Transverse Beam Dynamics

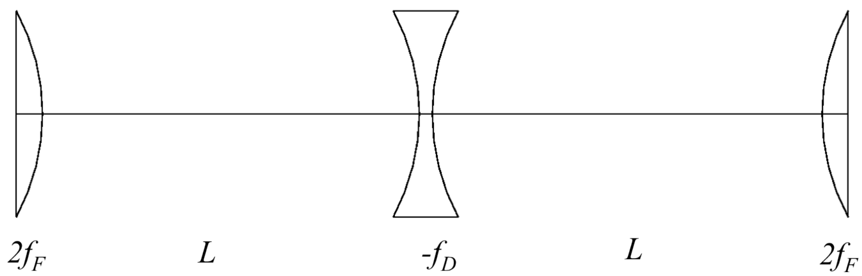
JUAS 2017 - tutorial 3 (solutions)

1 Exercise: chromaticity in a FODO cell

Consider a ring made of N_{cell} identical FODO cells with equally spaced quadrupoles. Assume that the two quadrupoles are both of length l_q, but their strengths may differ.

1.1 Calculate the maximum and the minimum betatron function in the FODO cell. (*Use the thin-lens approximations*)

Answer: First we calculate the transfer matrix for a FODO cell (see figure below). We start from the center of the focusing quadrupole where the betatron function is maximum:



This exercise considers a general case where f_F is not necessarily equal to f_D. Using the thin lens approximation for the FODO cell with drifts of length L we get the following matrix:

$$\begin{aligned} M_{cell} &= \begin{pmatrix} 1 & 0 \\ -\frac{1}{2f_F} & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1}{f_D} & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{2f_F} & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 - L(\frac{1}{f_F} - \frac{1}{f_D} + \frac{L}{2f_F f_D}) & 2L + \frac{L^2}{f_D} \\ \frac{1}{f_D} - \frac{1}{f_F}(1 - \frac{L}{2f_F} + \frac{L}{f_D} - \frac{L^2}{4f_F f_D}) & 1 - L(\frac{1}{f_F} - \frac{1}{f_D} + \frac{L}{2f_F f_D}) \end{pmatrix} \end{aligned} \tag{1}$$

REMEMBER that in terms of betatron functions and phase advance the matrix of a FODO cell is given by:

$$M_{cell} = \begin{pmatrix} cos\mu + \alpha \sin\mu & \beta \sin\mu \\ -\gamma \sin\mu & \cos\mu - \alpha \sin\mu \end{pmatrix} \tag{2}$$

Since β is maximum at the center of the focusing quadrupole: $\alpha = -\beta'/2 = 0$, and we can also write:

$$M_{cell} = \begin{pmatrix} cos\mu & \beta \sin\mu \\ -\frac{\sin\mu}{\beta} & \cos\mu \end{pmatrix}$$

Then, doing Eq. (1) equal to Eq. (2) we obtain:

$$\cos\mu = \frac{1}{2}\mathrm{tr}(M_{cell}) = 1 + \frac{L}{f_D} - \frac{L}{f_F} - \frac{L^2}{2f_D f_F} = 1 - 2\sin^2\frac{\mu}{2}$$

or

$$2\sin^2\frac{\mu}{2} = \frac{L}{f_F} - \frac{L}{f_D} + \frac{L^2}{2f_D f_F} \tag{3}$$

Here we have applied the following trigonometric relation: $\cos\mu = \cos(\frac{\mu}{2}+\frac{\mu}{2}) = \cos^2\frac{\mu}{2} - \sin^2\frac{\mu}{2} = 1 - 2\sin^2\frac{\mu}{2}$.

