Transverse Beam Dynamics

JUAS 2017 - tutorial 2 (solutions)

1 Exercise: local radius, rigidity

We wish to design an electron ring with a radius of R=200 m. Let us assume that only 50% of the circumference is occupied by bending magnets:

- What will be the local radius of bend ρ in these magnets if they all have the same strength?

$$2\pi\rho = 50\% \cdot 2\pi R \longrightarrow \rho = 100 \text{ m}$$

- If the momentum of the electrons is 12 GeV/c, calculate the rigidity $B\rho$ and the field in the dipoles.

Using the rigidity definition:

$$B\rho = 3.3356 \cdot p[\text{GeV}/c] = 40.03 \text{ T}\cdot\text{m}$$

and therefore $B = 0.4$ T.

2 Exercise: particle momentum, geometry of a storage ring and thin lenses

The LHC storage ring at CERN will collide proton beams with a maximum momentum of $p = 7$ TeV/c per beam. The main parameters of this machine are:

Circumference	$C_0 = 26658.9$ m	
Particle momentum	$p = 7$ TeV/c	
Main dipoles	$B = 8.392$ T	$l_B = 14.2$ m
Main quadrupoles	$G = 235$ T/m	$l_q = 5.5$ m

- Calculate the magnetic rigidity of the design beam, the bending radius of the main dipole magnets in the arc and determine the number of dipoles that is needed in the machine.

The beam rigidity is obtained in the usual way by the golden rule:

$$B\rho = \frac{p}{e} = \frac{1}{0.299792} \cdot p[\text{GeV}/c] = 3.3356 \cdot p[\text{GeV}/c] = 3.3356 \cdot 7000 \text{ Tm} = 23349 \text{ T}\cdot\text{m}$$

and knowing the magnetic dipole field we get

$$\rho = \frac{3.3356 \cdot 7000\text{Tm}}{8.392\text{T}} = 2782 \text{ m}$$

The bending angle for one LHC dipole magnet:

$$\theta = \frac{l_B}{\rho} = \frac{14.2\text{m}}{2782\text{m}} = 5.104 \text{ mrad}$$

and as we want to have a closed storage ring we require an overall bending angle of 2π:

$$N = \frac{2\pi}{\theta} = 1231 \text{ Magnets}$$

- Calculate the k-strength of the quadrupole magnets and compare its focal length to the length of the magnet. Can this magnet be treated as a thin lens?

We can use the beam rigidity (or the particle momentum) to calculate the normalized quadrupole strength:

$$k = \frac{G}{B\rho} = \frac{G}{p/e} = 0.299792 \cdot \frac{G}{p[\mathrm{GeV/c}]} = 0.299792 \cdot \frac{235\mathrm{T/m}}{7000\mathrm{GeV/c}} = 0.01\ \mathrm{m}^{-2}$$

an the focal length:

$$f = \frac{1}{k \cdot l_q} = 18.2\ \mathrm{m} > l_q$$

The focal length of this magnet is still quite bigger than the magnetic length l_q. So it is valid to treat that quadrupole in thin lens approximation.

- Compute and compare the matrices for the quadrupole for the thin (drift-kick-drift) and thick cases.

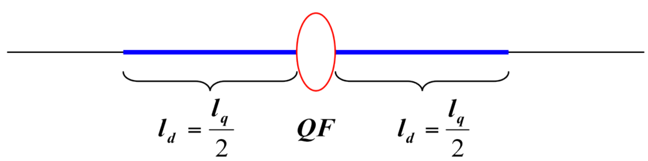
The matrix of a focusing quadrupole is given by

$$M_{QF} = \begin{pmatrix} \cos(\sqrt{|k|}l_q) & \frac{1}{\sqrt{|k|}}\sin(\sqrt{|k|}l_q) \\ -\sqrt{|k|}\sin(\sqrt{|k|}l_q) & \cos(\sqrt{|k|}l_q) \end{pmatrix} = \begin{pmatrix} 0.8525 & 5.22 \\ -0.0522 & 0.8525 \end{pmatrix}$$

In thin lens approximation we replace the matrix above by the expression

$$M_{QF} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{|f|} & 1 \end{pmatrix} \text{ with the focal length } f = \frac{1}{|kl_q|} = 18.2\ \mathrm{m}$$

The thin lens description has to be completed by the matrix of a drift space of half the quadrupole length in front and after the thin lens quadrupole. The appropriate description is therefore



So we write

$$M_{thinlens} = \begin{pmatrix} 1 & \frac{l_q}{2} \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & \frac{l_q}{2} \\ 0 & 1 \end{pmatrix}$$

Multiplying out we get

$$M_{thinlens} = \begin{pmatrix} 1 + \frac{l_q}{2}kl_q & \frac{l_q}{2}(2 + kl_q\frac{l_q}{2}) \\ kl_q & 1 + kl_q \end{pmatrix}$$

