K < 0$:

$$M_{\mathrm{QD}} = \begin{pmatrix} \cosh\left(\sqrt{|K|}L\right) & \frac{1}{\sqrt{|K|}}\sinh\left(\sqrt{|K|}L\right) & 0 \\ \sqrt{|K|}\sinh\left(\sqrt{|K|}L\right) & \cosh\left(\sqrt{|K|}L\right) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Dispersion propagation through the lattice

- The equation:

$$D(s) = S(s)\int_0^s \frac{1}{\rho(t)} C(t)\,\mathrm{d}t - C(s)\int_0^s \frac{1}{\rho(t)} S(t)\,\mathrm{d}t$$

allows to compute **the dispersion inside a magnet**, which does not depend on the dispersion that might have been generated by the upstreams magnets.

- At the exit of a magnet of length L_{m} the dispersion reaches the value $D(L_{\mathrm{m}})$
- The dispersion (also indicated as η, with its derivative η') propagates from there, through the rest of the machine, just like any other particle:

$$\begin{pmatrix} \eta \\ \eta' \\ 1 \end{pmatrix}_s = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \eta \\ \eta' \\ 1 \end{pmatrix}_0$$

Periodic dispersion

In a periodic lattice, also the dispersion must be periodic.

That is, for $\begin{pmatrix} \eta \\ \eta' \\ 1 \end{pmatrix}$ we need to have:

$$\begin{pmatrix} \eta \\ \eta' \\ 1 \end{pmatrix} = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \eta \\ \eta\prime \\ 1 \end{pmatrix}$$

Let's rewrite this in 2×2 form:

$$\begin{pmatrix} \eta \\ \eta' \end{pmatrix} = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} \begin{pmatrix} \eta \\ \eta' \end{pmatrix} + \begin{pmatrix} D \\ D' \end{pmatrix}$$

$$\begin{pmatrix} 1-C & -S \\ -C' & 1-S' \end{pmatrix} \begin{pmatrix} \eta \\ \eta' \end{pmatrix} = \begin{pmatrix} D \\ D' \end{pmatrix}$$

The solution is:

$$\begin{pmatrix} \eta \\ \eta' \end{pmatrix} = \frac{1}{(1-C)(1-S')-C'S} \begin{pmatrix} 1-S' & S \\ C' & 1-C \end{pmatrix} \begin{pmatrix} D \\ D' \end{pmatrix}$$

Dispersion function in a FODO lattice

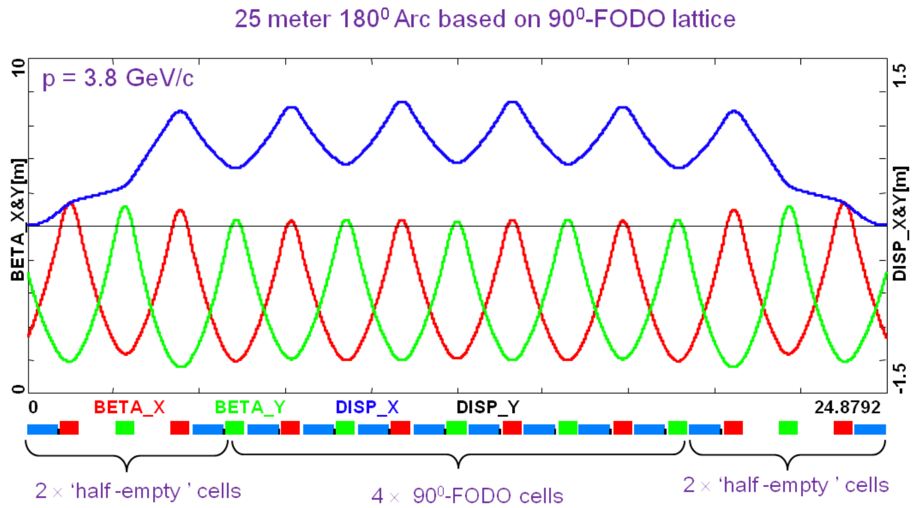
The dispersion function in a FODO cell is a periodic function with maxima at the focusing quadrupoles and minima at the defocusing quadrupoles:

$$D^{\pm} = \frac{L\phi\left(1 \pm \frac{1}{2}\sin\frac{\mu}{2}\right)}{4\sin^2\frac{\mu}{2}}$$

where:

- L is the total length of the cell
- ϕ is the total bending angle of the cell
- μ is the phase advance of the cell

Example of dispersion function in a FODO lattice



Aperture radius: r = 15 cm

12 × Dipoles:	field: 3.9 Tesla	length: 85 cm
15 × Quads:	gradient : 25 Tesla/m (3.8 Tesla at the pole)	length: 50 cm