The maximum for the betatron function β_{max} occurs at the focusing quadrupole. Since Eq. (1) is for a periodic cell starting at the center of the focusing quadrupole, the m_{12} component of the matrix gives us

$$\beta_{max}\sin\mu = 2L + \frac{L^2}{f_D}$$

Rearranging things:

$$\beta_{max} = \frac{2L + \frac{L^2}{f_D}}{\sin\mu} \tag{4}$$

On the other hand, the minimum for the betatron function occurs at the defocusing quadrupole position. Therefore, interchanging f_F with $-f_D$ for a FODO cell gives:

$$\beta_{min} = \frac{2L - \frac{L^2}{f_F}}{\sin\mu} \tag{5}$$

1.2 Calculate the natural chromaticities for this machine.

Answer:

Let us remember the definition of natural chromaticity. The so-called "natural" chromaticity is the chromaticity that derives from the energy dependence of the quadrupole focusing, i.e. the chromaticity arising only from quadrupoles. The chromaticity is defined in the following way:

$$\xi = \frac{\Delta Q}{\Delta p/p_0} \tag{6}$$

where ΔQ is the tune shift due to the chromaticity effects and $\Delta p/p_0$ is the momentum offset of the beam or the particle with respect to the nominal momentum p_0.

The natural chromaticity is defined as (remember from Lecture 4):

$$\xi_N = -\frac{1}{4\pi}\oint \beta(s)k(s)ds \tag{7}$$

Sometimes, especially for small accelerators, the chromaticity is normalized to the machine tune Q and defined also as:

$$\xi' = \frac{\Delta Q/Q}{\Delta p/p_0} \tag{8}$$

$$\xi'_N = -\frac{1}{4\pi Q}\oint \beta(s)k(s)ds \tag{9}$$

For this exercise, either you decide to use Eq. (7) or Eq. (9) it is fine! From now on let us use Eq. (7):

$$\begin{aligned}
\xi_N &= -\frac{1}{4\pi}\oint \beta(s)k(s)ds \\
&= -\frac{1}{4\pi} \times N_{cell}\int_{cell} \beta(s)k(s)ds \\
&= -\frac{N_{cell}}{4\pi} \sum_{i\in\{quads\}} \beta_i(kl_q)_i
\end{aligned}$$

Here we have used the following approximation valid for thin lens:

$$\int_{cell} \beta(s)k(s)ds \simeq \sum_{i\in\{quads\}} \beta_i (kl_q)_i$$

where we sum over each quadrupole i in the cell. In the case of the FODO cell we have two half focusing quadrupoles and one defocusing quadrupole. Taking into account that $(kl_q)_i = 1/f_i$, we have:

$$\begin{aligned}
\xi_N &\simeq -\frac{N_{cell}}{4\pi} \sum_{i\in\{quads\}} \beta_i (kl_q)_i \\
&= -\frac{N_{cell}}{4\pi}\left[\beta_{max}\left(\frac{1}{2f_F}\right) + \beta_{min}\left(-\frac{1}{f_D}\right) + \beta_{max}\left(\frac{1}{2f_F}\right)\right] \\
&= -\frac{N_{cell}}{4\pi}\left[\beta_{max}\left(\frac{1}{f_F}\right) + \beta_{min}\left(-\frac{1}{f_D}\right)\right] \\
&= -\frac{N_{cell}}{4\pi\sin\mu}\left[\left(2L + \frac{L^2}{f_D}\right)\frac{1}{f_F} - \left(2L - \frac{L^2}{f_F}\right)\frac{1}{f_D}\right] \\
&= -\frac{N_{cell}L}{2\pi\sin\mu}\left[\frac{1}{f_F} - \frac{1}{f_D} + \frac{L}{f_F f_D}\right]
\end{aligned}$$

Here we have used the expressions (4) and (5) for β_{max} and β_{min}.

1.3 Show that for short quadrupoles, if $f_F \simeq f_D$,

$$\xi_N \simeq -\frac{N_{cell}}{\pi}\tan\frac{\mu}{2}$$

Answer: If $f_F \simeq f_D$, we have

$$\begin{aligned}
\xi_N &\simeq -\frac{N_{cell}}{2\pi\sin\mu}\frac{L^2}{f_F f_D} \\
&= -\frac{N_{cell}}{4\pi\sin\frac{\mu}{2}\cos\frac{\mu}{2}} 4\sin^2\frac{\mu}{2}
\end{aligned}$$

where we have used $\sin\mu = \sin(\frac{\mu}{2}+\frac{\mu}{2}) = 2\sin\frac{\mu}{2}\cos\frac{\mu}{2}$
and considering Eq. (3): $4\sin^2\frac{\mu}{2} = \frac{L^2}{f_F f_D}$
we finally obtain

$$\xi_N \simeq -\frac{N_{cell}}{\pi}\tan\frac{\mu}{2}$$

Q.E.D.!

