Synchrotron radiation

R. Bartolini

John Adams Institute for Accelerator Science, University of Oxford

and

Diamond Light Source

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Contents

Effects of synchrotron radiation on electron beam dynamics

The electrons radiate energy: the equations of motion have a dissipative term; the system is non conservative and Liouville's theorem does not apply;

The emission of radiation leads to damping of the betatron and synchrotron oscillations

Radiation is not emitted continuously but in individual photons. The energy emitted is a random variable with a known distribution (from the theory of synchrotron radiation seen in previous lectures)

This randomness introduce fluctuations which tend to increase the betatron and synchrotron oscillations

Damping and growth reach an equilibrium in an electron synchrotron. This equilibrium defines the characteristics of the electron beam (e.g. emittance, energy spread, bunch size, etc)

Effects of synchrotron radiation on electron beam dynamics

We will now look at the effect of radiation damping on the three planes of motion

We will use two equivalent formalisms:

damping from the equations of motion in phase space

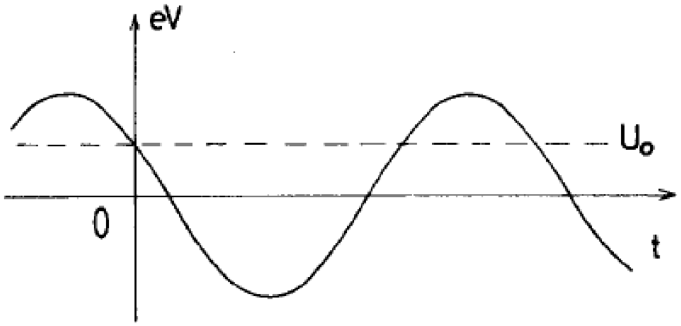
damping as a change in the Courant-Snyder invariant

The system is non-conservative hence the Courant-Snyder invariant – i.e. the area of the ellipse in phase space, is no longer a constant of motion

We will then consider the effect of radiation quantum excitation on the three planes of motion

We will use the formalism of the change of the Courant-Snyder invariant

From the lecture on longitudinal motion



A particle in an RF cavity changes energy according to the phase of the RF field found in the cavity

$$\Delta E = eV(t) = eV_o \sin(\varphi_s - \omega_{RF} t)$$

On the other hand, a particle lose energy because of synchrotron radiation, interaction with the vacuum pipe, etc. Assume that for each turn the energy losses are U_0

The synchronous particle is the particle that arrives at the RF cavity when the voltage is such that it compensate exactly the average energy losses U_0

$$\Delta E = U_0 = eV_0 \sin(\varphi_s)$$

Negative RF slope ensure stability for $\alpha > 0$ (above transition)

Veksler 1944 MacMillan 1945: **the principle of phase stability**

We describe the longitudinal dynamics in terms of the variables (ε, τ)

energy deviation ε w.r.t the synchronous particle

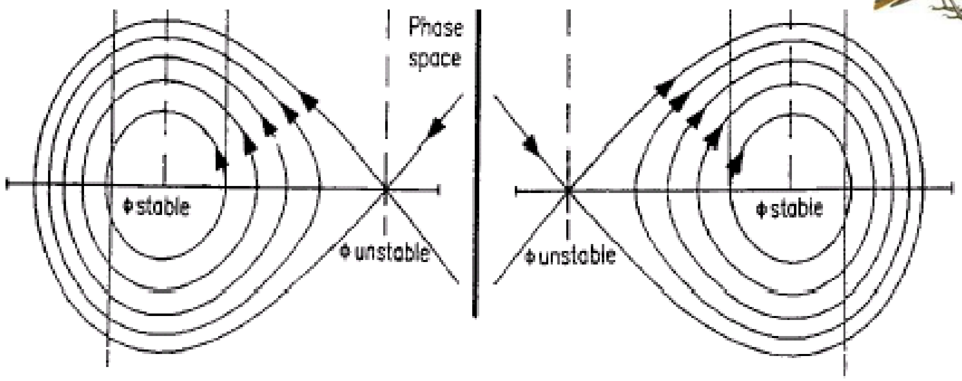
and τ time delay w.r.t. the synchronous particle

RF buckets recap.

Equations for the RF bucket

$$\varepsilon' = \frac{qV_0}{L}\left[\sin(\phi_s - \omega\tau) - \sin\phi_s\right]$$

$$\tau' = -\frac{\alpha_c}{E_s}\varepsilon$$



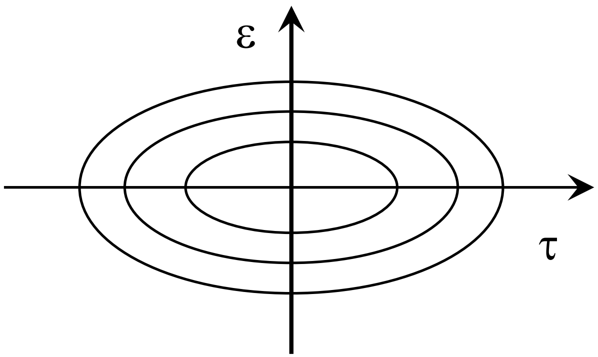
$\alpha > 0$ above transition $\alpha < 0$ below transition

Aide-memoire for stable motion: above transition the head goes up in energy, below transition the head goes down in energy

Linearised equations for the motion in the RF bucket: the phase space trajectories become ellipses

$$\varepsilon' = \frac{e}{T_0}\frac{dV}{d\tau}\tau$$

$$\tau' = -\frac{\alpha_c}{E_s}\varepsilon$$



$$\omega_s^2 = \frac{\alpha_c e\dot{V}}{T_0 E_0}$$ angular synchrotron frequency

Radiation damping: Longitudinal plane (I)

In presence of synchrotron radiation losses, with energy loss per turn U_0, the RF fields will compensate the loss per turn and the synchronous phase will be such that

$$U_0 = eV_0 \sin(\varphi_s)$$

The energy loss per turn U_0 depends on energy E. The rate of change of the energy will be given by two terms

$$\frac{\Delta E}{T_0} = \frac{eV(t) - U_0(E)}{T_0}$$

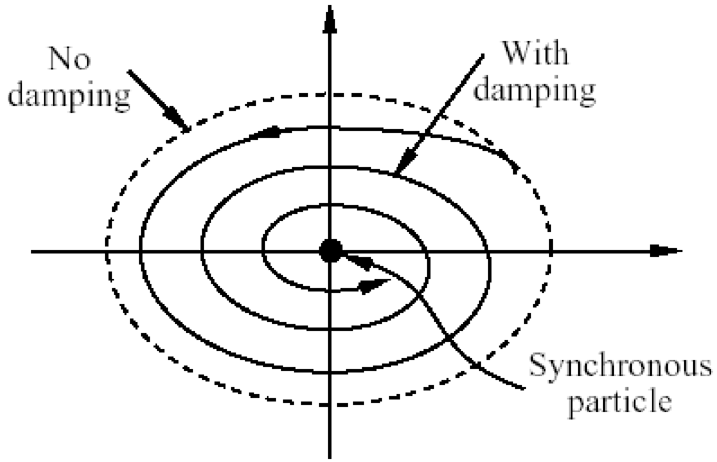
Assuming $\Delta E << E$ and $\tau << T_0$ we can expand

$$\frac{d\varepsilon}{dt} = \frac{\left(U_0(0) + e\frac{dV}{d\tau}\tau\right) - \left(U_0(0) + \frac{dU_0}{dE}\varepsilon\right)}{T_0} = \frac{e}{T_o}\frac{dV}{d\tau}\tau - \frac{1}{T_0}\frac{dU_0}{dE}\varepsilon$$

$$\frac{d\tau}{dt} = -\alpha_c \frac{\varepsilon}{E_s}$$

additional term responsible for damping

Radiation damping: Longitudinal plane (II)



The derivative $\frac{dU_0}{dE} \quad (>0)$

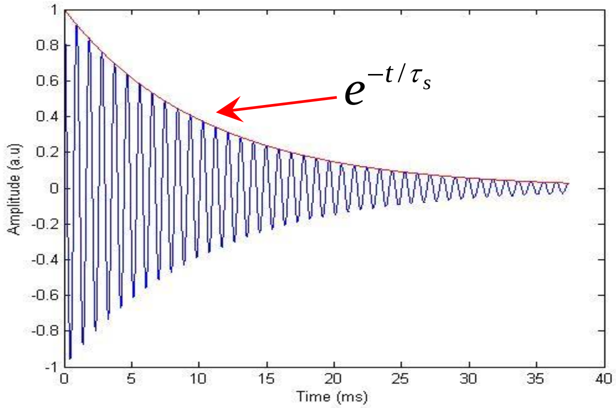
is responsible for the damping of the longitudinal oscillations

Combining the two equations for (ε, τ) in a single second order differential equation

$$\frac{d^2\varepsilon}{dt^2} + \frac{2}{\tau_s}\frac{d\varepsilon}{dt} + \omega_s^2\varepsilon = 0 \quad \longrightarrow \quad \varepsilon = Ae^{-t/\tau_s}\sin\left(\sqrt{\omega_s^2 - \frac{4}{\tau_s^2}}\,t + \varphi\right)$$

$$\omega_s^2 = \frac{\alpha e\dot{V}}{T_0 E_0}$$ angular synchrotron frequency

$$\frac{1}{\tau_s} = \frac{1}{2T_0}\frac{dU_0}{dE}$$ longitudinal damping time



Computation of dU_0/dE

We have to compute the dependence of U_0 on energy the E (or rather on the energy deviation ε)