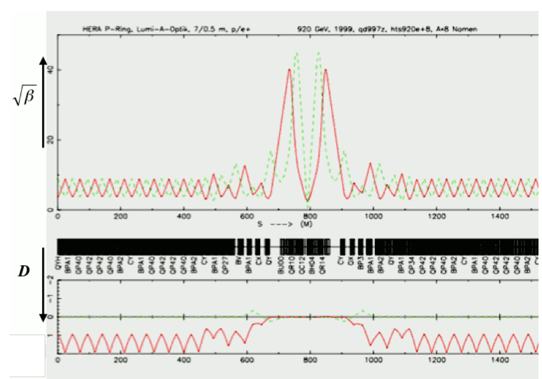
Impact of dispersion on the beam size

In this example from the HERA storage ring (DESY) we see the Twiss parameters and the dispersion near the interaction point. In the periodic region,

$$x_\beta(s) = 1\dots2\ \text{mm}$$
$$D(s) = 1\dots2\ \text{m}$$
$$\Delta P/P_0 \approx 1\cdot 10^{-3}$$

Remember:

$$x(s) = x_\beta(s) + D(s)\,\frac{\Delta P}{P_0}$$



Beware: the dispersion contributes to the beam size:

$$\sigma_x = \sqrt{\sigma_{x_\beta}^2 + \text{std}\left(D\cdot\frac{\Delta P}{P_0}\right)^2} = \sqrt{\epsilon_{\text{geometric}}\cdot\beta + D^2\cdot\frac{\sigma_P^2}{P_0^2}}$$

- We need to suppress the dispersion at the IP !
- We need a special insertion section: a *dispersion suppressor*
- Remember: $\epsilon_{\text{geometric}} = \dfrac{\epsilon_{\text{normalised}}}{\beta_{\text{rel}}\gamma_{\text{rel}}}$

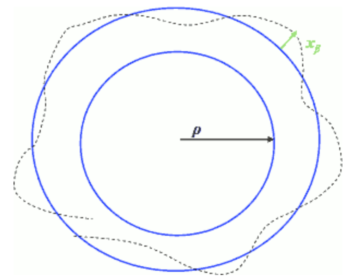
The momentum compaction factor

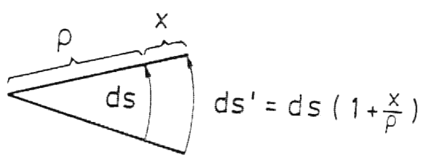
The dispersion function relates the momentum error of a particle to the horizontal orbit coordinate

The general solution of the equation of motion is

$$x(s) = x_\beta(s) + D(s)\frac{\Delta P}{P_0}$$

The dispersion changes also the length of the off-energy orbit.





particle with offset x w.r.t. the design orbit:

$$\frac{ds'}{ds} = \frac{\rho + x}{\rho} \quad \rightarrow \quad ds' = \left(1 + \frac{x}{\rho}\right) ds$$

The circumference change is ΔC, that is $C' = \oint \left(1 + \frac{x}{\rho}\right) ds = C + \Delta C$

We define the "momentum compaction factor" α_P, such that:

$$\frac{\Delta C}{C} = \alpha_P \frac{\Delta P}{P_0} \qquad \rightarrow \text{to the lowest order in } \Delta P/P_0: \quad \alpha_P = \frac{1}{C}\oint \frac{D(s)}{\rho} ds \approx \frac{1}{Q_x^2}$$

Summary

inhomogeneous Hill's equation $\quad x'' + K(s)\,x = \frac{1}{\rho}\frac{\Delta P}{P_0}$

...and its solution $\quad x(s) = x_\beta(s) + D(s)\frac{\Delta P}{P_0}$

dispersion function $\quad D(s)$

Part 5.

Imperfections

Fringe fields

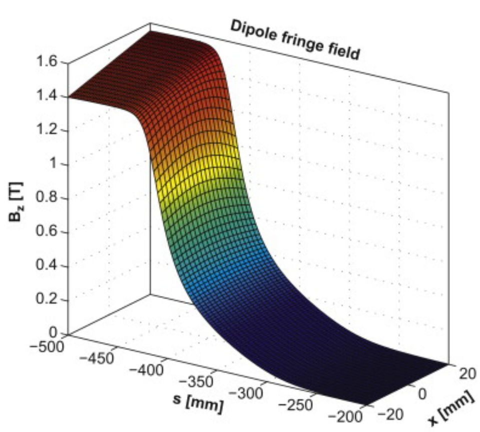
- Hard-edge model:

$$x''(s) + \left(\frac{1}{\rho^2} + k\right) x(s) = 0$$

this equation is not really correct (because it violates the Maxwell equations at the magnet edges!)