1.4 Design the FODO cell such that it has: phase advance $\mu = 90$ degrees, a total length of 10 m, and a total bending angle of 5 degrees. What are β_{max}, β_{min}, D_{max}, D_{min}?

Solution:

Lattice parameters: $L = 10$ m, $\theta = 5$ degrees$= 0.087266$ rad, $f = \frac{1}{\sqrt{2}}\frac{L}{2} = 3.535$ m

Maximum and minimum betatron functions:

$$\beta_{max} = \frac{L + \frac{L^2}{4f}}{\sin\mu} = L + \frac{L^2}{4f} = 17.07 \text{ m}, \quad \beta_{min} = \frac{L - \frac{L^2}{4f}}{\sin\mu} = L - \frac{L^2}{4f} = 2.93 \text{ m}$$

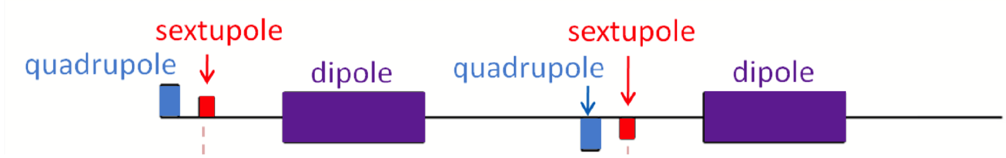
Maximum and minimum dispersion:

$$D_{max} = \frac{L\theta\left(1 + \frac{1}{2}\sin\frac{\mu}{2}\right)}{4\sin^2\frac{\mu}{2}} = \frac{f}{L}\left(4f + \frac{L}{2}\right)\theta = 0.59060 \text{ m}, \quad D_{min} = \frac{L\theta\left(1 - \frac{1}{2}\sin\frac{\mu}{2}\right)}{4\sin^2\frac{\mu}{2}} = \frac{f}{L}\left(4f - \frac{L}{2}\right)\theta = 0.28207 \text{ m}$$

1.5 Add two sextupoles at appropriate locations to correct horizontal and vertical chromaticities. (hints: use 1 sextupole for the horizontal plane and 1 for the vertical plane; do not consider geometric aberrations).

Solution:

By locating sextupoles with strength $K_s > 0$ where β_x is large and β_y is small, we can correct the horizontal chromaticity with relatively little impact on the vertical chromaticity. Similarly, by locating sextupoles with $K_s < 0$ where β_y is large and β_x is small, we can correct the vertical chromaticity with relatively little impact on the horizontal chromaticity. See figure below.



Let us assume the case of a FODO lattice where $f_F = f_D = f$. Then the natural chromaticity of this fodo cell is given by the expression (exercise 1.3):

$$\xi_N \simeq -\frac{1}{\pi} \tan \frac{\mu}{2}$$

For $\mu = 90$ it is $\xi_N \simeq -1/\pi$ in both horizontal and vertical plane. Therefore, we need to adjust the strength of the sextupoles to cancel this chromaticity:

$$-\frac{1}{4\pi} \left[K_{2F} D_{max} \beta_{max} + K_{2D} D_{min} \beta_{min} \right] \simeq -\frac{1}{\pi}$$

where $K_{2F} = k_{2F} l_s$ is the normalized integrated strength of the sextupole located near the focusing quadrupole, and $K_{2D} = k_{2D} l_s$ the normalized integrated strength of the sextupole near the defocusing quadrupole (with l_s the effective length of the sextupole). For an effective cancellation of the chromaticity in both planes, usually $K_{2F} > 0$ and $K_{2D} < 0$. Substituting the values for the maximum and minimum dispersion and betatron function in terms of the total length of the lattice L and the focal length of the quadrupoles f, one obtains the following expression:

$$-\frac{1}{4\pi} \frac{f}{L} \theta \left[K_{2F} \left(4f + \frac{L}{2} \right) \left(L + \frac{L^2}{4f} \right) + K_{2D} \left(4f - \frac{L}{2} \right) \left(L - \frac{L^2}{4f} \right) \right] \simeq -\frac{1}{\pi}$$

Considering the same absolute value for the strength of the sextupoles, $K_{2F} = -K_{2D} = K_s$, we can write then:

$$\frac{3}{4\pi} K_s L f \theta = \frac{1}{\pi}$$

The strength of the sextupole is given then by:

$$K_s = \frac{4}{3 L f \theta}$$

Then, substituting all the numerical values for the lattice parameters:
$K_{2F} = 0.865\ \mathrm{m}^{-2}$
$K_{2D} = -0.865\ \mathrm{m}^{-2}$

1.6 If the gradient of all focusing quadrupoles in the ring is wrong by +10%, how much is the tune-shift with and without sextupoles?

