Transverse Beam Dynamics

JUAS 2017 - tutorial 1 (solutions)

1 Exercise: Wien Filter

A Wien Filter is a device that allows to select the particles in a charged beam according to the desired velocity.

- Write down the expression of the Lorentz force.

$$\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}$$

- How is the force originating from a uniform magnetic field oriented with respect to the particle velocity and the field? How should we orient an electric field if we want to compensate that force?

The magnetic force is oriented perpendicularly to both $\vec{v}$ and $\vec{B}$ and so should be the electric field.

- Assuming the magnetic field is 1 mT, what would be the required electric field (V/m) to select protons travelling with a velocity of $0.1c$?

$$E = vB = 0.1c\ 0.1\ \mathrm{mT} = 3\ \mathrm{kV/m}$$

- [Optional] Write down the equations of motion and their solutions. What are the implications of the oscillating terms?

See: "CONSTRUCTION OF A WIEN FILTER HEAVY ION ACCELERATOR", K. Jensen and E. Veje, Nuclear Instruments and Methods 122 (1974)

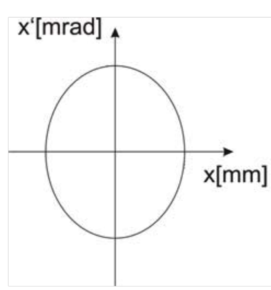
- [Optional] Could we use a Wien filter with a neutral beam (eg. neutron)? What other techniques may be employed in this case?

A Wien filter will not work with a neutral beam, but still it is possible to filter it by velocity. A possibility consists in having two rotating disks made of an absorbing matherial, presenting a radial slit. The velocity can be selected tuning the distance, the rotation frequency and the phase between the the discs.

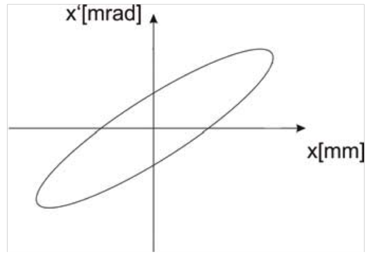
2 Exercise: understanding the phase space concept

1. Sketch the emittance ellipse of a particle beam in horizontal x-x' phase space (I) at the position of a transverse waist, (II) when the beam is divergent and (III) when the beam is convergent.

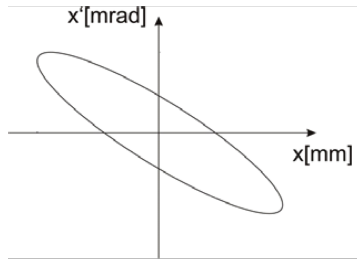
(I) Beam at the position of a transverse (x) waist – upright/round ellipse:



(II) Divergent beam – positive slope:



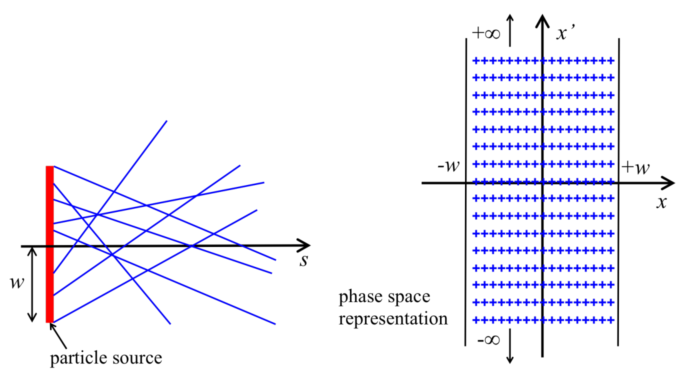
(III) Convergent beam – negative slope:



2. Phase Space Representation of a Particle Source:

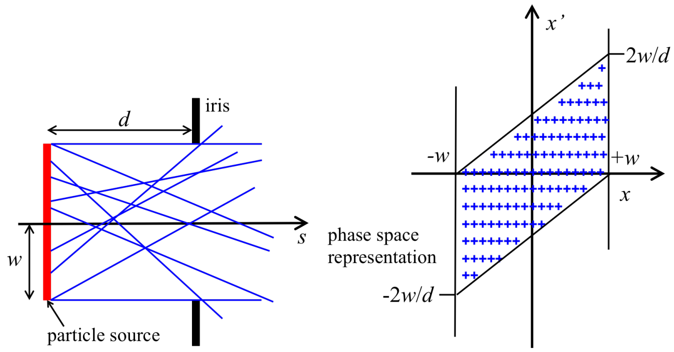
   - Consider a source at position s_0 with radius w emitting particles. Make a drawing of this setup in configuration space and in phase space. Which part of phase space can be occupied by the emitted particles?