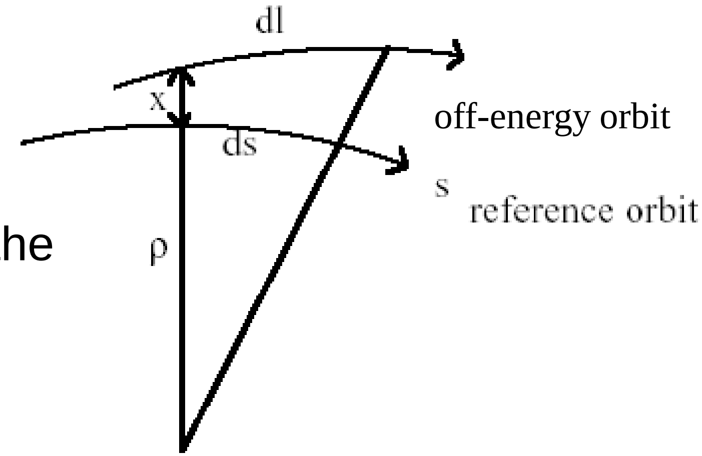
$$U(\varepsilon) = \oint P dt$$

The energy loss per turn is the integral of the power radiated over the time spent in the bendings. Both depend on the energy of the particle.

$$U(\varepsilon) = \oint P(\varepsilon) dt = \frac{1}{c}\oint P(\varepsilon) dl = \frac{1}{c}\oint P(\varepsilon)\left(1+\frac{x}{\rho}\right) ds$$

The time that an off-energy particle spends in the bending element dl is given by

$$dt = \frac{dl}{c} = \frac{1}{c}\left(1+\frac{x}{\rho}\right) ds$$



This is an elementary geometric consideration on the arc length of the trajectory for different energies

Computation of dU_0/dE

Using the dispersion function

$$U(\varepsilon) = \frac{1}{c}\oint P\left(1 + \frac{D}{\rho}\frac{\varepsilon}{E_0}\right)ds \qquad U(0) = U_0$$

Computing the derivative w.r.t. ε at $\varepsilon = 0$ we get [**Sands**]

$$\frac{dU}{d\varepsilon} = \frac{1}{c}\oint\left(\frac{dP}{d\varepsilon} + \frac{D}{\rho}\frac{P}{E_0}\right)ds$$

To compute $dP/d\varepsilon$ we use the result obtained in the lecture on synchrotron radiation, whereby the instantaneous power emitted in a bending magnet with field B by a particle with energy E is given by

$$P = \frac{e^2}{6\pi\varepsilon_0 c}\left|\dot{\vec{\beta}}\right|^2\gamma^4 = \frac{e^2}{6\pi\varepsilon_0 c}\left|\dot{\vec{\beta}}\right|^2\frac{E}{E_o^4}^4 = \frac{e^2}{6\pi\varepsilon_0 m^2 c^3}\left|\frac{d\vec{p}}{dt}\right|^2\gamma^2 = \frac{e^2 c}{6\pi\varepsilon_0}\frac{\gamma^4}{\rho^2} = \frac{e^4}{6\pi\varepsilon_0 m^4 c^5}E^2B^2$$

Watch out! There is an implicit dependence of ρ or B on E. Off energy particles have different curvatures ρ or can experience different B if B varies with x

Computation of dU_0/dE

and since P is proportional to E^2B^2 we can write [Sands]

$$\frac{dP}{d\varepsilon} = \frac{2P_0}{E_0} + \frac{2P_0}{B_0}\frac{dB}{d\varepsilon} = \frac{2P_0}{E_0} + \frac{2P_0}{B_0}\frac{D}{E_0}\frac{dB}{dx}$$

check this as an exercise !

we get

$$\frac{dU}{d\varepsilon} = \frac{1}{c}\oint\left(\frac{2P_0}{E_0} + \frac{2P_0}{B_0}\frac{dB}{dx}\frac{D}{E_0} + \frac{P_0 D}{\rho E_0}\right)ds$$

and using

$$k\rho = \frac{1}{B_0}\frac{dB}{dx} \qquad U_0 = \frac{1}{c}\oint P_0 ds$$

We have the final result

$$\frac{dU(\varepsilon)}{d\varepsilon} = \frac{2U_0}{E_0} + \frac{1}{cE_0}\oint P_0 D\left(\frac{1}{\rho} - 2k\rho\right)ds$$

Radiation damping: Longitudinal plane (III)

The longitudinal damping time reads

$$\frac{1}{\tau_\varepsilon} = \frac{1}{2T_0}\frac{dU}{dE} = \frac{1}{2T_0}\frac{U_0}{E_0}(2+\wp) \qquad \wp = \frac{\oint \frac{D}{\rho}\left(\frac{1}{\rho^2} - 2k\rho\right)ds}{\oint \frac{1}{\rho^2}ds}$$

$\wp$ depends only on the magnetic lattice; typically it is a small positive quantity

$$\tau_\varepsilon = \frac{2T_0E_0}{U_0(2+\wp)} \approx \frac{E_0T_0}{U_0}$$

τ_ε is approximately the time it takes an electron to radiate all its energy (with constant energy loss U_0 per turn)

For separated function magnets with constant dipole field:

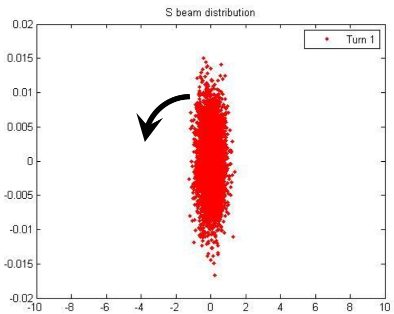
$$\frac{1}{\tau_\varepsilon} = \frac{U_0}{2E_0T_0}\left(2+\frac{\alpha R}{\rho}\right) \qquad \frac{1}{\tau_\varepsilon} \propto \frac{\gamma^3}{\rho R}$$

Tracking example: longitudinal plane

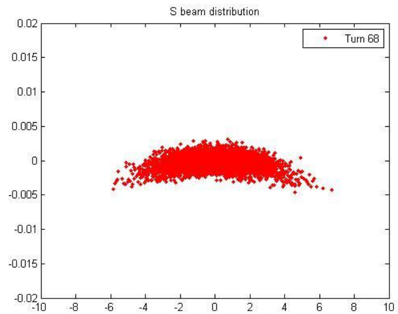
Consider a storage ring with a synchrotron tune of 0.0037 (273 turns);

and a radiation damping of 6000 turns:

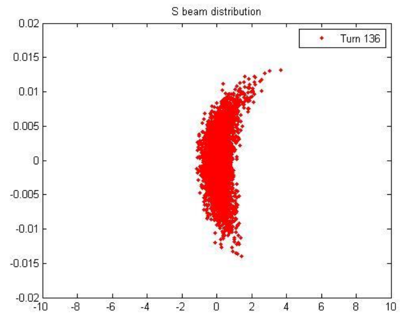
start



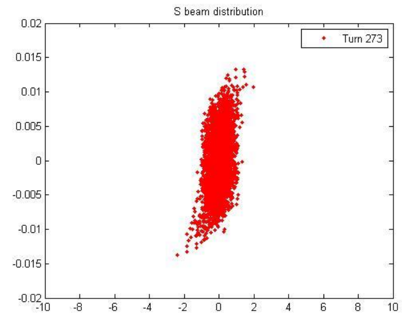
¼ of synch period



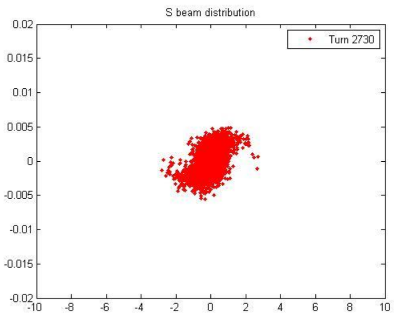
½ of synch period



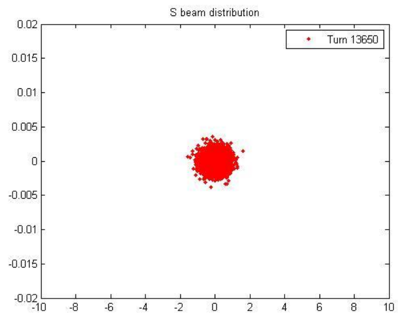
1 synch period



10 synch periods



50 synch periods



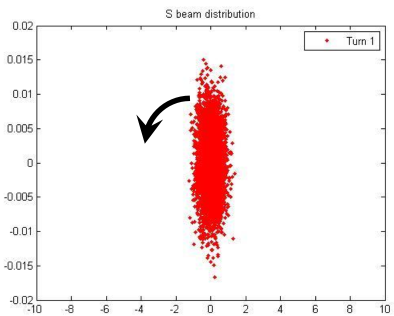
After 50 synchrotron periods (~2 radiation damping time) the longitudinal phase space distribution has almost reached the equilibrium and is matched to the RF bucket

Tracking example: longitudinal plane

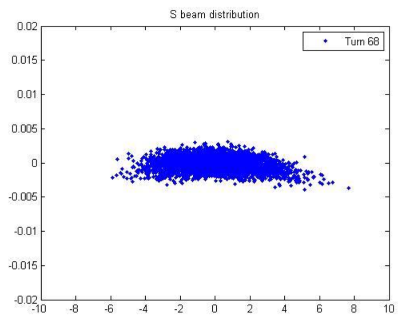
Consider a storage ring with a synchrotron tune of 0.0037 (273 turns);

negligible radiation damping:

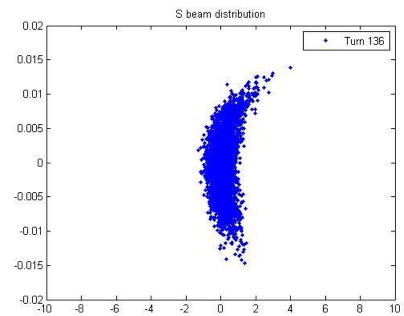
start



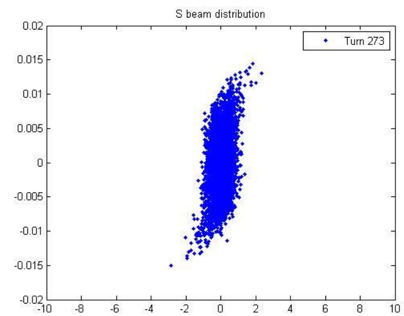
¼ of synch period



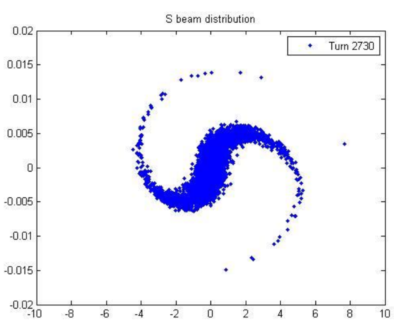
½ of synch period



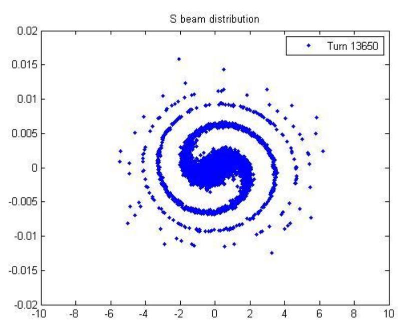
1 synch period



10 synch periods



50 synch periods



After 50 synchrotron periods the longitudinal phase space distribution is completely filamented (decoherence).

