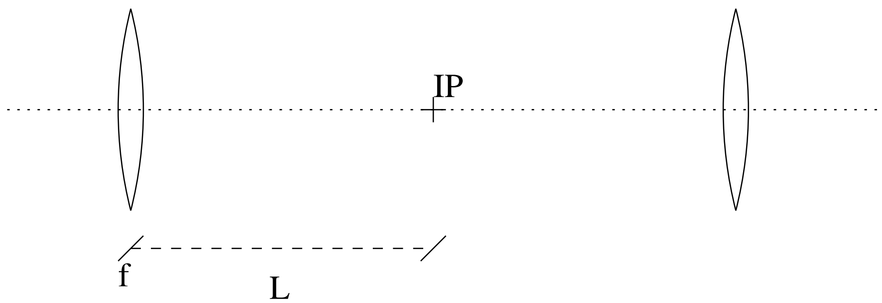
Transverse Beam Dynamics

JUAS 2017 - tutorial 4 (solutions)

1 Exercise: Low-Beta Insertion

Consider the following low-beta insertion around an interaction point (IP). The quadrupoles are placed with mirror-symmetry with respect to the IP:



The beam enters the quadrupole with Twiss parameters $\beta_0 = 20$ m and $\alpha_0 = 0$. The drift space has length $L = 10$ m.

(i) Determine the focal length of the quadrupole in order to locate the waist at the IP.

(ii) What is the value of $\beta^\star$?

(iii) What is the phase advance between the quadrupole and the IP?

Solution.

$$M = \begin{pmatrix} 1 - \frac{L}{f} & L \\ -\frac{1}{f} & 1 \end{pmatrix}$$

$$\begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}_{\mathrm{IP}} = M \cdot \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}_0 \cdot M^T$$

$$\begin{pmatrix} \beta_{\mathrm{IP}} & 0 \\ 0 & 1/\beta_{\mathrm{IP}} \end{pmatrix} = M \cdot \begin{pmatrix} \beta_0 & 0 \\ 0 & 1/\beta_0 \end{pmatrix} \cdot M^T$$

We get a system of equations:

$$\begin{cases} \beta_{\mathrm{IP}} = & \beta_0 \left(1 - \frac{L}{f}\right)^2 + \frac{L^2}{\beta_0} \\ \frac{1}{\beta_{\mathrm{IP}}} = & \frac{\beta_0}{f^2} + \frac{1}{\beta_0} \end{cases}$$

multiplying them:

$$1 = \left(\beta_0 \left(1 - \frac{L}{f}\right)^2 + \frac{L^2}{\beta_0}\right)\left(\frac{\beta_0}{f^2} + \frac{1}{\beta_0}\right)$$

and solving for f:

$$f = \frac{\beta_0 \sqrt{(\beta_0^2 - 4L^2)} + \beta_0^2}{2L}$$

from which one finds:

$$f = 20 \text{ m}$$

and substituting back into one of the equations in the system:

$$\beta_{\mathrm{IP}} = 10 \text{ m}.$$

The phase advance can be computed remembering that

$$M_{0\to s} = \begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}}\left(\cos\psi_s + \alpha_0 \sin\psi_s\right) & \sqrt{\beta_s\beta_0}\sin\psi_s \\ \frac{(\alpha_0-\alpha_s)\cos\psi_s-(1+\alpha_0\alpha_s)\sin\psi_s}{\sqrt{\beta_s\beta_0}} & \sqrt{\frac{\beta_0}{\beta_s}}\left(\cos\psi_s - \alpha_s\sin\psi_s\right) \end{pmatrix}$$

In this case, $\alpha_0 = \alpha_{\mathrm{IP}} = 0$,

$$\mathrm{Trace}\,(M) = \frac{3}{2} = \left(\sqrt{\frac{\beta^\star}{\beta_0}} + \sqrt{\frac{\beta^0}{\beta^\star}}\right)\cos\Delta\mu$$

$$\Delta\mu = \arccos\left(\frac{3}{2}\cdot\frac{1}{\sqrt{\frac{\beta^\star}{\beta_0}}+\sqrt{\frac{\beta^0}{\beta^\star}}}\right) = \arccos\left(\frac{3}{2}\cdot\frac{1}{2.1213}\right) = 45 \text{ degrees}$$

Alternatively, given that the system:

$$M = Q\cdot D\cdot D\cdot Q$$

is indeed periodic, one can say:

$$M = \begin{pmatrix} 1-\frac{2L}{f} & 2L \\ \frac{2L}{f^2}-\frac{2}{f} & 1-\frac{2L}{f} \end{pmatrix}$$

$$\cos\Delta\mu_{\mathrm{twice}} = \frac{1}{2}\mathrm{Trace}\,(M) = \frac{1}{2}\mathrm{Trace}\left(2-\frac{4L}{f}\right) = 0$$

$$\Delta\mu_{\mathrm{twice}} = 90 \text{ degrees} \qquad \Rightarrow \Delta\mu = 45 \text{ degrees}$$