     Particles are emitted from the entire source surface $x \in [-w, +w]$ with a trajectory slope $\varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, i.e. the particles can have any $x' \in \mathbb{R}$. The occupied phase space area is infinite.



   - Any real beam emerging from a source like the one above will be clipped by aperture limitations of the vacuum chamber. This can be modelled by assuming that a distance d away from the source there is an iris with an opening with radius $R = w$. Make a drawing of this setup in configuration and phase space. Which part of phase space is occupied by the beam at a location after the iris?

     Particles with angle $x' = 0$ are emitted from the entire source surface $x \in [-w, +w]$ and arrive behind the iris opening. For $x = \pm w$ there is a maximum angle $x' = \pm 2w/d$ that will still be accepted by the iris. This leads to a parallelogram in phase space. Such a beam has a specific emittance given by the occupied phase space area.



3 Stability condition

Given a periodic lattice with generic transport map M,

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

under which condition the matrix M provides stable motion after N turns (with $N \to \infty$)?
Hint: the motion is stable when all elements of M^N are finite.

- But... what does this imply, for M ? Remember:

$$\det(M) = ad - bc = 1$$
$$\text{trace}(M) = a + d$$

If we diagonalize M, we can rewrite it as:

$$M = U \cdot \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \cdot U^T$$

where U is some unitary matrix, λ_1 and λ_2 are the eigenvalues.

What happens if we consider N turns?

$$M^N = U \cdot \begin{pmatrix} \lambda_1^N & 0 \\ 0 & \lambda_2^N \end{pmatrix} \cdot U^T$$

Notice that λ_1 and λ_2 can be complex numbers: $\rho e^{i\theta}$with $\rho, \theta \in \mathrm{R}$, but ρ must be 1 since we require that λ^N is bounded for each N.

Given that $\det(M) = 1$, then

$$\lambda_1 \cdot \lambda_2 = 1 \quad \to \lambda_1 = \frac{1}{\lambda_2} \quad \to \lambda_{1,2} = e^{\pm i\theta}$$

Now we can find the eigenvalues through the characteristic equation:

$$\det(M - \lambda I) = \det \begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix} = 0$$

$$\lambda^2 - (a + d)\,\lambda + (ad - bc) = 0$$
$$\lambda^2 - \text{trace}(M)\,\lambda + 1 = 0$$

$$\text{trace}(M) = \lambda + 1/\lambda =$$
$$= e^{i\theta} + e^{-i\theta} = 2\cos\theta$$

From which derives the stability condition:

$$\text{since } \theta \in \mathrm{R} \quad \to \quad |\text{trace}(M)| \leq 2$$

4 Exercise: stability condition

Consider a lattice composed by a single 2 meters long quadrupole, with $f = 1$ m

- Prove that if the quadrupole is defocusing, then a lattice is not stable
- Prove that if the quadrupole is focusing, then the lattice is stable

Solution:

- Let's work in thin-lens approximation

$$M_{\text{QD}} = \begin{pmatrix} 1 & L_{\text{quad}}/2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & L_{\text{quad}}/2 \\ 0 & 1 \end{pmatrix}$$

  which can be computed to be

$$M_{\text{QD}} = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$$

  has trace $\text{Tr}\left(M_{\text{QD}}\right) = 4$, which does not fulfill the stability requirement:

$$\left|\text{Tr}\left(M_{\text{QD}}\right)\right| \leq 2$$

- In the case of a focusing quadrupole:

$$M_{\text{QF}} = \begin{pmatrix} 1 & L_{\text{quad}}/2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & L_{\text{quad}}/2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

  which clearly satisfies the stability criterion.