Solution:

If the gradient of the focusing quadrupole has and offset of 10%, then the corresponding quad. strength offset is also 10%. We calculate the number of cells of a ring made of these fodo cells, $N_{cell} = 72$ cells, and then we calculate the total tune-shift in both planes:
$\Delta Q_x = N_{cell} \frac{\Delta K_F \beta_{max}}{4\pi} = 9.78$

$\Delta Q_y = N_{cell} \frac{\Delta K_F \beta_{min}}{4\pi} = 1.68$
When the sextupoles correct for the chromaticity, the particles have, in principle, no tune-shift with energy.

2 Exercise: Double-Bend Achromat (DBA) lattice

A DBA can be made from two dipoles with a horizontally focusing quadrupole between them. The transfer matrix through the achromat is of the form:

$$M_{\text{DBA}} = M_{\text{bend}} M_{\text{drift}} M_{1/2\text{F}} M_{1/2\text{F}} M_{\text{drift}} M_{\text{bend}}$$

Note that this magnet configuration does not produce vertical focusing, therefore will not be enough to create a stable lattice. A full DBA typically comprises additional quadrupole doublets before and after the bending section, but for sake of simplicity we will neglect them.

2.1 Use the thin lens approximation for quadrupoles and small angle approximation for bends to find the dispersion in the middle of the quadrupole. Write the focal length in terms of the drift and bend parameters.

Let us consider the 3×3 transfer matrices of each element of the lattice (using the thin lens approximation and small angle approximation for the bending magnets) for the beam coordinates x, $x'$and $\Delta p/p$:

$$M_{\text{bend}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix}, \quad M_{\text{drift}} = \begin{pmatrix} 1 & L & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_{1/2\text{F}} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2f} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Assuming the initial dispersion vector $(\eta_0, \eta'_0, 1) = (0, 0, 1)$ and propagating it to the center of the quadrupole:

$$\begin{pmatrix} \eta_c \\ 0 \\ 1 \end{pmatrix} = M_{1/2\text{F}} M_{\text{drift}} M_{\text{bend}} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Here we take into account that η'=0 at the center of a quadrupole. After matrix multiplication we obtain:

$$\begin{pmatrix} \eta_c \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & L & L\theta \\ -\frac{1}{2f} & 1 - \frac{L}{2f} & \theta\left(1 - \frac{L}{2f}\right) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Therefore, one obtains:

$$\eta_c = L\theta$$

$$1 - \frac{L}{2f} = 0 \Rightarrow L = 2f$$

2.2 Show that the dispersion is again zero ($\eta = \eta' = 0$) after the bend.

Propagating the dispersion vector from the center of the quadrupole to the end of the lattice:

$$\begin{pmatrix} \eta_{end} \\ \eta'_{end} \\ 1 \end{pmatrix} = M_{\text{bend}} M_{\text{drift}} M_{1/2\text{F}} \begin{pmatrix} \eta_c \\ 0 \\ 1 \end{pmatrix},$$

$$\begin{pmatrix} \eta_{end} \\ \eta'_{end} \\ 1 \end{pmatrix} = \begin{pmatrix} 1 - \frac{L}{2f} & L & 0 \\ -\frac{1}{2f} & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \eta_c \\ 0 \\ 1 \end{pmatrix}$$

and taking into account $L = 2f$, we obtain:

$$\eta_{end} = (2f - L)\theta = 0,$$

$$\eta'_{end} = \theta - \frac{1}{2f}\eta_c = \theta - \frac{1}{2f}(2f\theta) = 0$$

2.3 Compute the parameters L, f for a DBA 10 meters long, bending the beam by an angle of 1 radians. What is the dispersion in the centre? How much a particle with 1% energy deviation will be displaced at the centre of the cell?