2 Exercise: beam size and luminosity

An e^+e^- collider has an interaction Point (IP) with $\beta_x^* = 0.5$ m and $\beta_y^* = 0.1$ cm. The peak luminosity available by a e^+e^- collider can be written as:

$$L = \frac{N_{\mathrm{b}} N_{e^-} N_{e^+} f_{\mathrm{rev}}}{4\pi\sigma_x^*\sigma_y^*}\ [\mathrm{cm}^{-2}\mathrm{s}^{-1}]$$

where $N_{\mathrm{b}} = 80$ is the number of bunches per beam (we assume the same number of bunches for both the e^- and the e^+ beams), $N_{e^-} = N_{e^+} = 5\times 10^{11}$ is the number of particles per bunch (we assume the same number for both e^- and e^+ bunches), and f_{rev} is the revolution frequency. The horizontal and vertical normalised beam emittances are respectively: $\epsilon_{x,N} = 2.2$ mm and $\epsilon_{y,N} = 4.7$ μm.

- Compute the revolution frequency f_{rev}, knowing that the circumference is 80 km and that the beam moves nearly at the speed of light

Solution. The revolution period is given by $T_{rev} = \text{circumference}/c = 80\text{km/c}$, and therefore the revolution frequency is:

$$f_{rev} = 1/T_{rev} = c/80 \text{ km} \simeq 3.75 \text{ kHz}$$

- Calculate the beam transverse beam sizes σ_x^* and σ_y^* at the IP, and the luminosity L for two different beam energies: 45 GeV and 120 GeV

Solution. For 45 GeV beam energy: in this case the Lorentz factor is $\gamma = 88062.622$, and $\sigma_x^* = \sqrt{\beta_x^*\epsilon_{x,N}/\gamma} \simeq 111.76$ μm and $\sigma_y^* = \sqrt{\beta_y^*\epsilon_{y,N}/\gamma} \simeq 0.23$ μm, and the luminosity is $L \simeq 2.32\times 10^{34}$ cm^{-2}s^{-1}

For 120 GeV beam energy: in this case the Lorentz factor is $\gamma = 234833.66$, and $\sigma_x^* = \sqrt{\beta_x^*\epsilon_{x,N}/\gamma} \simeq 68.56$ μm and $\sigma_y^* = \sqrt{\beta_y^*\epsilon_{y,N}/\gamma} \simeq 0.14$ μm, and the luminosity is $L \simeq 6.22\times 10^{34}$ cm^{-2}s^{-1}

- What are the beam divergences (horizontal and vertical) at the IP for the 45 GeV case?

Solution. Where $\alpha = 0$ we have

$$\sigma^*_{x'} = \sqrt{\frac{\epsilon_{x,N}}{\gamma \beta^*_x}} = ...$$

- What is the value of the betatron function at position $s = 0.5$ m from the IP?

Solution. We know that the betatron function in the drift space of a low beta region (where we have the interaction point) depends on the longitudinal coordinate as follows:

$$\beta(s) = \beta^* + \frac{s^2}{\beta^*}$$

Therefore, $\beta_x(0.5\text{m}) = 1$ m, and $\beta_y(0.5\text{m}) = 250$ m

3 Exercise: Bump and Orbit Control

Given two kickers located at the two ends of a FODO cell with phase advance 45 degrees (the two kickers are located at L_{cell} distance from each other), compute the strengths of such kickers (in radians) in order to give the beam, initially at $(x_i,\, x'_i) = (0,\, 0)$, an arbitrary offset at the end of the cell while preserving its angle, $\left(x_f,\, x'_f\right) = (x_{\text{arbitrary}},\, 0)$.

Solution

The transfer matrix of a periodic cell is:

$$M = \begin{pmatrix} \cos\varphi + \alpha \sin\psi & \beta \sin\varphi \\ -\gamma \sin\varphi & \cos\varphi - \alpha \sin\varphi \end{pmatrix}$$

Substituting the value for the phase advance we get the matrix to apply to the beam right after the first kick k_1:

$$\begin{pmatrix} x_f \\ x'_f \end{pmatrix} = \frac{\sqrt{2}}{2} \begin{pmatrix} 1+\alpha & \beta \\ -\gamma & 1-\alpha \end{pmatrix} \begin{pmatrix} 0 \\ k_1 \end{pmatrix} = \frac{\sqrt{2}}{2} \begin{pmatrix} \beta k_1 \\ (1-\alpha) k_1 \end{pmatrix}$$

From this we see that to achieve an arbitrary x_f we need:

$$k_1 = \frac{\sqrt{2} x_f}{\beta}$$

The second kick, k_2, has only to remove the final tilt:

$$k_2 = -x'_f = -\frac{(1-\alpha)}{\sqrt{2}} k_1$$

Notice that one can reduce the strength of the kickers by placing them close to a focusing quadrupoles, where β is maximum.