With the parameters in the example we get finally

$$M_{thinlens} = \begin{pmatrix} 0.848 & 5.084 \\ -0.055 & 0.848 \end{pmatrix}$$

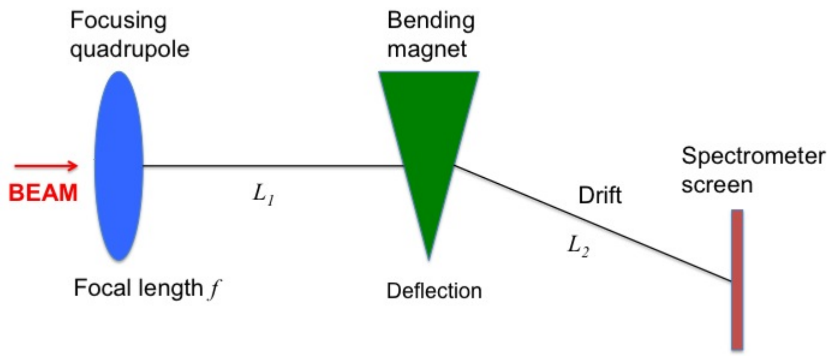
which is still quite close to the result of the exact calculation above.

3 Exercise: A spectrometer line in CTF3

The CTF3 (CLIC Test Facility 3) experiment at CERN consists of a linac that injects very short electron bunches into an isochronous ring.

A spectrometer line made of one quadrupole and one bending magnet is located at the end of the linac where the particle momentum is 350 MeV/c. The goal of the spectrometer is to measure the energy before injecting the electrons in the ring.

The spectrometer line is sketched on the figure below. It is made of a focusing quadrupole of focal length f, a drift space of length L_1, a bending magnet of deflection angle θ in the horizontal plane, and a drift space of length L_2. We assume that the spectrometer line starts at the quadrupole and ends at the end of the second drift. We neglect the focusing effect of the dipole.



3.1 If the effective length of the dipole is $l_B = 0.43$ m, what should be the magnetic field (in Tesla) inside the dipole to deflect the electrons by an angle of 35 degrees?

Answer: One has $\theta = \frac{l}{\rho}$ and $B\rho = 3.356\,p$: $B = \dfrac{3.356\,p\,\theta}{l} = 1.66$ T.

3.2 Starting from the general horizontal 3×3 transfer matrix of a sector dipole of deflection angle θ, show that the transfer matrix of a dipole in the thin lens approximation is

$$M_{dipole} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix}$$

Which approximations are done?
Hint: A sector dipole has the following 3×3 transfer matrix:

$$M_{\text{dipole}} = \begin{pmatrix} \cos\theta & \rho\sin\theta & \rho(1-\cos\theta) \\ -\frac{\sin\theta}{\rho} & \cos\theta & \sin\theta \\ 0 & 0 & 1 \end{pmatrix}$$

Answer: We need to compute the limit for $l \to 0$ while keeping $\theta = \frac{l}{\rho} = \text{const}$. Remember that, if θ is a small angle, $\cos\theta \approx 1$, $\sin\theta \approx \theta$. Besides the trivial elements, such as m_{11}, m_{22}, and m_{23}, the others read:

$$m_{12}: \quad \lim_{l\to 0} \rho\sin\theta = \lim_{l\to 0} \frac{\sin\theta}{\frac{1}{\rho}} = \lim_{l\to 0} l \cdot \underbrace{\frac{\sin\frac{l}{\rho}}{\frac{l}{\rho}}}_{\text{const}} = 0$$

$$m_{13}: \quad \lim_{l\to 0} \rho\,(1-\cos\theta) = \lim_{l\to 0} \rho\left(1-\cos\frac{l}{\rho}\right) = \lim_{l\to 0} l \cdot \underbrace{\frac{1-\cos\frac{l}{\rho}}{\frac{l}{\rho}}}_{\text{const}} = 0$$

$$m_{21}: \quad \lim_{l\to 0} -\frac{\sin\theta}{\rho} = \lim_{\rho\to\infty} -\frac{\sin\theta}{\rho} = 0$$

therefore, in thin-lens approximation the matrix of a dipole magnet, is

$$M_{\text{dipole}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix}.$$

3.3 In the thin lens approximation, derive the horizontal extended 3×3 transfer matrix of the spectrometer line. Show that it is:

$$M_{spectro} = \begin{pmatrix} \frac{f-L_1-L_2}{f} & L_1+L_2 & L_2\theta \\ -\frac{1}{f} & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix}$$

Answer: For the spectrometer line, one has

$$M_{\text{spectro}} = M_{\text{Drift2}} \cdot M_{\text{Dipole}} \cdot M_{\text{Drift1}} \cdot M_{\text{Quad}}$$

therefore:

$$M_{\text{spectro}} = \begin{pmatrix} 1 & L_2 & L_2\theta \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 - \frac{L_1}{f} & L_1 & 0 \\ -\frac{1}{f} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

3.4 Assuming $D = D' = 0$ at the entrance of the quadrupole, what is the dispersion and its derivative at the end of the spectrometer line? Give the numerical value of D' at the end of the spectrometer line for the angle of 35 degrees.