Any injection mismatch will blow up the beam

Transverse plane: vertical oscillations (I)

We now want to investigate the radiation damping in the vertical plane.

Because of radiation emission the motion in phase space is no longer conservative and symplectic, i.e. the area of the ellipse defining the Courant-Snyder invariant is changing along one turn. We want to investigate this change.

We assume to simplify the calculations that we are in a section of the ring where ($\alpha_z = 0$), then

$$z = A\cos(\phi(s) + \phi_0) \qquad z' = -\frac{A}{\beta}\sin(\phi(s) + \phi_0)$$

The ellipse in the vertical phase space is upright. The Courant-Snyder invariant reads

$$A^2 = z^2 + (\beta z')^2$$

Transverse plane: vertical oscillations (II)

Effect of the emission of a photon:

The photon is emitted in the direction of the momentum of the electron (remember the cone aperture is $1/\gamma$)

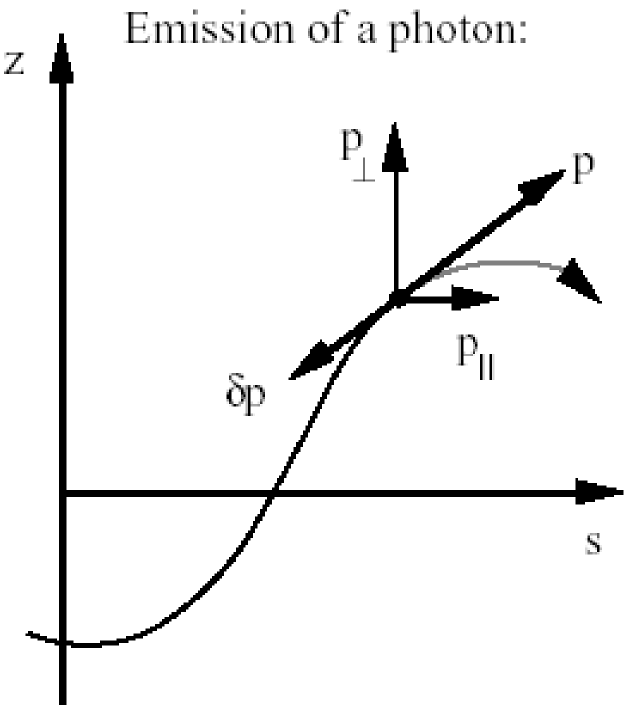
The momentum p is changed in modulus by dp but it is not changed in direction

neither z nor z' change

and

the oscillation pattern is not affected since $D_z = 0$

(see later case where $Dx \neq 0$ as for the horizontal plane)



Therefore the Courant-Snyder invariant does not change as result of the emission of a photon

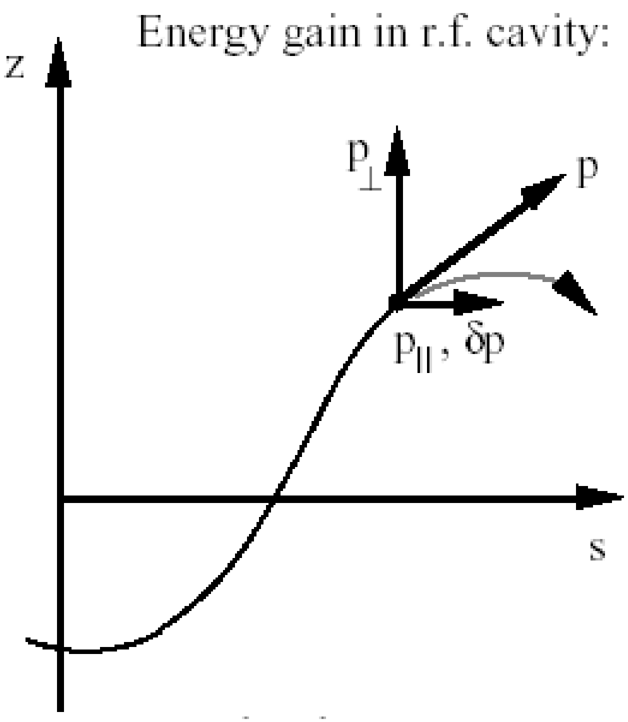
… however the RF cavity must replenish the energy lost by the electron

Transverse plane: vertical oscillations (III)

In the RF cavity the particle sees a longitudinal accelerating field therefore only the longitudinal component is increased to restore the energy

The momentum variation is no longer parallel to the momentum

this leads to a reduction of the betatron oscillations amplitude



The angle $z' = \frac{P_\perp}{p_\parallel}$ changes because

$$z' + \delta z' = \frac{p_\perp}{p_\parallel + \delta p} \approx z'\left(1 - \frac{\delta p}{p}\right)$$

$$\delta z' = -z' \frac{\delta \varepsilon}{E_0}$$

$\delta\varepsilon$ acquired in the RF cavity

Transverse plane: vertical oscillations (IV)

After the passage in the RF cavity the expression for the vertical invariant becomes

$$A^2 = z^2 + (\beta z')^2 \qquad (A + \delta A)^2 = z^2 + [\beta(z' + \delta z')]^2$$

$$A\delta A = \beta^2 z' \, \delta \, z' = -\beta^2 z'^2 \frac{\delta\varepsilon}{E_0}$$

The change in the Courant-Snyder invariant depends on the angle z' for this particular electron. Let us consider now all the electrons in the phase space travelling on the ellipse, and therefore having all the same invariant A

For each different z' the change in the invariant will be different. However averaging over the electron phases, assuming a uniform distribution along the ellipse, we have

$$< z'^2 >_{e-} = \frac{A^2}{2\beta^2} \qquad \text{and therefore} \qquad < \delta A >_{e-} = -\frac{A}{2}\frac{\delta\varepsilon}{E_0}$$

The average invariant decreases.

Transverse plane: vertical oscillations (V)

Let us consider now all the photons emitted in one turn. The total energy lost is

$$U_0 = \sum_C \varepsilon$$

The RF will replenish all the energy lost in one turn.

Summing the contributions $\delta\varepsilon$, we find that in one turn:

$$<< \delta A >_{e^-} >_{ph} = -\frac{A}{2}\frac{U_0}{E_0} \qquad \text{we write} \qquad \frac{\Delta A}{A} = -\frac{U_0}{2E_0} \qquad -\frac{1}{A}\frac{dA}{dt} = \frac{U_0}{2E_0T_0} = \frac{1}{\tau_z}$$

The average invariant decreases exponentially with a damping time τ_z

$\tau_z \approx$ half of longitudinal damping time always dependent on $1/\gamma^3$.

This derivation remains true for more general distribution of electron in phase space with invariant A (e.g Gaussian)

The synchrotron radiation emission combined with the compensation of the energy loss with the RF cavity causes the damping.

Transverse plane: vertical oscillations (VI)

The betatron oscillations are damped in presence of synchrotron radiation

$$z(t) = z_0 e^{-t/\tau_z} \sin\left(\omega_\beta t + \varphi\right)$$

Since the emittance of a bunch of particles is given by the average of the square of the betatron amplitude of the particles in the bunch taken ofver the bunch distribution in phase space

$$\varepsilon_z = \frac{< z^2 >}{\beta_z}$$

the emittance decays with a time constant which is half the radiation damping time

$$\varepsilon_z(t) = \varepsilon_z(0) \exp\left(-\frac{2t}{\tau_z}\right)$$

Transverse plane: horizontal oscillations (I)

The damping of the horizontal oscillation can be treated with the same formalism used for the vertical plane, e.g.

- consider the electron travelling on an ellipse in phase space with invariant A
- compute the change in coordinates due to the emission of one photon
- compute the change of coordinates due to the passage in the RF
- averaging over all electron with the same invariant
- compute the change in the average invariant for all photons emitted in one turn

The new and fundamental difference is that in the horizontal plane we do not neglect the dispersion, i.e. $D_x \neq 0$

The reference orbit changes when a quantum is emitted because of D_x in the bendings. The electron will oscillate around its new off-energy orbit. In details:

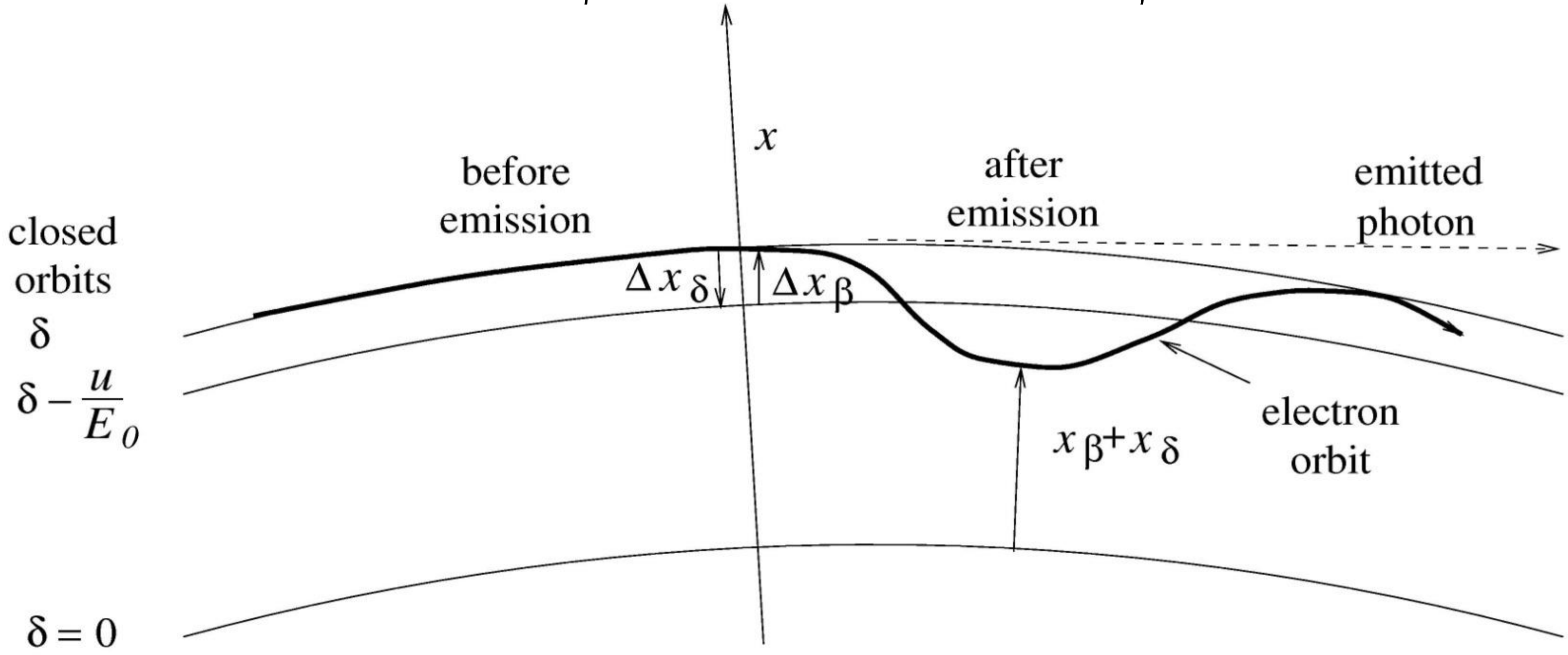
Transverse plane: horizontal oscillations (II)

After the emission of a photon, the physical position and the angle of the electron do not change. However they must be referenced to a new orbit:

This is the off-energy orbit corresponding to the new energy of the electron

With respect to the off-energy orbit, the emission of a photon appears as an offset (and an angle)

$\delta x = 0$, $\delta x' = 0$ but $\delta x_\beta + \delta x_\delta = 0$ (and likewise $\delta x'_\beta + \delta x'_\delta = 0$)



Transverse plane: horizontal oscillations (III)

We follow the same line as done for the vertical plane. The equations of motion in the horizontal plane (α_x = 0) are

$$x = A\cos(\phi(s) + \phi_0) \qquad x' = -\frac{A}{\beta}\sin(\phi(s) + \phi_0)$$

Invariant in the horizontal plane

$$A^2 = x^2 + (\beta x')^2$$

After the photon emission position and angle do not change but with respect to the new (off energy) orbit

$$x_\beta = x + \delta x_\beta \qquad x'_\beta = x' + \delta x'_\beta$$

and we have said that

$$\delta x_\beta = -\delta x_\varepsilon = -D\frac{\delta\varepsilon}{E_0}$$ and similarly $$\delta x'_\beta = -\delta x'_\varepsilon = -D'\frac{\delta\varepsilon}{E_0}$$

The new invariant in the horizontal plane (with respect to the new orbit) reads

$$(A + \delta A)^2 = (x + \delta x_\beta)^2 + [\beta(x' + \delta x'_\beta)^2]$$

Transverse plane: horizontal oscillations (IV)

The change in the Courant-Snyder invariant due to δx_β and $\delta x'_\beta$ to first order in $\delta\varepsilon$ reads

$$A\delta A = x_\beta \delta x_\beta + \beta^2 x_\beta' \delta x_\beta' = -(D x_\beta + \beta^2 D' x_\beta') \frac{\delta\varepsilon}{E_0}$$

As before the change in the Courant Snyder invariant depends on the specific betatron coordinates x_β and x'_β of the electron .

We want to average of all possible electron in an ellipse with the same Courant-Snyder invariant and get

$$< A\delta A >_{e^-} = -D \frac{< x_\beta \delta\varepsilon >_{e^-}}{E_0} + \beta^2 D' \frac{< x_\beta ' \delta\varepsilon >_{e^-}}{E_0}$$

If for each photon emission the quantity $\delta\varepsilon$ is independent on x_β and x'_β, then averaging the previous expression over the phases of the betatron oscillations would give zero.

However, in the horizontal plane $\delta\varepsilon$ depends on x_β in two ways [Sands]

Transverse plane: horizontal oscillations (V)

Let us compute the dependence of the energy $\delta\varepsilon$ of the photon emitted in the horizontal plane on x_β [Sands].

Assuming that the emission of photon is described as a continuous loss of energy (no random fluctuations in the energy of the photon emitted), we have

$$\delta\varepsilon = -P\,dt$$

both P and dt depend on the betatron coordinate of the electron

$$dt = \frac{dl}{c} = \frac{1}{c}\left(1 + \frac{x_\beta}{\rho}\right)ds$$

And, since $P \propto B^2$, to the first order in x_β

$$P(x_\beta) = P + 2\frac{P}{B}\frac{dB}{dx}x_\beta$$

Transverse plane: horizontal oscillations (V)

The energy change reads

$$\delta\varepsilon = -Pdt = -\frac{P}{c}\left[1+\left(\frac{2}{B}\frac{dB}{dx}+\frac{1}{\rho}\right)x_\beta\right]\delta s$$

Substituting in

$$A\delta A = -(D\,x_\beta + \beta^2 D'\,x_\beta'\,)\frac{\delta\varepsilon}{E_0}$$

We get

$$A\delta A = \frac{2P_0}{U_s c}[\gamma x_\beta D + \alpha(x_\beta D' + x'_\beta\, D) + \beta x'_\beta\, D']\left(1 - 2k\rho\, x_\beta + \frac{x_\beta}{\rho}\right)\delta s$$

The change in the Courant-Snyder invariant depends on the position and angle x_β and x'_β'for this particular electron. Let us consider now all the electrons in the phase space travelling on the ellipse, and therefore having all the same invariant A

Transverse plane: horizontal oscillations (VI)

For each different x_β and x'_β the change in the invariant will be different. However averaging over the electron phases, assuming a uniform distribution along the ellipse, we have

$$\frac{<\delta A>_{e^-}}{A} = \frac{1}{2cE_0} P_0 D \left(\frac{1}{\rho} - 2k\rho \right) \delta s \qquad k\rho = \frac{1}{B_0} \frac{dB}{dx}$$

The average invariant can now increase or decrease depending on the sign of the previous term, i.e. depending on the lattice.

Let us consider all the photons emitted in one turn. The total energy lost is

$$U_0 = \sum_C \varepsilon$$

Summing the contributions $\delta\varepsilon$ in one turn, we find that in one turn:

$$\frac{<<\delta A>_{e^-}>_{ph}}{A} = \frac{1}{2cE_0} \oint P_0 D \left(\frac{1}{\rho} - 2k\rho \right) \delta s = \frac{U_0}{E_0} \frac{\wp}{2}$$

Transverse plane: horizontal oscillations (VII)

The change in the horizontal average invariant due to the emission of a photon

$$\frac{1}{A}\frac{dA}{dt} = \frac{U_0 \wp}{2E_0 T_0}$$

$\wp > 0$ gives an anti-damping term

As in the vertical plane we must add the contribution due to the RF that will replenish all the energy lost.

Adding the RF contribution (as before assuming $D_x = 0$ at the RF cavities)

$$-\frac{1}{A}\frac{dA}{dt} = \frac{U_0}{2E_0 T_0}(1-\wp) = \frac{1}{\tau_x}$$

The average horizontal invariant decreases (or increases) exponentially with a damping time τ_z .$\tau_z \approx$ half of longitudinal damping time always dependent on $1/\gamma^3$.

This remains true for more general distribution of electron in phase space with invariant A (e.g Gaussian)

Transverse plane: horizontal oscillations (VIII)

As in the vertical plane, the horizontal betatron oscillations are damped in presence of synchrotron radiation

$$x(t) = x_0 e^{-t/\tau_x} \sin\left(\omega_\beta t + \varphi\right)$$

Since the emittance of a bunch of particle is given by the average of the square of the betatron amplitude of the particles in the bunch

$$\varepsilon_x = \frac{< x^2 >}{\beta_x}$$

the emittance decays with a time constant which is half the radiation damping time

$$\varepsilon_x(t) = \varepsilon_x(0) \exp\left(-\frac{2t}{\tau_x} \right)$$

Damping partition numbers (I)

The results on the radiation damping times can be summarized as

$$\frac{1}{\tau_i} = \frac{J_i U_0}{2E_0 T_0}$$

Jx = 1 - $\wp$;   Jz = 1;   Jε = 2 + $\wp$;

The J_i are called damping partition numbers, because the sum of the damping rates is constant for any $\wp$ (any lattice)

Jx + Jz + Jε = 4   (Robinson theorem)

Damping in all planes requires $-2 < \wp < 1$

Fixed U_0 and E_0 one can only trasfer damping from one plane to another

Adjustment of damping rates

Modification of all damping rates:

Increase losses U_0

Adding damping wigglers to increase U_0 is done in damping rings to decrease the emittance

Repartition of damping rates on different planes:

Robinson wigglers: increase longitudinal damping time by decreasing the horizontal damping (reducing dU/dE)

Change RF: change the orbit in quadrupoles which changes $\wp$ and reduces τ_x

Robinson wiggler at CERN

Example: damping rings

Damping rings are used in linear colliders to reduce the emittance of the colliding electron and positron beams:

The emittance produced by the injectors is too high (especially for positrons beams).