$$L = 5 \text{ m}$$

$$f = 2.5 \text{ m}$$

$$D = L \cdot \theta = 5 \text{ m}$$

$$x = \delta D = 0.01 * 5 \text{ m} = 5 \text{ cm}$$

3 Exercise: geometry of a storage ring, thin lens, tune, dispersion

Consider a proton synchrotron accumulator made of identical cells. The relevant ring parameters are given in the following table:

Proton kinetic energy	2 GeV
Cell type	Symmetric triplet$^{(*)}$
Ring circumference	960 m
Integrated quadrupole gradient ($\int Gdl$)	1.5 T

$^{(*)}$Note: A thin lens symmetric triplet cell consists of a thin lens defocusing quadrupole of focal length $-f$, followed with a drift space of length L_1, a thin lens focusing quadrupole of focal length f, a drift of length L_2, a thin lens dipole of horizontal bending angle θ, a drift of length L_2, a thin lens focusing quadrupole of focal length f, a drift of length L_1, and a thin lens defocusing quadrupole of focal length $-f$ (see Figure below).

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Hint:
The 3×3 horizontal transfer matrix for one symmetric triplet cell is (in the thin lens approximation):

$$M_{triplet} = \begin{pmatrix} \frac{f^3+2L_1^2L_2-2L_1f(L_1+L_2)}{f^3} & \frac{2(f-L_1)(L_1f+L_2f-L_1L_2)}{f^2} & (L_1+L_2-\frac{L_1L_2}{f})\theta \\ \frac{2L_1(L_1L_2-L_1f-f^2)}{f^4} & \frac{f^3+2L_1^2L_2-2L_1f(L_1+L_2)}{f^3} & \frac{(f^2+L_1f-L_1L_2)}{f^2}\theta \\ 0 & 0 & 1 \end{pmatrix}$$

for the transport of a vector $\begin{pmatrix} x \\ x' \\ \Delta p/p_0 \end{pmatrix}$, where $\Delta p/p_0$ is the momentum offset with respect to the design momentum p_0.

3.1 Compute the focal length f of the quadrupoles. The proton rest mass is 938 MeV.

Assuming a longitudinally constant gradient for the quadrupole $\int Gdl = G \cdot l$, where l is the length of the quadrupole

The quadrupole strength:

$$k = \frac{G}{B\rho} = \frac{1.5\text{T}}{(B\rho)l}$$

Knowing $f = \frac{1}{kl}$, we obtain

$$f = \frac{B\rho}{1.5\text{T}},$$

so we calculate the particle momentum in order to calculate the rigidity $B\rho = \frac{p}{e}$.

We have the information of the kinetic energy, therefore we can use the relativistic formula to obtain the momentum pc [GeV]

The total energy is given by

$$E = \sqrt{m_0^2c^4 + p^2c^2},$$

where $m_0 = 938$ MeV is the rest mass of the particle (in this case protons)

Knowing that the kinetic energy $E_k = E - m_0c^2 = \sqrt{m_0^2c^4 + p^2c^2} - m_0c^2$, and after some trivial algebra we obtain:

$$p = 2.78GeV/c$$

$$B\rho \approx \frac{1}{0.3}p[\text{GeV/c}]{=}9.27Tm$$

and with this information we can finally obtain the focal length:

$$f = \frac{B\rho}{1.5\text{T}} = 6.18m$$

3.2 Given the numerical values $L_1 = 1.5$ m and $L_2 = 6$ m:

- Compute the horizontal and vertical transfer matrices of a triplet cell (take into account that the sign of the focal length changes when going from horizontal to vertical plane).

Let's calculate first the bending angle of the dipole:

The length of the cell is $L_{cell} = 2L_1 + 2L_2 = 15$ m

Knowing that the circumference of the machine is 960 m and that we have a thin dipole per cell:

Number of cells:$N = \frac{960}{15} = 64$

and therefore:

$\theta = \frac{2\pi}{64} = 0.098$ rad.