- Bending and focusing forces -even inside a magnet- depend on the position s

$$x''(s) + \left\{\frac{1}{\rho^2(s)} + k(s)\right\} x(s) = 0$$



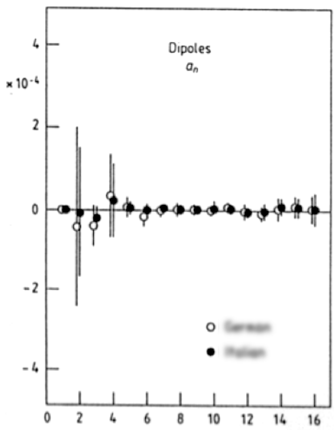
Fringe field of a dipole magnet (in this case: a combined dipole + quadrupole magnet, notice the slope of the field along the x axis)

Magnetic imperfections

High-order multipolar components and misalignments

Taylor expansion of the B field:

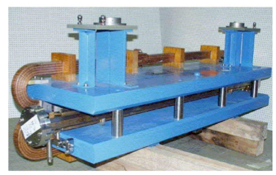
$$B_y(x) = \underbrace{B_{y0}}_{\text{dipole}} + \underbrace{\frac{\partial B_y}{\partial x}}_{\text{quad}} x + \underbrace{\frac{1}{2}\frac{\partial^2 B_y}{\partial x^2}}_{\text{sextupole}} x^2 + \underbrace{\frac{1}{3!}\frac{\partial^3 B_y}{\partial x^3}}_{\text{octupole}} x^3 + \dots \qquad \text{divide by } B_{y0}$$



There can be undesired multipolar components, due to small fabrication defects

Or also errors in the windings, in the gap h, ...

remember: $B = \dfrac{\mu_0 n I}{h}$



Moreover: **"feed-down"** effect $\Rightarrow$ a misalign magnet of order n, behaves like a magnet of order n, plus a magnet of order $n-1$ overlapped

Dipole magnet errors

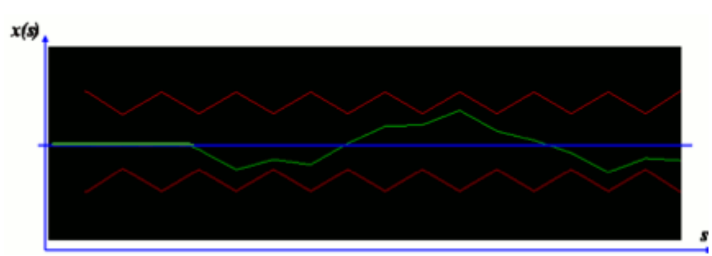
Let's imagine to have a magnet with $B = B_0 + \Delta B$. This will give an additional kick to each particle, and will distort the ideal design orbit

$$F_x = ev\,(B_0 + \Delta B)\,; \qquad \Delta x' = \Delta B ds / B\rho$$

A dipole error will cause a distortion of the closed orbit, that will „run around" the storage ring, being observable everywhere. If the distortion is small enough, it will still lead to a closed orbit.

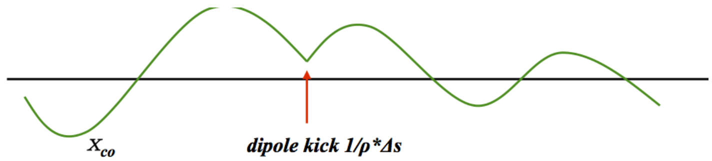
Example: 1 single dipole error

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = M_{\text{lattice}} \begin{pmatrix} 0 \\ \Delta x' \end{pmatrix}_0$$



In order to have bounded motion the tune Q must be non-integer, $Q \neq 1$. We see that even for particles with reference momentum P_0 an integer Q value is forbidden, since small field errors are always present.

Orbit distortion for a single dipole field error



We consider a single thin dipole field error at the location $s = s_0$, with a kick angle $\Delta x'$.

$$X_- = \begin{pmatrix} x_0 \\ x_0' + \Delta x' \end{pmatrix}, \quad X_+ = \begin{pmatrix} x_0 \\ x_0' \end{pmatrix}$$

are the phase space coordinates before and after the kick located at s_0. The closed-orbit condition becomes

$$M_{\text{Lattice}} \begin{pmatrix} x_0 \\ x_0' \end{pmatrix} = \begin{pmatrix} x_0 \\ x_0' + \Delta x' \end{pmatrix}$$

The resulting closed orbit at s_0 is

$$x_0 = \frac{\beta_0 \Delta x'}{2 \sin \pi Q} \cos \pi Q; \quad x_0' = \frac{\Delta x'}{2 \sin \pi Q} (\sin \pi Q - \alpha_0 \cos \pi Q)$$

where Q is the tune. The orbit at any other location s is

$$x(s) = \frac{\sqrt{\beta_s \beta_0} \Delta x'}{2 \sin \pi Q} \cos (\pi Q - |\mu_s - \mu_0|)$$