In presence of synchrotron radiation losses the emittance is damped according to

$$\varepsilon_{fin} = \varepsilon_{eq} + (\varepsilon_{in} - \varepsilon_{eq}) \cdot e^{-2T/\tau_x}$$

The time it takes to reach an acceptable emittance will depend on the transverse damping time

The emittance needs to be reduced by large factors in a short store time T. If the natural damping time is too long, it must be decreased.

This can be achieved by introducing damping wigglers. Note that damping wigglers also generate a smaller equilibrium emittance ε_{eq}.

Example: damping rings

Using ILC parameters

$\varepsilon_i = 0.01$ m $\varepsilon_f = 10$ nm $\varepsilon_f / \varepsilon_i = 10^{-6}$

The natural damping time is T ~ 400 ms while it is required that $T/\tau_x \sim 15$, i.e. a damping time $\tau_x \sim 30$ ms (dictated by the repetition rate of the following chain of accelerators – i.e. a collider usually)

Damping wigglers reduce the damping time by increasing the energy loss per turn

$$\frac{1}{\tau_i} = \frac{J_i U_0}{2 E_0 T_0}$$

With the ILC damping ring data

E = 5 GeV, $\rho = 106$ m, C = 6700 m,

we have

$U_0 = 520$ keV/turn $\tau_x = 2ET_0/U_0 = 430$ ms

Example: damping rings

The damping time τ_x has to be reduced by a factor 17 to achieve e.g. 25 ms.

Damping wigglers provide the extra synchrotron radiation energy losses without changing the circumference of the ring.

The energy loss of a wiggler E_w with peak field B and length L and are given by

$$E_w = \frac{2}{3}\frac{r_e e^2}{m^3 c^4} E^2 B_w^2 L_w = \frac{8\pi^2 r_e}{3mc^2}\frac{E^2 K^2}{\lambda_u^2} L_w$$

or in practical units the energy loss per electron reads

$$E_w(\mathrm{eV}) = 0.07257 \frac{E[\mathrm{GeV}]^2 K^2}{\lambda_u[\mathrm{m}]^2} L_w[\mathrm{m}] \qquad K = \frac{e\lambda_u B_w}{2\pi mc} = \frac{e\lambda_u B_0}{2\sqrt{2}\pi mc}$$

A total wiggler length of 220 m will provide the required damping time.

Radiation integrals

Many important properties of the stored beam in an electron synchrotron are determined by integrals taken along the whole ring:

$$I_1 = \oint \frac{D_x}{\rho} ds \qquad I_2 = \oint \frac{1}{\rho^2} ds \qquad I_3 = \oint \frac{1}{\rho^3} ds$$

$$I_4 = \oint \frac{D_x}{\rho}\left(\frac{1}{\rho^2} - 2k\right) ds \qquad I_5 = \oint \frac{H}{\rho^3} ds \qquad H = \gamma D_x^2 + 2\alpha D_x D'_x + \beta D'^2_x$$

In particular

$$U_0 = \frac{e^2}{6\pi\varepsilon_0}\gamma^4 \oint \frac{1}{\rho^2} ds = \frac{e^2}{6\pi\varepsilon_0}\gamma^4 I_2$$

Energy loss per turn

$$J_x = 1 - \frac{I_4}{I_2} \qquad J_\varepsilon = 2 + \frac{I_4}{I_2}$$

Damping partition numbers

$$\tau_x = \frac{3T_0}{r_0\gamma^3}\frac{1}{I_2 - I_4} \qquad \tau_z = \frac{3T_0}{r_0\gamma^3}\frac{1}{I_2} \qquad \tau_\varepsilon = \frac{3T_0}{r_0\gamma^3}\frac{1}{2I_2 + I_4}$$

Damping times

Summary

Synchrotron radiation losses and RF energy replacement generate a damping of the oscillation in the three planes of motion

The damping times depend on the energy as $1/\gamma^3$ and on the magnetic lattice parameters (stronger for light particles)

The damping times can be modified, but at a fixed energy losses, the sum of the damping partition number is conserved regardless of the lattice type

Radiation damping combined with radiation excitation determine the equilibrium beam distribution and therefore emittance, beam size, energy spread and bunch length.

Quantum nature of synchrotron emission

The radiated energy is emitted in quanta: each quantum carries an energy $u = \hbar\omega$;

The emission process is instantaneous and the time of emission of individual quanta are statistically independent;

The distribution of the energy of the emitted photons can be computed from the spectral distribution of the synchrotron radiation;

The emission of a photon changes suddenly the energy of the emitting electron and perturbs the orbit inducing synchrotron and betatron oscillations.

These oscillations grow until reaching an equilibrium when balanced by radiation damping

Quantum excitation prevents reaching zero emittance in both planes with pure damping.

From the lecture on synchrotron radiation

Total radiated power

$$P = \frac{e^2 c}{6\pi\varepsilon_0} \frac{\gamma^4}{\rho^2}$$

Frequency distribution of the power radiated

$$\frac{dI}{d\omega} = \frac{\sqrt{3}e^2\gamma}{4\pi\varepsilon_0 c} \frac{\omega}{\omega_c} \int_{\omega/\omega_c}^{\infty} K_{5/3}(x)dx = \frac{2e^2\gamma}{9\varepsilon_0 c} S(\xi)$$

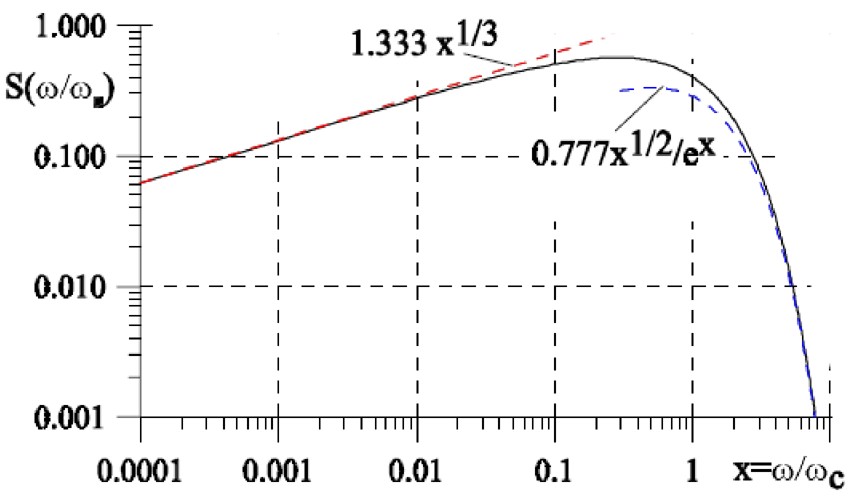
$$\xi = \frac{\omega\rho}{3c\gamma^3}\left(1+\gamma^2\theta^2\right)^{3/2}$$

Critical frequency

$$\omega_c = \frac{3}{2}\frac{c}{\rho}\gamma^3$$

Critical angle at the critical frequency

$$\theta_c = \frac{1}{\gamma}$$



Energy distribution of photons emitted by synchrotron radiation (I)

Energy is emitted in quanta: each quantum carries an energy $u = \hbar\omega$

From the frequency distribution of the power radiated $$\frac{dP}{d\omega} = \frac{P_\gamma}{\omega_c} S\left(\frac{u}{u_c}\right)$$

We can get the energy distribution of the photons emitted per second:

n(u) number of photons emitted per unit time with energy in u, u+du
u·n(u) energy of photons emitted per unit time with energy in u, u+du

u·n(u) must be equal to the power radiated in the frequency range du/ħ at u/ ħ

$$u \cdot n(u)\,du = \frac{dP(u/\hbar)}{du/\hbar} du/\hbar$$

$$n(u) = \frac{1}{\hbar u} \frac{dP}{du/\hbar}\left(\frac{u}{u_c}\right) \qquad n(u) = \frac{P_\gamma}{u_c^2} \frac{u_c}{u} S\left(\frac{u}{u_c}\right) \qquad u_c = \hbar\omega_c$$

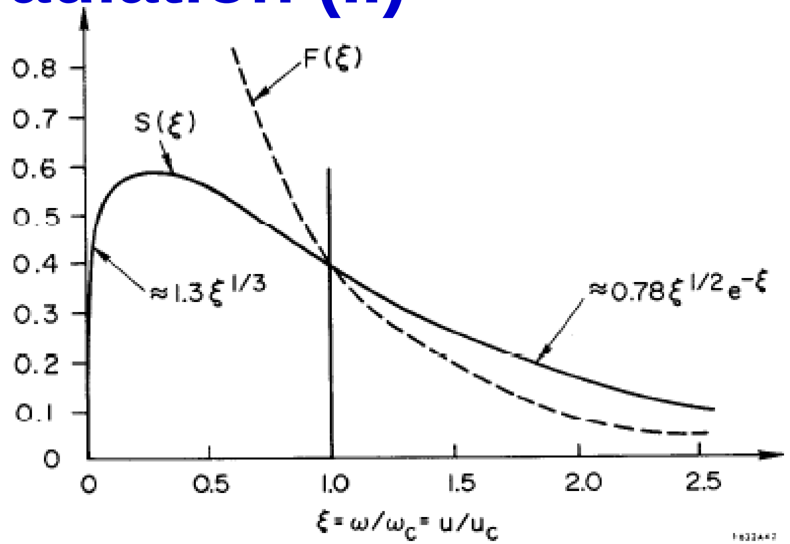
Energy distribution of photons emitted by synchrotron radiation (II)

Introducing the function F(ξ)

$$F(\xi) = \frac{1}{\xi} S(\xi)$$

we have

$$n(u) = \frac{P_\gamma}{u_c^2} F\left(\frac{u}{u_c}\right)$$



Using the energy distribution of the rate of emitted photons one can compute:

Total number of photons emitted per second

$$N_\gamma = \int_0^\infty n(u)\,du = \int_0^\infty \frac{P_\gamma}{u_c^2} F\left(\frac{u}{u_c}\right) du = \frac{P_\gamma}{u_c} \int_0^\infty F(\xi)\,d\xi = \frac{15\sqrt{3}}{8} \frac{P_\gamma}{u_c}$$

Mean energy of photons emitted per second

$$\langle u \rangle = \frac{1}{N_\gamma} \int_0^\infty u \cdot n(u)\,du = \frac{8}{15\sqrt{3}} u_c \approx 0.32 u_c$$

Mean square energy of photons emitted per second

$$\langle u^2 \rangle = \frac{1}{N_\gamma} \int_0^\infty u^2 n(u)\,du = \frac{11}{27} u_c^2 \approx 0.41 \cdot u_c^2$$

Quantum fluctuations in energy oscillations (IV)

Let us consider again the change in the invariant for linearized synchrotron oscillations

$$A^2 = \varepsilon^2 + \left(\frac{U_s \omega_s}{\alpha}\right)^2 \tau^2$$

After the emission of a photon of energy u we have

$$\varepsilon \to \varepsilon - u \qquad \tau \to \tau$$

The time position τ w.r.t. the synchronous particle does not change

$$dA^2 = -2\varepsilon u + u^2$$

We do not discard the u^2 term since it is a random variable and its average over the emission of n(u)du photons per second is not negligible anymore.