4 Exercise: measurement of Twiss parameters

One of the possible ways to determine experimentally the Twiss parameters at a given point makes use of a so-called quadrupole scan. One can measure the transverse size of the beam in a profile monitor, called Wire Beam Scanner (WBS), located at a distance L downstream a focusing quadrupole, as a function of the normalized gradient in this quadrupole. This allows to compute the emittance of the beam, as well as the β and the α functions at the entrance of the quadrupole.

Let's consider a quadrupole Q with a length of $l = 20$ cm. This quadrupole is installed in an electron transport line where the particle momentum is 300 MeV/c. At a distance $L = 10$ m from the quadrupole the transverse beam size is measured with a WBS, for various values of the current I_Q. The maximum value of the quadrupole gradient G is obtained for a current of 100 A, and is $G = 1$ T/m. G is proportional to the current.

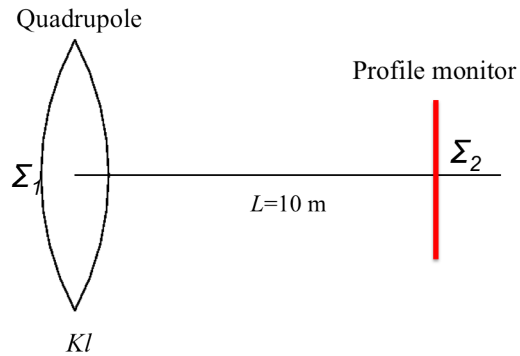
Advice: use thin-lens approximation.

4.1 How does the normalized focusing strength K vary with I_Q?

Answer:

If G proportional to I_Q: $G = C \cdot I_Q$ where C is the proportionality coefficient. We know that $G = 1$ T/m when $I_Q = 100$ A, therefore $C = 0.01$ T/(A·m).

4.2 Let Σ_1 and Σ_2 be the 2×2 matrices with the Twiss parameters, $\Sigma = \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}$, at the quadrupole entrance and at the wire scanner, respectively.



It is worth explaining that the matrix Σ multiplied by the emittance ϵ is the covariance matrix of the beam distribution:

$$\Sigma\epsilon = \begin{pmatrix} \beta\epsilon & -\alpha\epsilon \\ -\alpha\epsilon & \gamma\epsilon \end{pmatrix} = \begin{pmatrix} \langle x^2 \rangle & \langle xx' \rangle \\ \langle x'x \rangle & \langle x'^2 \rangle \end{pmatrix}$$

The transverse beam size of the beam is given by $\sigma_x = \sqrt{\langle x^2 \rangle} = \sqrt{\beta_x \epsilon_x}$ (horizontal beam size), and $\sigma_y = \sqrt{\langle y^2 \rangle} = \sqrt{\beta_y \epsilon_y}$ (vertical beam size). Here we will simply use the following notation: $\sigma_1 = \sqrt{\beta_1 \epsilon}$ for the beam size (horizontal or vertical) at position 1, and $\sigma_2 = \sqrt{\beta_2 \epsilon}$ for the beam size (horizontal or vertical) at position 2.

- Give the expression Σ_2 as function of α_1, β_1, and γ_1

Answer:

The matrix Σ propagates from position 1 to position 2 as follows:

$$\Sigma_2 = M \Sigma_1 M^T$$

where M is the transfer matrix of the system and M^T its transpose. We have:

$$\begin{aligned} \Sigma_2 = \begin{pmatrix} \beta_2 & -\alpha_2 \\ -\alpha_2 & \gamma_2 \end{pmatrix} &= \begin{pmatrix} 1 - KlL & L \\ -Kl & 1 \end{pmatrix} \begin{pmatrix} \beta_1 & -\alpha_1 \\ -\alpha_1 & \gamma_1 \end{pmatrix} \begin{pmatrix} 1 - KlL & -Kl \\ L & 1 \end{pmatrix} \\ &= \begin{pmatrix} \beta_1 L^2 (Kl)^2 + 2L(\alpha_1 L - \beta_1) Kl + \beta_1 - 2\alpha_1 L + \gamma_1 L^2 & \beta_1 L (Kl)^2 + (2\alpha_1 L - \beta_1) Kl + \gamma_1 L - \alpha_1 \\ \beta_1 L (Kl)^2 + (2\alpha_1 L - \beta_1) Kl + \gamma_1 L - \alpha_1 & \beta_1 (Kl)^2 + 2\alpha_1 Kl + \gamma_1 \end{pmatrix} \end{aligned} \tag{1}$$