(see the references for a demonstration)

Orbit distortion for distributed dipole field errors

One single dipole field error

$$x(s) = \frac{\sqrt{\beta_s \beta_0} \Delta x'}{2 \sin \pi Q} \cos (\pi Q - |\mu_s - \mu_0|)$$

Distributed dipole field errors

$$x(s) = \frac{\sqrt{\beta_s}}{2 \sin \pi Q} \sum_i \sqrt{\beta_i} \Delta x_i' \cos (\pi Q - |\mu_s - \mu_i|)$$

- orbit distortion is visible at any position s in the ring, even if the dipole error is located at one single point s_0
- the β function describes the sensitivity of the beam to external fields
- the β function acts as amplification factor for the orbit amplitude at the given observation point
- there is a singularity at the denominator when Q integer $\Rightarrow$ it's called resonance

Quadrupole errors: tune shift

Orbit perturbation described by a thin lens quadrupole:

$$M_{\text{Perturbed}} = \underbrace{\begin{pmatrix} 1 & 0 \\ \Delta k \mathrm{d}s & 1 \end{pmatrix}}_{\text{perturbation}} \underbrace{\begin{pmatrix} \cos\mu_0 + \alpha \sin\mu_0 & \beta \sin\mu_0 \\ -\gamma \sin\mu_0 & \cos\mu_0 - \alpha \sin\mu_0 \end{pmatrix}}_{\text{ideal ring}}$$

Let's see how the tunes changes: one-turn map

$$M_{\text{Perturbed}} = \begin{pmatrix} \cos\mu_0 + \alpha \sin\mu_0 & \beta \sin\mu_0 \\ \Delta k \mathrm{d}s \left(\cos\mu_0 + \alpha \sin\mu_0\right) - \gamma \sin\mu_0 & \Delta k \mathrm{d}s \beta \sin\mu_0 + \cos\mu_0 - \alpha \sin\mu_0 \end{pmatrix}$$

Remember the rule for computing the tune:

$$2\cos\mu = \mathrm{trace}\,(M) = 2\cos\mu_0 + \Delta k \mathrm{d}s \beta \sin\mu_0$$

Quadrupole errors: tune shift (cont.)

We rewrite $\cos\mu = \cos\left(\mu_0 + \Delta\mu\right)$

$$\cos\left(\mu_0 + \Delta\mu\right) = \cos\mu_0 + \frac{1}{2}\Delta k \mathrm{d}s \beta \sin\mu_0$$

from which we can compute that

$$\Delta\mu = \frac{\Delta k\, \mathrm{d}s\, \beta}{2} \quad \text{shift in the phase advance}$$

$$\Delta Q = \oint_{\text{quads}} \frac{\Delta k\left(s\right)\beta\left(s\right)\mathrm{d}s}{4\pi} \quad \text{tune shift}$$

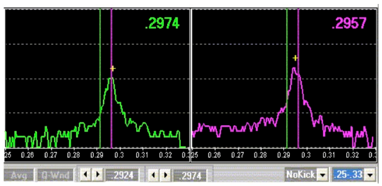
Important remarks:

- the tune shift if proportional to the β-function at the location of the quadrupole
  - field quality, power supply tolerances etc. are much tighter at places where β is large
- β is a measurement of the sensitivity of the beam

Quadrupole errors: tune shift example

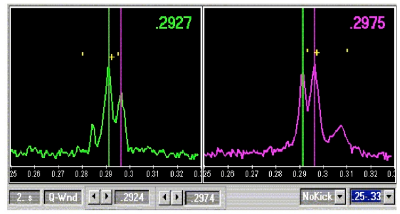
Deliberate change of a quadrupole strength in a synchrotron:

$$\Delta Q = \oint_{\text{quads}} \frac{\Delta K\,(s)\,\beta\,(s)\,\mathrm{d}s}{4\pi} \approx \frac{\Delta K\,(s)\;L_{\text{quad}}\,\overline{\beta}}{4\pi}$$



The tune is measured permanently

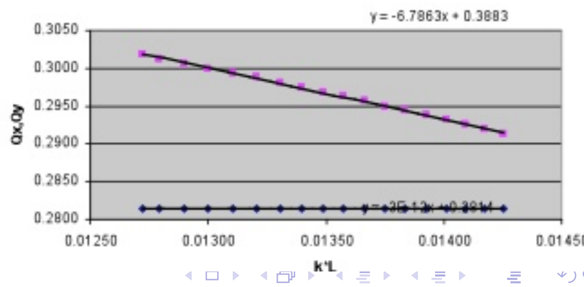
$\Rightarrow$



We change the strength of "trim" quads to fix Q

Horizontal axis is a scan of K_1 (quad integrated focusing strength):

- tune shift is proportional to β through $\Delta Q \propto \Delta K \cdot \beta$
- En passant, we use this to measure β.