Notice that now also the Courant Snyder invariant becomes a random variable!

Quantum fluctuations in energy oscillations (II)

We want to compute the average of the random variable A over the distribution of the energy of the photon emitted

$$< dA^2 >_p = -2\varepsilon N_\gamma < u >_p + N_\gamma < u^2 >_p$$

Quantum excitation

Radiation damping

We have to compute the averages of u and u^2 over the distribution n(u)du of number of photons emitted per second.

As observed the term with the square of the photon energy (wrt to the electron energy) is not negligible anymore

Quantum fluctuations in energy oscillations (VI)

Following [Sands] the excitation term can be written as

$$N_\gamma < u^2 >_p = \int_0^\infty u^2 n(u) du = \frac{55}{24\sqrt{3}} r_0 \hbar m c^4 \frac{\gamma^7}{\rho^3}$$

and depends on the location in the ring. We must average over the position in the ring, by taking the integral over the circumference.

$$\left\langle N_\gamma < u^2 >_p \right\rangle = \frac{55}{24\sqrt{3}} r_0 \hbar m c^4 \gamma^7 \frac{1}{\rho^2 R}$$

The contribution from the term linear in u, after the average over the energy distribution of the photon emitted, and the average around the ring reads

$$< \mathrm{N}_\gamma < \mathrm{u} >_\gamma >_{\Delta t = T_0} = \oint \mathrm{N}_\gamma < \mathrm{u} >_\gamma \frac{\mathrm{ds}}{\mathrm{c}} = \mathrm{N}_\gamma \mathrm{T}_0 < \mathrm{u} >_\gamma = \mathrm{U} = \sum_{\mathrm{ring}} \mathrm{u} = \mathrm{U}_0 + \frac{\partial \mathrm{U}}{\partial \varepsilon} \varepsilon$$

Using these expressions…

Quantum fluctuations in energy oscillations (VII)

The change in the invariant averaged over the photon emission and averaged around one turn in the ring now reads

$$< dA^2 >_p = -2\varepsilon N_\gamma < u >_p + N_\gamma < u^2 >_p$$

The change in the invariant still depends on the energy deviation of the initial particle. We can average in phase space over a distribution of particle with the same invariant A. A will become the averaged invariant

The linear term in u generates a term similar to the expression obtained with the radiative damping. We have the differential equation for the average of the longitudinal invariant

$$\frac{d < A^2 >}{dt} = -\frac{2 < A^2 >}{\tau_\varepsilon} + \left\langle N_\gamma < u^2 >_p \right\rangle$$

Quantum fluctuations in energy oscillations (VIII)

The average longitudinal invariant decreases exponentially with a damping time τ_ε and reaches an equilibrium at

$$< A^2 > = \frac{\tau_\varepsilon}{2} \left\langle N_\gamma < u^2 >_p \right\rangle$$

This remains true for more general distribution of electron in phase space with invariant A (e.g Gaussian)

The variance of the energy oscillations is for a Gaussian beam is related to the Courant-Snyder invariant by

$$\sigma_\varepsilon^2 = < \varepsilon^2 > = \frac{< A^2 >}{2}$$

Quantum fluctuations in energy oscillations (IX)

The equilibrium value for the energy spread reads

$$\sigma_\varepsilon^2 = \frac{55}{32\sqrt{3}} \hbar m c^3 \gamma^4 \frac{I_3}{J_\varepsilon I_2} = \frac{55}{32\sqrt{3}} \hbar m c^3 \gamma^4 \frac{I_3}{2I_2 + I_4}$$

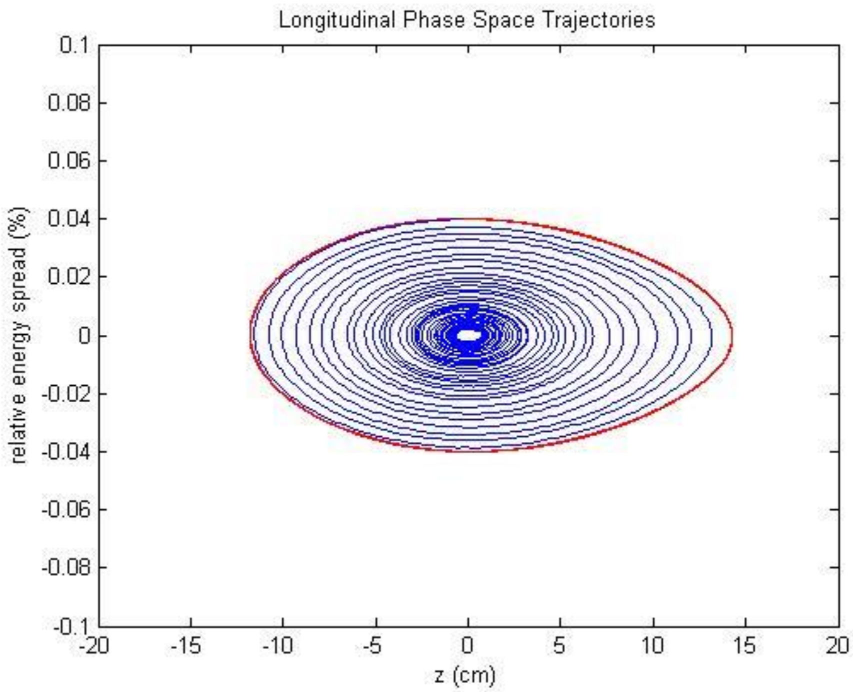
For a synchrotron with separated function magnets

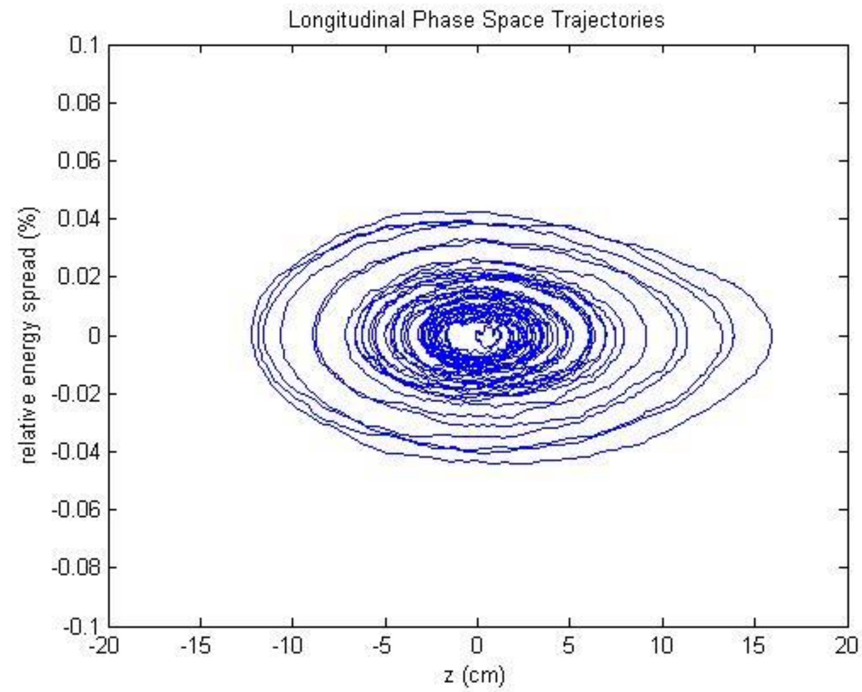
$$\boxed{\frac{\sigma_\varepsilon^2}{E_0^2} = \frac{55}{64\sqrt{3}} \frac{\hbar}{mc} \frac{\gamma^2}{\rho}}$$

The relative energy spread depends only on energy and the lattice (namely the curvature radius of the dipoles)

A tracking example

synchrotron period 200 turns; damping time 6000 turns;





Diffusion effect off

Diffusion effect on

Quantum fluctuations in horizontal oscillations (I)

Invariant for linearized horizontal betatron oscillations

$$A^2 = \gamma x^2 + 2\alpha x x' + \beta x'^2$$

after the emission of a photon of energy u we have

$$\delta x_\beta = -\delta x_\varepsilon = -D_x \frac{u}{U_s} \qquad \text{and} \qquad \delta x_\beta' = -\delta x_\varepsilon' = -D_x' \frac{u}{U_s}$$

Neglecting for the moment the linear part in u, that gives the horizontal damping, the modification of the horizontal invariant reads

$$dA^2 = (\gamma D_x^{\;2} + 2\alpha D_x D_x' + \beta D_x'^2)\frac{u^2}{U_s^2}$$

Defining the function

$$H(s) = \gamma D_x^{\;2} + 2\alpha D_x D_x' + \beta D_x'^2$$ Dispersion invariant

As before we have to compute the effect on the invariant due to the emission of a photon, averaging over the photon distribution, over the betatron phases and over the location in the ring [see Sands]:

Quantum fluctuations in horizontal oscillations (II)

We obtain

$$\frac{d < A^2 >}{dt} = \frac{\left\langle N_\gamma < u^2 >_p H \right\rangle}{E_0^2}$$