Answer: At the entrance of the line, $D = 0$ and $D' = 0$. If M is the transfer matrix of a system the dispersion D at exit is the element m_{13} of M, whereas D' is the element m_{23}:

$$D = L_2\theta,$$
$$D' = \theta = 35 \text{ degrees} = 0.61.$$

3.5 What is the difference between a periodic lattice and a beam transport lattice (or transfer line) as concerns the betatron function ?

Answer: In a periodic lattice the β-functions are periodic and contained in the (periodic) transfer matrix of the lattice. In transfer line one needs to know the initial conditions in order to calculate the β-functions at any point (using the transfer matrix).

3.6 The Courant-Snyder invariant allows to trace the Twiss parameters α, β, and γ through a transfer line.

Remember from the lecture:

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_s = \begin{pmatrix} C^2 & -2SC & S^2 \\ -CC' & SC' + S'C & -SS' \\ C'^2 & -2S'C' & S'^2 \end{pmatrix} \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_0$$

An alternative way to transport the Twiss parameters is through the σ matrix:

$$\sigma_i = \begin{pmatrix} \beta_i & -\alpha_i \\ -\alpha_i & \gamma_i \end{pmatrix}$$

This matrix multiplied by the emittance ϵ gives the so-called beam matrix (which has already been introduced during the lecture):

$$\Sigma_i = \begin{pmatrix} \beta_i\epsilon & -\alpha_i\epsilon \\ -\alpha_i\epsilon & \gamma_i\epsilon \end{pmatrix}$$

If σ_1 is the matrix at the entrance of the transfer line, the matrix σ_2 at the exit of the transfer line is given by

$$\sigma_2 = M\sigma_1 M^T$$

where M is the 2×2 transfer matrix of the line extracted from the extended 3×3 transfer matrix (see question 1.3), and M^T the transpose matrix of M.

Assuming $\alpha_1 = 0$, derive the betatron function β_2 at the end of the spectrometer line in terms of L_1, L_2, f and β_1.

Hint: For the calculations, write M as $M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$ and replace the values of the matrix elements only at the end.

Answer: One has $\sigma_2 = M\sigma_1 M^T$. If $\alpha_1 = 0$, then $\sigma_1 = \begin{pmatrix} \beta_1 & 0 \\ 0 & 1/\beta_1 \end{pmatrix}$

$$\sigma_2 = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \begin{pmatrix} \beta_1 & 0 \\ 0 & 1/\beta_1 \end{pmatrix} \begin{pmatrix} m_{11} & m_{21} \\ m_{12} & m_{22} \end{pmatrix}$$

$$\sigma_2 = \begin{pmatrix} \beta_1 m_{11}^2 + m_{12}^2/\beta_1 & \beta_1 m_{11} m_{21} + m_{12} m_{22}/\beta_1 \\ \beta_1 m_{11} m_{21} + m_{12} m_{22}/\beta_1 & \beta_1 m_{21}^2 + m_{22}^2/\beta_1 \end{pmatrix}$$

Therefore:

$$\beta_2 = \beta_1 m_{11}^2 + m_{12}^2/\beta_1$$

Since $m_{11} = \frac{f-L_1-L_2}{f}$ and $m_{12} = L_1 + L_2$, one has:

$$\beta_2 = \beta_1 \left(1 - \frac{L_1 + L_2}{f}\right)^2 + \frac{(L_1 + L_2)^2}{\beta_1}.$$

3.7 Given the numerical values $L_1 = 1$ m, $L_2 = 2$ m, $\beta_1 = 10$ m, $\alpha_1 = 0$, compute the betatron function β_2 at the end of the spectrometer line as a function of the focal length f of the quadrupole.

Answer: If $L_1 = 1$ m, $L_2 = 2$ m, and $\beta_1 = 10$ m, then $\beta_2 = 0.9 + 10\left(1 - \frac{3}{f}\right)^2$.

3.8 Find the value of the focal length f such that the betatron function β_2 at the end of the spectrometer line is minimum. Give the minimum value of β_2.

Answer: To have β_2 minimum one needs $\left(1 - \frac{3}{f} = 0\right)$.Therefore, $f = 3$ m.

3.9 In the presence of dispersion, what is the particle deviation from the design orbit due to the different particle momentum $p \neq p_0$ (p_0 is the design momentum)? Why is it important to minimize the β function in the spectrometer?

Answer: With dispersion, the deviation from the design orbit is $\Delta_s = D\frac{\Delta p}{p_0}$.

Measuring Δ_s allows to determine Δp and therefore p, if one has calibrated the spectrometer at p_0. It is important to minimize β_2 in order to have the best possible resolution for Δ_s: on the spectrometer screen, we want a spot with small transverse size for an accurate measurement.