Having the information L_1, L_2, f and θ we can calculate the elements of the matrix of the triplet cell (Use the expression from the **Hint**):

$$M_{triplet} = \begin{pmatrix} \frac{f^3+2L_1^2L_2-2L_1f(L_1+L_2)}{f^3} & \frac{2(f-L_1)(L_1f+L_2f-L_1L_2)}{f^2} & (L_1+L_2-\frac{L_1L_2}{f})\theta \\ \frac{2L_1(L_1L_2-L_1f-f^2)}{f^4} & \frac{f^3+2L_1^2L_2-2L_1f(L_1+L_2)}{f^3} & \frac{(f^2+L_1f-L_1L_2)}{f^2}\theta \\ 0 & 0 & 1 \end{pmatrix}$$

For the horizontal plane:

$$M_{triplet(H)} = \begin{pmatrix} 0.525 & 9.153 & 0.592 \\ -0.079 & 0.525 & 0.099 \\ 0 & 0 & 1 \end{pmatrix}$$

For the vertical plane:

Here we have to take into account that in our example we consider only horizontal bending magnets, and no bend in vertical ($\theta = 0$ in vertical). In addition we have to consider $f \to -f$,

$$M_{triplet(V)} = \begin{pmatrix} 0.296 & 22.26 & 0 \\ -0.04 & 0.296 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- Compute the horizontal and vertical machine tunes.

In this example we are considering the propagation of 3D vectors $(x, x', \Delta p/p_0)$:

$$\begin{pmatrix} x \\ x' \\ \Delta p/p_0 \end{pmatrix}_s = M \begin{pmatrix} x \\ x' \\ \Delta p/p_0 \end{pmatrix}_0 = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \\ \Delta p/p_0 \end{pmatrix}_0$$

REMEMBER: In terms of the betatron functions these elements of the matrix M between two points can be defined as follows:

$$M = \begin{pmatrix} \sqrt{\frac{\beta_x}{\beta_{x0}}}(\cos\phi_x + \alpha_{x0}\sin\phi_x) & \sqrt{\beta_x\beta_{x0}}\sin\phi_x & D_x \\ \frac{(\alpha_{x0}-\alpha_x)\cos\phi_x-(1+\alpha_{x0}\alpha_x)\sin\phi_x}{\sqrt{\beta_x\beta_{x0}}} & \sqrt{\frac{\beta_{x0}}{\beta_x}}(\cos\phi_x - \alpha_x\sin\phi_x) & D'_x \\ 0 & 0 & 1 \end{pmatrix}$$

The terms $m_{13} = D_x$ and $m_{23} = D'_x$, are the horizontal dispersion and the derivative of the horizontal dispersion over s, respectively.

We are dealing with a symmetric cell, so:
$\beta_x(s_0 + L_{cell}) = \beta_x(s_0) = \beta_{x0}$
$\alpha_x(s_0 + L_{cell}) = \alpha_x(s_0) = \alpha_{x0}$
$D_x(s_0 + L_{cell}) = D_x(s_0)$

These periodicity conditions are valid for the case of our triplet, FODO cells, and other symmetric cells. The Twiss parameters and the dispersion at the entrance of the cell are equal to those at the exit of the cell. Considering this we can rewrite the transport matrix of our triplet cell as:

$$M = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos\mu_x + \alpha_{x0}\sin\mu_x & \beta_{x0}\sin\mu_x & D_x \\ -\frac{(1+\alpha_{x0}^2)\sin\mu_x}{\beta_{x0}} & \cos\mu_x - \alpha_{x0}\sin\mu_x & D'_x \\ 0 & 0 & 1 \end{pmatrix}$$

where μ_x is the horizontal phase advance of the cell (Remember that we are calculating first the transport in the horizontal phase, and we have to proceed in a similar way with the vertical case).

Since we have calculated the matrix elements corresponding to our triplet cell, then we can calculate the phase advance comparing the elements $M_{triplet(H)} = M$:

$$\cos\mu_x = \frac{1}{2}(m_{11} + m_{22}) = 0.525 \longrightarrow \mu_x = 1.018\text{rad}$$

In order not to get confused with the concepts, it is necessary to remember that in circular machines the transfer matrix for a complete turn can also be written as:

$$\begin{pmatrix} \cos\tilde{\mu_x} + \alpha_{x0}\sin\tilde{\mu_x} & \beta_{x0}\sin\tilde{\mu_x} & D_x \\ -\frac{(1+\alpha_{x0}^2)\sin\tilde{\mu_x}}{\beta_{x0}} & \cos\tilde{\mu_x} - \alpha_{x0}\sin\tilde{\mu_x} & D'_x \\ 0 & 0 & 1 \end{pmatrix}$$

But in this case $\tilde{\mu_x}$ is the phase advance after one turn.