The linear term in u averaged over the betatron phases gives the horizontal damping

$$\frac{d < A^2 >}{dt} = -\frac{2 < A^2 >}{\tau_x}$$

Combining the two contributions we have the following differential equation for the average of the invariant in the longitudinal plane

$$\frac{d < A^2 >}{dt} = -\frac{2 < A^2 >}{\tau_x} + \frac{\left\langle N_\gamma < u^2 >_p H \right\rangle}{E_0^2}$$

Quantum fluctuations in horizontal oscillations (III)

At equilibrium

$$< A^2 >= \frac{\tau_x}{2} \frac{\left\langle N_\gamma < u^2 >_p H \right\rangle}{E_0^2}$$

The variance of the horizontal oscillations is

$$\sigma_x^2 =< x^2 >= \beta_x \frac{< A^2 >}{2} = \beta_x \varepsilon_x$$

Therefore we get the emittance

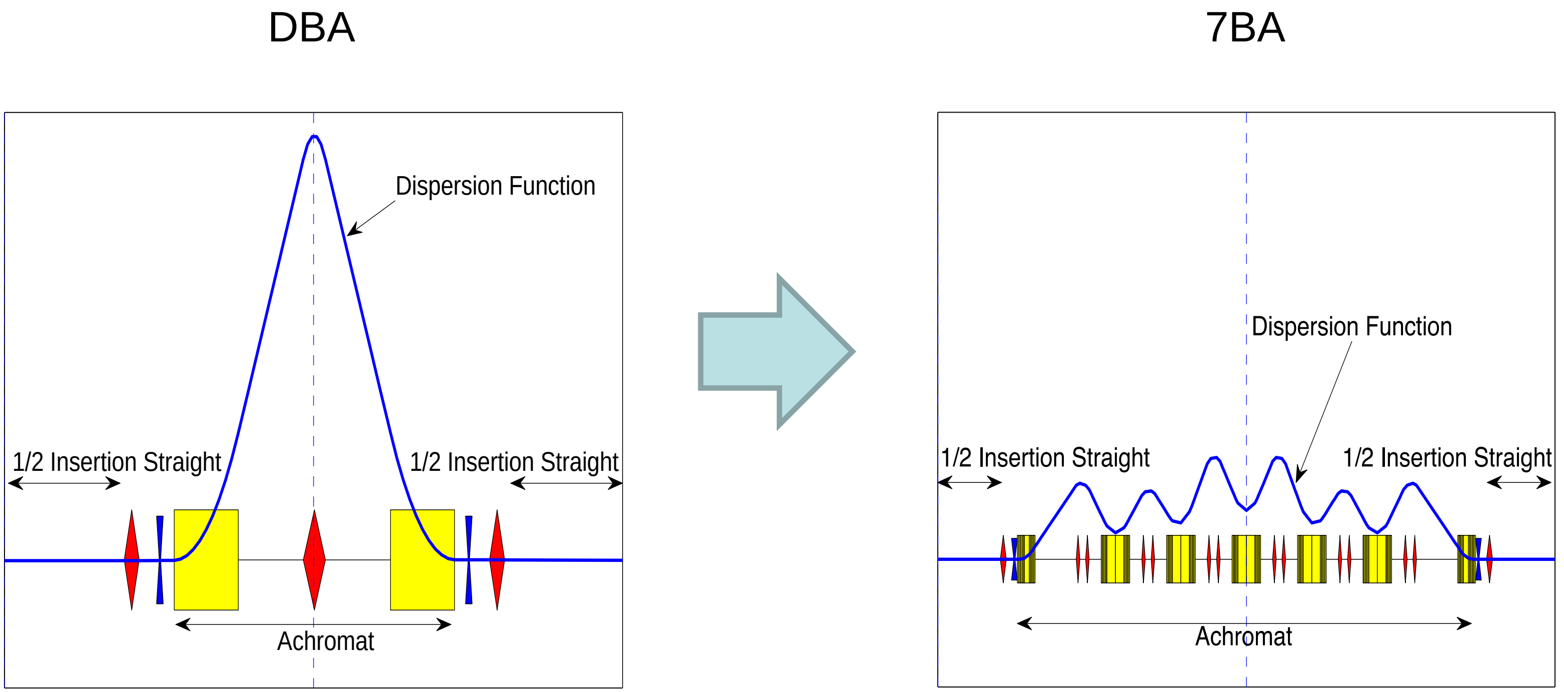
$$\varepsilon_x = \frac{< A^2 >}{2} = \frac{\tau_x}{4E_0^2 L} \oint N_\gamma < u^2 >_p H(s) ds$$

The emittance depends on the dispersion function at bendings, where radiation emission occurs

$$\boxed{\varepsilon_x = \frac{55}{32\sqrt{3}} \frac{\hbar}{mc} \frac{\gamma^2}{J_x} \frac{\left\langle H / \rho^3 \right\rangle}{\left\langle 1/ \rho^2 \right\rangle}}$$

Low emittance lattices strive to minimise $<H/\rho^3>$ and maximise J_x

Low emittance with multiple bend achromats



Simplified explanation

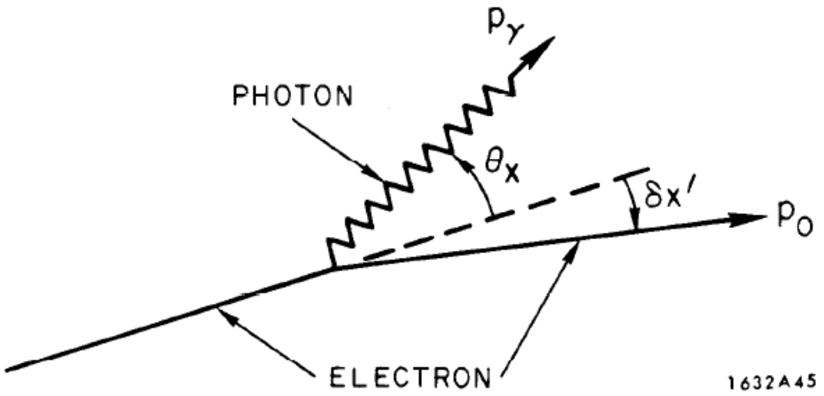
– Emittance is driven by randomness of photon emission in presence of dispersive (energy-dependent) orbits – electron recoils randomly
– Breaking up dipoles and putting focusing (quadrupoles) between the parts allows reducing the amplitude of dispersive orbits – smaller electron recoils

Quantum fluctuations in vertical oscillations (I)

With zero dispersion the previous computation will predict no quantum fluctuations i.e. zero vertical emittance.

However a small effect arises due to the fact that photons are not exactly emitted in the direction of the momentum of the electrons

The electron must recoil to preserve the total momentum



Invariant for linearized vertical betatron oscillations

$$A^2 = \gamma z^2 + 2\alpha z z' + \beta z'^2$$

after the emission of a photon of energy u the electron angle is changed by

$$\delta z' = \frac{u}{E_0}\theta_z$$

Quantum fluctuations in vertical oscillations (II)

the modification of the vertical invariant after the emission of a photon reads

$$dA^2 = \frac{u^2}{E_0^2}\theta_z^2\beta_z(s)$$

Averaging over the photon emission, the betatron phases and the location around the ring:

$$< u^2\theta_z^2 > \approx < u^2 >< \theta_z^2 > \qquad < \theta_z^2 > = \frac{1}{2\gamma^2}$$

At equilibrium

$$< A^2 > = \frac{\tau_z}{2}\frac{\left\langle N < u^2 >_p \beta_z \right\rangle}{E_0^2}$$

$$\varepsilon_z = \frac{55}{64\sqrt{3}}\frac{\hbar}{mc}\frac{< \beta_z / \rho^3 >}{J_z < 1/\rho^2 >}$$

In practice this effect is very small: the vertical emittance is given by vertical dispersion errors and linear coupling between the two planes of motion.

Related beam quantities: beam size

The horizontal beam size has contributions from the variance of betatron oscillations and from the energy oscillations via the dispersion function: Combining the two contributions we have the bunch size:

$$\sigma_x = \left\{ \varepsilon_x \beta_x(s) + D_x^2(s) \left(\frac{\sigma_\varepsilon}{U_s} \right)^2 \right\}^{1/2}$$

The vertical beam size has contributions from the variance of betatron oscillations but generally not from the energy oscillations (Dz = 0). However the contribution from coupling is usually dominant

$$\sigma_z = \left(\varepsilon_z \beta_z(s) \right)^{1/2} \qquad \varepsilon_z = k \varepsilon_x$$

In 3rd generation light sources the horizontal emittance is few nm and the coupling k is easily controlled to 1% or less, e.g. for Diamond

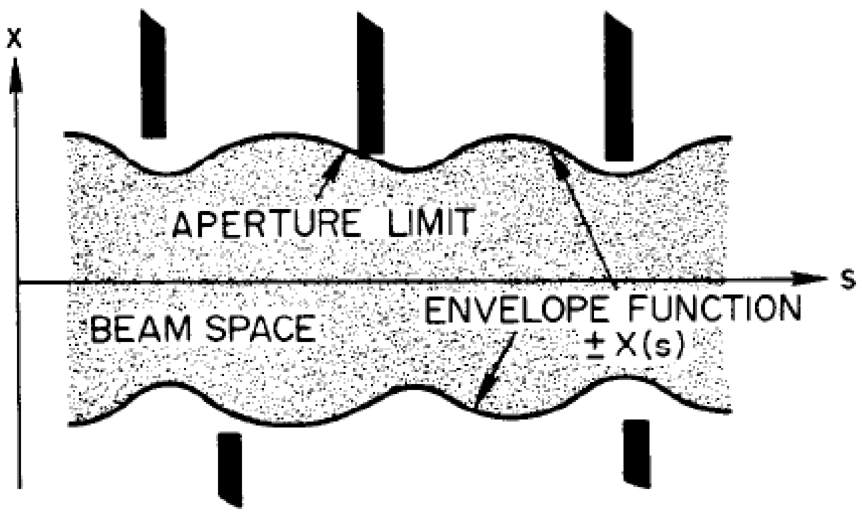
ε_x = 2.7 nm; k = 1% $\rightarrow$ ε_y = 27 pm;