Tune shift correction

Errors in the quadrupole fields induce tune shift:

$$\Delta Q = \oint_{\text{quads}} \frac{\Delta k\,(s)\,\beta\,(s)\,\mathrm{d}s}{4\pi}$$

Cure: we compensate the quad errors using other (correcting) quadrupoles

- If you use only one correcting quadrupole, with $1/f = \Delta k_1 L$
  - it changes both Q_x and Q_y:

$$\Delta Q_x = \frac{\beta_{1x}}{4\pi f_1} \quad \text{and} \quad \Delta Q_y = -\frac{\beta_{1y}}{4\pi f_1}$$

- We need to use two independent correcting quadrupoles:

$$\Delta Q_x = \frac{\beta_{1x}}{4\pi f_1} + \frac{\beta_{2x}}{4\pi f_2} \qquad \begin{pmatrix} \Delta Q_x \\ \Delta Q_y \end{pmatrix} = \frac{1}{4\pi} \begin{pmatrix} \beta_{1x} & \beta_{2x} \\ \beta_{1y} & \beta_{2y} \end{pmatrix} \begin{pmatrix} 1/f_1 \\ 1/f_2 \end{pmatrix}$$

$$\Delta Q_y = -\frac{\beta_{1y}}{4\pi f_1} - \frac{\beta_{2y}}{4\pi f_2}$$

- Solve by inversion:

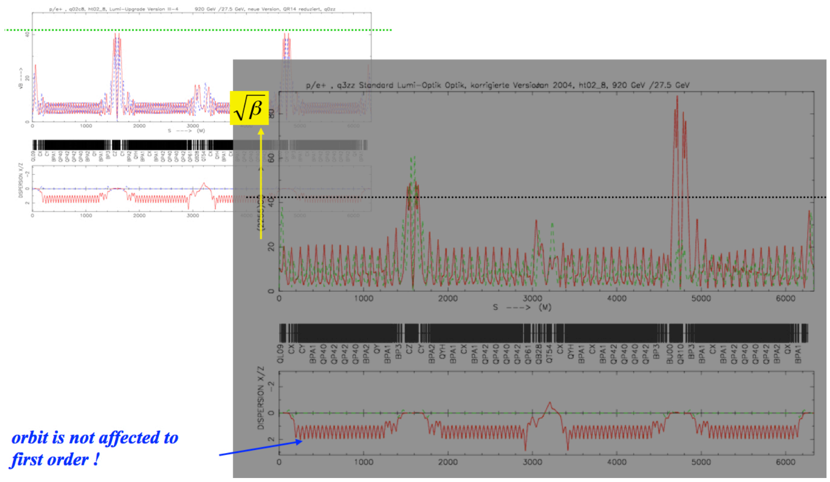
$$\begin{pmatrix} 1/f_1 \\ 1/f_2 \end{pmatrix} = \frac{4\pi}{\beta_{1x}\beta_{2y} - \beta_{2x}\beta_{1y}} \begin{pmatrix} \beta_{2y} & -\beta_{2x} \\ -\beta_{1y} & \beta_{1x} \end{pmatrix} \begin{pmatrix} \Delta Q_x \\ \Delta Q_y \end{pmatrix}$$

Quadrupole errors: beta beat

A quadrupole error at s_0 causes distortion of β-function at s: $\Delta\beta(s)$ due to the errors of all quadrupoles:

$$\frac{\Delta\beta_s}{\beta_s} = \frac{1}{2\sin 2\pi Q}\sum_i \beta_i \Delta k_i \cos\left(2\pi Q - 2\left(\mu_i - \mu_s\right)\right)$$

Note: Unstable betatron motion if tune is half integer!

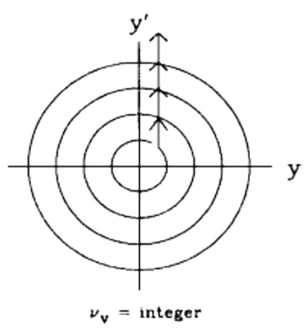


This imperfection can be corrected with an appropriate distribution of tuneable sextupoles.

Tunes and resonances

The particles – oscillating under the influence of the external magnetic fields – can be excited in case of resonant tunes to infinite high amplitudes.

There is particle loss within a short number of turns.