The tune is defined as the number of betatron oscillations per turn:

$$Q_{x,y} \equiv \frac{\tilde{\mu}_{x,y}}{2\pi} = \frac{N\mu_{x,y}}{2\pi}$$

where $N = 64$ is the number of periodic cells in the total machine.

The horizontal tune is

$$Q_x = \frac{N\mu_x}{2\pi} = \frac{64 \cdot 1.018}{2\pi} = 10.37$$

If we proceed in a similar way for the vertical plane with the transport matrix $M_{triplet(V)}$, we obtain:

$$Q_y = \frac{N\mu_y}{2\pi} = \frac{64 \cdot 1.27}{2\pi} = 12.94$$

- Compute the horizontal and vertical betatron functions at the entrance of a triplet cell.

$$\beta_{x0} \sin\mu_x = m_{12} = 9.153 \longrightarrow \beta_{x0} = 10.75$$

$$\beta_{y0} \sin\mu_y = m_{12} = 22.26 \longrightarrow \beta_{y0} = 23.31$$

- Compute the horizontal and vertical dispersion functions at the entrance of a triplet cell.

$$D_x = m_{13} = 0.592$$

$$D_y = 0$$

4 Exercise: normalized phase space

Let us consider the following phase space vector: (x, x'). The transformation to a normalized phase space (X, X') is given by:

$$\begin{pmatrix} X \\ X' \end{pmatrix} = \begin{pmatrix} 1/\sqrt{\beta_x} & 0 \\ \alpha_x/\sqrt{\beta_x} & \sqrt{\beta_x} \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}$$

The normalization process of the phase space is illustrated in the figure below:

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If we know that the transfer matrix between two points 1 and 2 (with phase advance ϕ_x between them) in the phase space (x, x') is given by:

$$M_{12} = \begin{pmatrix} \sqrt{\frac{\beta_{x2}}{\beta_{x1}}}(\cos\phi_x + \alpha_{x1}\sin\phi_x) & \sqrt{\beta_{x1}\beta_{x2}}\sin\phi_x \\ \frac{(\alpha_{x1}-\alpha_{x2})\cos\phi_x - (1+\alpha_{x1}\alpha_{x2})\sin\phi_x}{\sqrt{\beta_{x2}\beta_{x1}}} & \sqrt{\frac{\beta_{x1}}{\beta_{x2}}}(\cos\phi_x - \alpha_{x2}\sin\phi_x) \end{pmatrix}$$

Obtain the transfer matrix between two points 1 and 2 in the normalized phase space.

- Solution: if one writes

$$M_{12} = U_2^{-1} \cdot R \cdot U_1$$

  with U_1 the transformation into normalized coordinates for the Twiss parameters at 1, and U_2 its inverse for the Twiss parameters at 2: i.e.,

$$U_1 = \begin{pmatrix} \frac{1}{\sqrt{\beta_1}} & 0 \\ \frac{\alpha_1}{\sqrt{\beta_1}} & \sqrt{\beta_1} \end{pmatrix}; \qquad U_2^{-1} = \begin{pmatrix} \sqrt{\beta_2} & 0 \\ -\frac{\alpha_2}{\sqrt{\beta_2}} & \frac{1}{\sqrt{\beta_2}} \end{pmatrix}$$

It can be shown that the matrix M_{12} can be written as:

$$M_{12} = \begin{pmatrix} \sqrt{\beta_2} & 0 \\ -\frac{\alpha_2}{\sqrt{\beta_2}} & \frac{1}{\sqrt{\beta_2}} \end{pmatrix} \begin{pmatrix} \cos\Delta\phi & \sin\Delta\phi \\ -\sin\Delta\phi & \cos\Delta\phi \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{\beta_1}} & 0 \\ \frac{\alpha_1}{\sqrt{\beta_1}} & \sqrt{\beta_1} \end{pmatrix}$$

with

$$R = \begin{pmatrix} \cos\Delta\phi & \sin\Delta\phi \\ -\sin\Delta\phi & \cos\Delta\phi \end{pmatrix}.$$