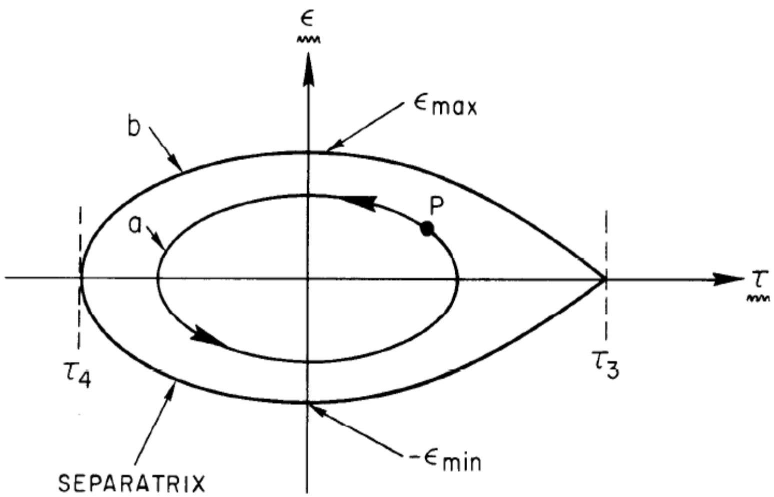
σ_x = 120 μm σ_y = 6 μm

Quantum lifetime (I)

Electrons are continuously stirred by the emission of synchrotron radiation photons

It may happen that the induced oscillations hit the vacuum chamber or get outside the RF aperture:





The number of electron per second whose amplitudes exceed a given aperture and is lost at the wall or outside the RF bucket can be estimated from the equilibrium beam distribution [see Sands]

Quantum lifetime (II)

$$\frac{dN}{dt} = -\frac{N}{\tau_q}$$

Exponential decay of the number of particle stored

$$\xi = \frac{x_{\max}^2}{2\sigma_x^2} \qquad \tau_q = \frac{\tau_x}{2}\frac{\exp(\xi)}{\xi}$$

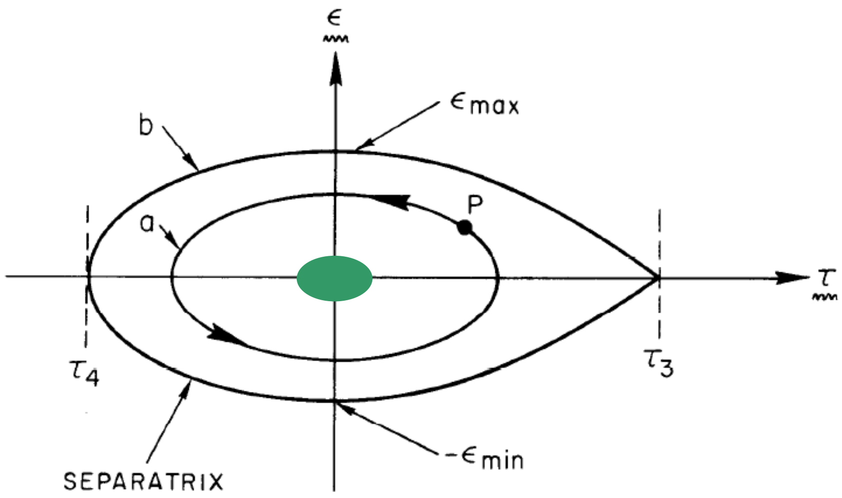
quantum lifetime for losses in the transverse plane

$$\xi = \frac{\varepsilon_{\max}^2}{2\sigma_\varepsilon^2} \qquad \tau_q = \frac{\tau_\varepsilon}{2}\frac{\exp(\xi)}{\xi}$$

quantum lifetime for losses in the longitudinal plane

Given the exponential dependence on the ratio between available aperture and beam size the quantum lifetime is typically very large for modern synchrotron light sources, e.g. Diamond

$$\xi = \frac{\varepsilon_{\max}^2}{2\sigma_\varepsilon^2} = \frac{(0.04)^2}{2\cdot(0.001)^2} = 800$$



Related beam quantities: bunch length

Bunch length from energy spread

$$\sigma_z = \frac{\alpha c}{2\pi f_s}\sigma_\varepsilon \propto \sqrt{\frac{\alpha \gamma^3}{dV_{RF}/dz}}$$

The bunch length also depends on RF parameters: voltage and phase seen by the synchronous particle

$\alpha = 1.7\cdot10^{-4}$; V = 3.3 MV; $\sigma_\varepsilon = 9.6\cdot10^{-4}$ → σ_z = 2.8 mm (9.4 ps)

σ_z depends on

the magnetic lattice (quadrupole magnets) via α

the RF slope

Shorten bunches decreasing α (low-alpha optics)

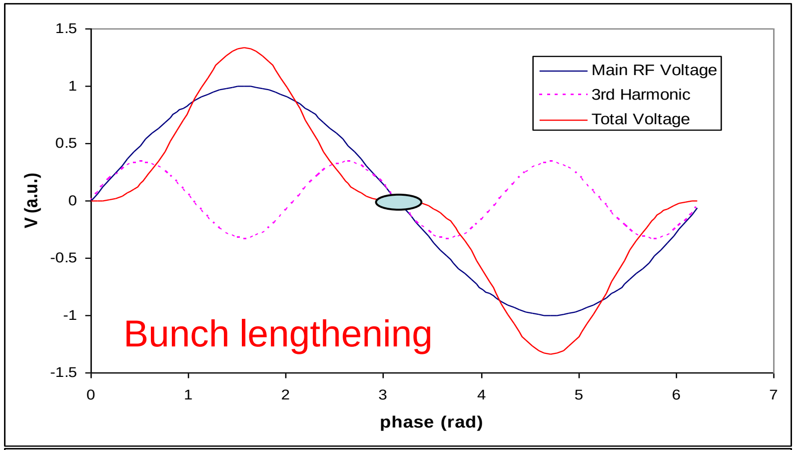
$$\alpha = \frac{1}{L}\oint \frac{D_x}{\rho}\,ds \approx 10^{-6}$$

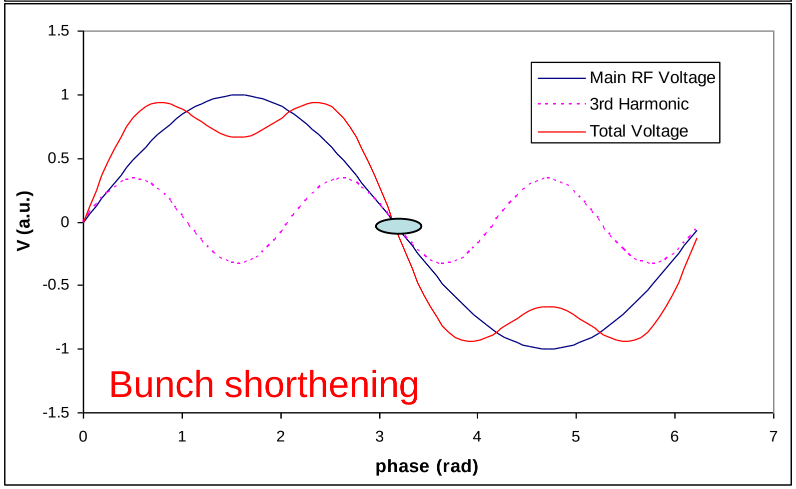
Shorten/Lengthen bunches increasing the RF slope at the bunch (Harmonic cavities)

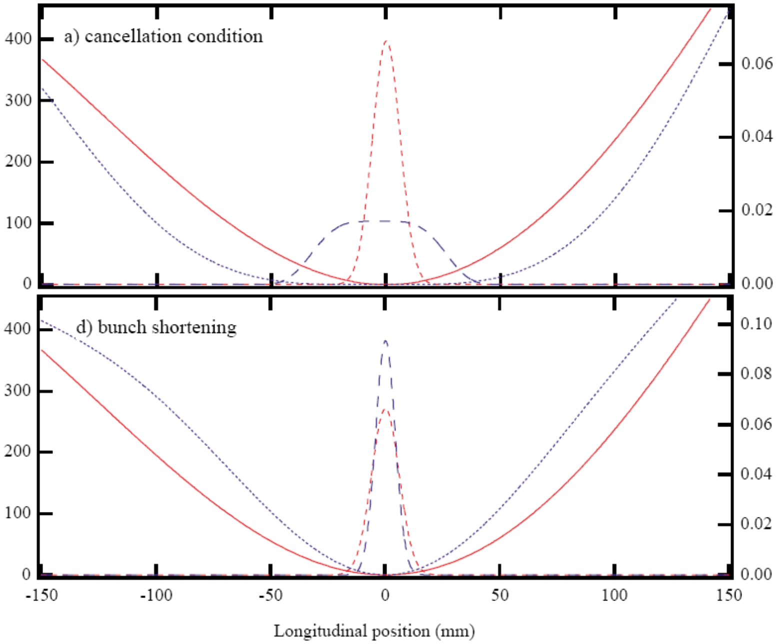
bunch length manipulation: harmonic cavities

RF cavities with frequency equal to an harmonic of the main RF frequency (e.g. 3rd harmonic) are used to lengthen or shorten the bunch

$$\sigma_z = \frac{\alpha c}{2\pi f_s}\sigma_\varepsilon \propto \sqrt{\frac{\alpha \gamma^3}{d V_{RF} / dz}}$$







Summary

The emission of synchrotron radiation occurs in quanta of discrete energy

The fluctuation in the energy of the emitted photons introduce a noise in the various oscillation modes causing the amplitude to grow

Radiation excitation combined with radiation damping determine the equilibrium beam distribution and therefore emittance, beam size, energy spread and bunch length.

The excitation process is responsible for a loss mechanism described by the quantum lifetime

The emittance is a crucial parameter in the operation of synchrotron light source. The minimum theoretical emittance depends on the square of the energy and the inverse cube of the number of dipoles