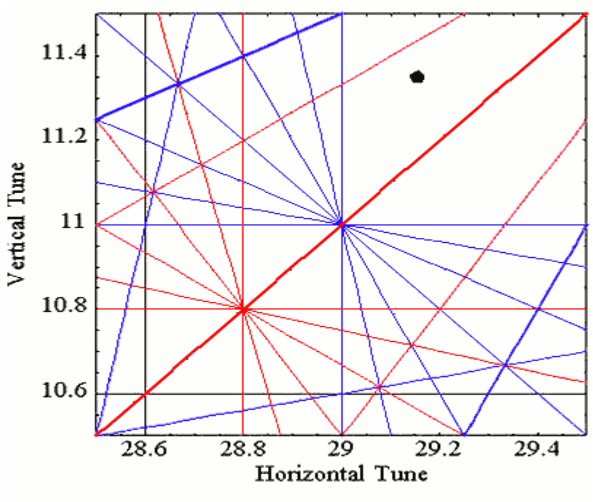
The cure:

1. **avoid** large magnet errors
2. **avoid forbidden tune values** in both planes

$$m\cdot Q_x + n\cdot Q_y \neq p$$

with m, n, p integer numbers

Resonance diagram

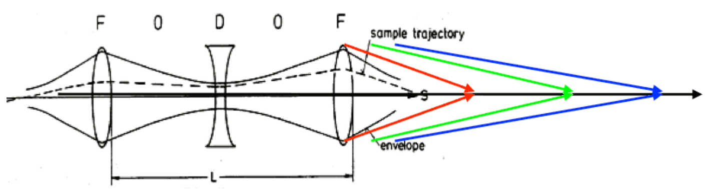


$\mathrm{m}\cdot Q_x + \mathrm{n}\cdot Q_y \neq \mathrm{p}$ where $|m|+|n|$ is the order of the resonance

A resonance diagram for the Diamond light source. The lines shown are the resonances and the black dot shows a suitable place where the machine could be operated.

Quadrupole errors: chromaticity, ξ

Is an error (optical aberration) that happens in quadrupoles when $\Delta P/P_0 \neq 0$:



The chromaticity ξ is the variation of tune ΔQ with the relative momentum error:

$$\Delta Q = \xi \frac{\Delta P}{P_0} \qquad \Rightarrow \qquad \xi = \frac{\Delta Q}{\Delta P/P_0}$$

Remember the quadrupole strength:

$$k = \frac{g}{P/q} \qquad \text{with } P = P_0 + \Delta P = P_0\,(1+\delta)$$

then

$$k = \frac{qg}{P_0 + \Delta P} = \frac{k_0}{1+\delta} \approx \frac{q}{P_0}\left(1 - \frac{\Delta P}{P_0}\right) g = k_0 + \Delta k$$

$$\Delta k = -\frac{\Delta P}{P_0} k_0$$

Quadrupole errors: chromaticity (cont.)

$$\Delta k = -\frac{\Delta P}{P_0} k_0$$

$\Rightarrow$ Chromaticity acts like a quadrupole error and leads to a *tune spread:*

$$\Delta Q_{\text{one quad}} = -\frac{1}{4\pi}\frac{\Delta P}{P_0} k_0 \beta(s)\,\mathrm{d}s \qquad \Rightarrow \Delta Q_{\text{all quads}} = -\frac{1}{4\pi}\frac{\Delta P}{P_0}\oint k(s)\,\beta(s)\,\mathrm{d}s$$

Therefore the definition of chromaticity ξ is

$$\xi = -\frac{1}{4\pi}\oint_{\text{quads}} k(s)\,\beta(s)\,\mathrm{d}s$$

The peculiarity of chromaticity is that it isn't due to external agents, it is generated by the lattice itself!

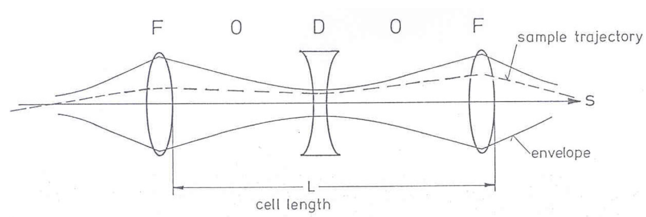
Remarks:

- ξ is a number indicating the size of the tune spot in the working diagram
- ξ is always created by the focusing strength k of **all** quadrupoles
- **natural chromaticity of a focusing quad is always negative**

In other words, because of chromaticity the tune is not a sharp point, but is a **spot**

Example: Chromaticity of the FODO cell

Consider a FODO cells like in figure, with two thin quads, each with focal length f, separated by length $L/2$, and total phase advance μ:



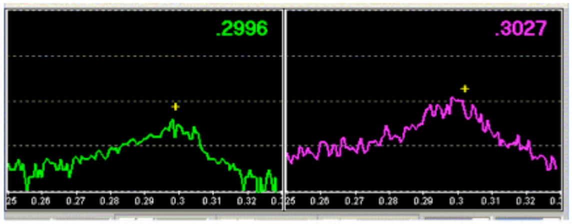
The natural chromaticity ξ_N of the cell is:

$$\begin{aligned}
\xi_N &= -\frac{1}{4\pi}\oint \beta(s)k(s)ds \\
&= -\frac{1}{4\pi}\int_{\text{cell}} \beta(s)\underbrace{k(s)ds}_{\frac{1}{f}} \\
&= -\frac{1}{4\pi}\left[\frac{\beta^+}{f} - \frac{\beta^-}{f}\right]
\end{aligned}$$

$$\begin{aligned}
&= -\frac{1}{4\pi\sin\mu}\left[\left(L + \frac{L^2}{4f}\right)\frac{1}{f} - \left(L - \frac{L^2}{4f}\right)\frac{1}{f}\right] \\
&= -\frac{1}{4\pi\sin\mu}\left[\frac{L}{f} - \frac{L}{f} + \frac{L^2}{2f^2}\right] \\
&= -\frac{1}{8\pi\sin\mu}\frac{L^2}{f^2} \simeq -\frac{1}{\pi}\tan\frac{\mu}{2}
\end{aligned}$$

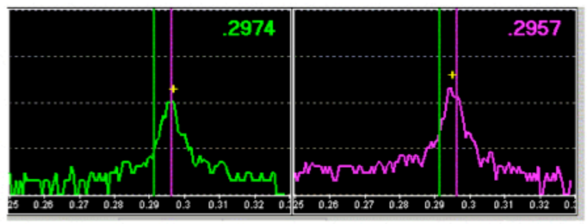
For N_{cell} cells, the total chromaticity is N_{cell} times the chromaticity of each cell $\xi_{N,N_{\text{cell}}} = -\frac{N_{\text{cell}}}{\pi}\tan\frac{\mu}{2}$.

Quadrupole errors: chromaticity



Tune signal for a nearly uncompensated cromaticity (Q' ≈ 20)

Ideal situation: cromaticity well corrected, (Q' ≈ 1)

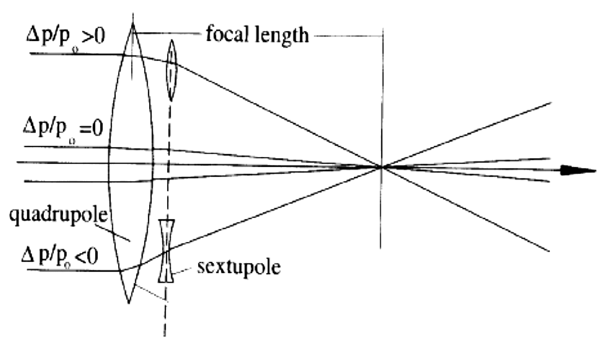


Chromaticity correction

Remember what is chromaticity: the quadrupole focusing experienced by particles changes with energy

- it induces tune shift, which can cause beam lifetime reduction due to resonances

Cure: we need additional, energy-dependent, focusing. This is given by sextupoles



- The sextupole magnetic field rises quadratically:

$$\begin{aligned} B_x &= \tilde{g}xy \\ B_y &= \frac{1}{2}\tilde{g}\left(x^2 - y^2\right) \end{aligned} \quad \Rightarrow \frac{\partial B_x}{\partial y} = \frac{\partial B_y}{\partial x} = \tilde{g}x \quad \text{a "gradient"}$$

it provides a linearly increasing quadrupole gradient

Chromaticity correction (cont.)

Now remember:

- Normalised quadrupole strength is

$$k = \frac{g}{P/q}\ [\mathrm{m}^{-2}]$$

- Sextupoles are characterised by a normalised sextupole strength k_2, which carries a focusing quadrupolar component k_1:

$$k_2 = \frac{\tilde{g}}{P/q}\ [\mathrm{m}^{-3}]; \qquad \tilde{k}_1 = \frac{\tilde{g}x}{P/q}\ [\mathrm{m}^{-2}]$$

Cure for chromaticity: we need sextupole magnets installed in the storage ring in order to increase the focusing strength for particles with larger energy

- A sextupole at a location with dispersion does the trick: $x = D \cdot \frac{\Delta P}{P_0}$

$$\tilde{k}_1 = \frac{\tilde{g}\left(D\frac{\Delta P}{P_0}\right)}{P/q}\ [\mathrm{m}^{-2}]$$

- for $x = 0$ it corresponds to an energy-dependent focal length

$$\frac{1}{f_{\mathrm{sext}}} = \tilde{k}_1 L_{\mathrm{sext}} = \overbrace{\underbrace{\frac{\tilde{g}}{P/q}}_{k_2} \underbrace{D\frac{\Delta P}{P_0}}_{[\mathrm{m}]}}^{\tilde{k}_1} \cdot L_{\mathrm{sext}} = k_2 D \cdot \frac{\Delta P}{P_0} \cdot L_{\mathrm{sext}}$$