- Show that β_2 can be written in the form: $\beta_2 = A_2 (Kl)^2 + A_1 (Kl) + A_0$
- Express A_0, A_1, and A_2 as a function of L, α_1, β_1, and γ_1

Answer:

We can see from Eq. (1) that:

$$\beta_2 = \beta_1 L^2 (Kl)^2 + 2L(\alpha_1 L - \beta_1) Kl + \beta_1 - 2\alpha_1 L + \gamma_1 L^2$$

and therefore:

$$
\begin{aligned}
A_2 &= \beta_1 L^2 \\
A_1 &= 2L(\alpha_1 L - \beta_1) \\
A_0 &= \beta_1 - 2\alpha_1 L + \gamma_1 L^2
\end{aligned}
$$

Hint for the next questions: show that if you express β_2 as

$$\beta_2 = B_0 + B_1 \left(Kl - B_2\right)^2$$

you have:
$B_0 = A_0 - A_1^2/4A_2^2 = L^2/\beta_1$
$B_1 = A_2 = L^2\beta_1$
$B_2 = -A_1/A_2 = 1/L - \alpha_1/\beta_1$

4.3 The transverse beam r.m.s. beam size is $\sigma = \sqrt{\epsilon\beta}$, where ϵ is the transverse emittance. Express σ_2 as a function of Kl and find its minimum, $(Kl)_{\mathbf{min}}$. Give the expression for $\frac{\mathrm{d}\sigma_2}{\mathrm{d}(Kl)}$.

As we have seen in the previous questions β_2 depends quadratically on Kl: $\beta_2 = B_0 + B_1 \left(Kl - B_2\right)^2$. Since ϵ is constant, if we want to minimize σ_2, we have to minimize β_2:

$$\frac{\mathrm{d}\beta_2}{\mathrm{d}(Kl)} = 0 \longrightarrow 2B_1(Kl - B_2) = 0 \longrightarrow (Kl)_{min} = B_2 = \frac{1}{L} - \frac{\alpha_1}{\beta_1} \tag{2}$$

We can write:

$$\sigma_2^2 = \beta_2\epsilon = \frac{L^2}{\beta_1}\left(1 + \beta_1^2(Kl - (Kl)_{min})^2\right)\epsilon$$

Why is this useful? By means of a quadrupole scan (changing the strength of the quadrupole) we look for the strength Kl which minimizes the value σ_2^2. We fit a parabola to the measurements σ_2^2 vs. Kl, and select then $\sigma_2^2((Kl)_{min})$. The minimum beam size is given by:

$$\mathrm{Min}(\sigma_2) = L\sqrt{\frac{\epsilon}{\beta_1}} = \sqrt{B_0\epsilon} \tag{3}$$

The derivative of σ_2 is: $\frac{\mathrm{d}\sigma_2}{\mathrm{d(Kl)}} = \frac{L^2\beta_1}{\sigma_2}(Kl - (kl)_{min})\epsilon$

4.4 How does σ_2 vary with Kl when $|Kl - (Kl)_{\mathbf{min}}| \gg 1/\beta_1$?

Under this condition:

$$\sigma_2^2 = \frac{L^2}{\beta_1}\left(1 + \beta_1^2(Kl - (Kl)_{min})^2\right)\epsilon \longrightarrow \sigma_2 \simeq L\sqrt{\beta_1\epsilon}(Kl - (Kl)_{min})$$

For $|Kl - (Kl)_{\mathrm{min}}| \gg 1/\beta_1$, σ_2 depends linearly on Kl, with slope $\frac{\mathrm{d}\sigma_2}{\mathrm{d(Kl)}} = L\sqrt{\beta_1\epsilon} = L\sigma_1$.

4.5 Deduce the values of α_1, β_1, and γ_1 from the measurement σ_2, as a function of the quadrupole current I_Q.

We know that

$$Kl = \frac{G \cdot l}{p/e} = \frac{C \cdot l \cdot I_Q}{p/e} = \frac{0.01[\mathrm{T/(Am)}]\cdot 0.2[\mathrm{m}]}{(0.3[\mathrm{GeV}]/0.3)[\mathrm{Tm}]} \cdot I_Q = 2 \times 10^{-3} \cdot I_Q$$

If we measure σ_2 as a function of the quadrupole current I_Q, from the minimum value we can get β_1 (Eq. (3)), and since from the measurement we obtain $(Kl)_{min} = 2 \times 10^{-3}(I_Q)_{min}$, using Eq. (2) we can calculate α_1. Once we know β_1 and α_1, it is then straightforward to calculate $\gamma_1 = (1 + \alpha_1^2)/\beta_1$.