Now the formula for the chromaticity rewrites:

$$\xi = \underbrace{-\frac{1}{4\pi}\oint k(s)\,\beta(s)\,\mathrm{d}s}_{\text{chromaticity due to quadrupoles}} + \underbrace{\frac{1}{4\pi}\oint k_2(s)\,D\beta(s)\,\mathrm{d}s}_{\text{chromaticity due to sextupoles}}$$

Design rules for sextupole scheme

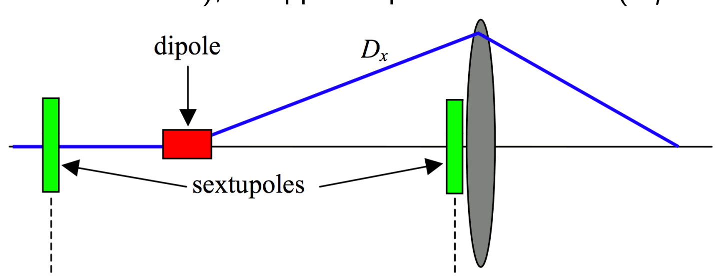
- Chromatic aberrations must be corrected in both planes $\Rightarrow$ you need at least two sextupoles, S_F and S_D (sextupole strengths)
- In each plane the sextupole fields contribute with different signs to the chromaticity ξ_x and ξ_y:

$$\xi_x = -\frac{1}{4\pi}\oint \beta_x(s)\left[\ \ k(s) - S_F D_x(s) + S_D D_x(s)\right] ds$$
$$\xi_y = -\frac{1}{4\pi}\oint \beta_y(s)\left[-k(s) + S_F D_x(s) - S_D D_x(s)\right] ds$$

- To minimise chromatic sextupoles strengths, sextupoles should be located near quadrupoles where $\beta_x D_x$ and $\beta_y D_x$ are large
- For optimal independent chromatic correction S_F should be located where the ratio β_x/β_y is large, S_D where β_y/β_x is large.

Example of chromaticity correction scheme

- *Chromatic aberrations* introduced by quadrupoles are locally cancelled by sextupoles placed near the quadrupoles, in dispersive regions (in straight sections dispersion is generated using an upstream bending magnet)
- Notice that the sextupoles affect also the on-momentum particles: i.e. they introduce geometric *aberrations*. These can be cancelled by adding one additional sextupoles (per each direction), in opposite phase with them ($\Delta\mu = \pi$)



The phase advance between the two sextupoles S_1 and S_2 must be π, so that:

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_1} \rightarrow \underbrace{\begin{matrix}\Delta\mu = \pi \\ \Updownarrow \\ M = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\end{matrix}}_{s_1 \rightarrow s_2} \rightarrow \begin{pmatrix} -x \\ -x' \end{pmatrix}_{s_2}$$

Summary of imperfections

Error	Effect	Cure
fabrication imperfections	unwanted multipolar components	better fabrication / multipolar corrector coils
transverse offsets	"feed-down" effect	better alignment / corrector kickers
roll effects	*couplings $x - y$*	*skew quads*
dipole kicks along the ring	orbit distortion $\propto \beta_{\text{kick location}}$, residual dispersion	corrector kickers
quad field errors	tune shift	trim special quadrupoles
chromaticity	tune spread	design / sextupoles
power supplies	closed orbit distortion tune shift / spread	try to correct / improve power supplies

Summary

orbit for an off-momentum particle	$x(s) = x_\beta(s) + D(s)\frac{\Delta P}{P_0}$
dispersion trajectory	$D(s) = S(s)\int_0^s \frac{1}{\rho(t)} C(t)\,\mathrm{d}t - C(s)\int_0^s \frac{1}{\rho(t)} S(t)\,\mathrm{d}t$
equations of motion with dispersion	$\begin{pmatrix} x \\ x' \\ \Delta P/P_0 \end{pmatrix}_s = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \\ \Delta P/P_0 \end{pmatrix}_0$
definition of momentum compaction, α_P	$\frac{\Delta C}{C} = \alpha_P \frac{\Delta P}{P_0}$
stability condition	$m \cdot Q_x + n \cdot Q_y \neq p$ with n, m, p integers
tune shift	$\Delta Q = \frac{1}{4\pi}\oint_{\text{quads}} \Delta k(s)\,\beta(s)\,\mathrm{d}s$
beta beat	$\frac{\Delta\beta(s)}{\beta(s)} = \frac{1}{2\sin 2\pi Q} \cdot \oint \beta(t)\,\Delta k(t)\cos(2\pi Q - 2(\mu(t) - \mu(s)))$
chromaticity	$\xi = \frac{\Delta Q}{\Delta P/P_0} = -\frac{1}{4\pi}\oint_{\text{quads}} k(s)\,\beta(s)\,\mathrm{d}s$