Introduction to RF

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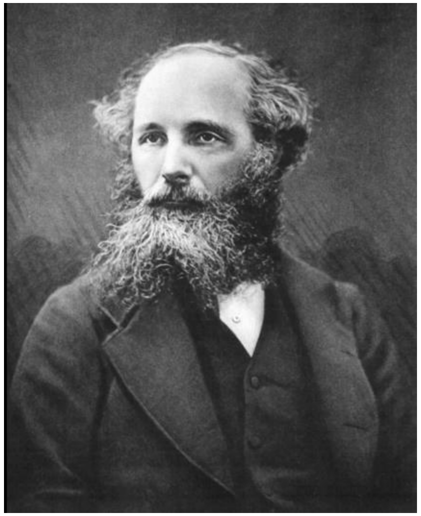
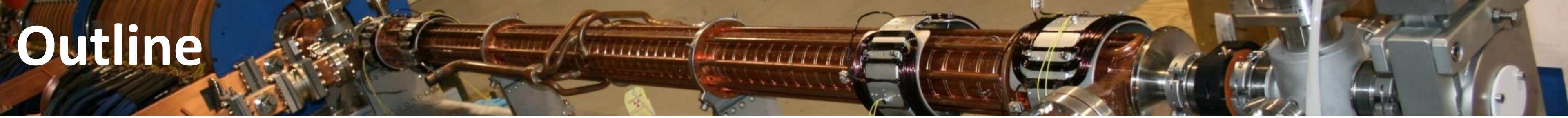
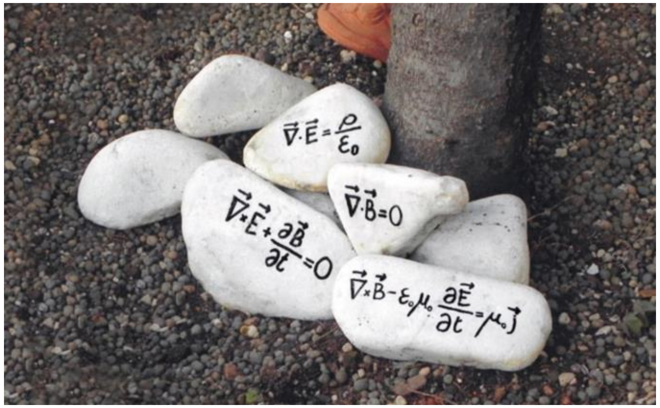




Outline

Maxwell equations



- General review
- The lumped element limit
- The wave equation
- Maxwell equations for time harmonic fields
- Fields in media and complex permittivity
- Boundary conditions and materials
- Plane waves

Boundary value problems for metallic waveguides

- The concept of mode
- Maxwell equations and vector potentials
- Cylindrical waveguides: TM, TE and TEM modes
- Solving Maxwell Equations in metallic waveguides
- Rectangular waveguide (detailed example)

Goal of the lecture

Show principles behind the practice discussed in the RF engineering module

math inside®

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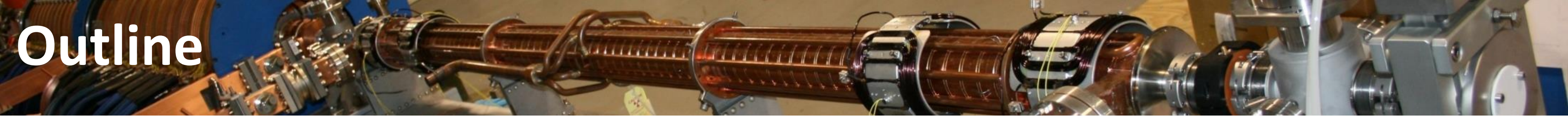

Outline

Maxwell equations

Schedule 2017	Monday Feb 13th	Tuesday Feb 14th	Wednesday Feb 15th	Thursday Feb 16th	Friday Feb 17th
09:00					
09:00–10:00	**Arrival and registration at ESI Office** **&** **Accommodation**	**Introduction to RF** lecture *A. Mostacci*	**Vacuum systems** lecture *V. Baglin*	**RF Engineering** lecture *F. Caspers*	**RF Engineering** lecture *F. Caspers*
10:00–10:15		**Coffee Break**	**Coffee Break**	**Coffee Break**	**RF Engineering** tutorial *F. Caspers / M. Wendt*
10:15–11:15		**Introduction to RF** lecture *A. Mostacci*	**Vacuum systems** lecture *V. Baglin*	**Vacuum systems** lecture *V. Baglin*	**Coffee Break**
11:15–12:15	**12:00 ESI WELCOME & BUILDING VISIT**	**Vacuum systems** lecture *V. Baglin*	**Vacuum systems** tutorial *V. Baglin / R. Kersevan*	**Vacuum systems** tutorial *V. Baglin / R. Kersevan*	***Bus leaves at 11:30 from JUAS***
12:15–14:00	**12:30 WELCOME LUNCH OFFERED BY ESI**	**BREAK**	**BREAK**	**BREAK**	***(Lunch at CERN, offered by ESI)***
14:00–15:00	**Presentation of JUAS & Presentation of students 2017** *E. Métral*	**Vacuum systems** lecture *V. Baglin*	**RF Engineering** lecture *F. Caspers*	**RF Engineering** lecture *F. Caspers*	**VISIT AT CERN**
15:00–16:00	**Introduction to CERN practical days** *Magnet,Superconductivity*	**RF Engineering** lecture *F. Caspers*	**RF Engineering** tutorial *F. Caspers / M. Wendt*	**RF Engineering** tutorial *F. Caspers / M. Wendt*	***AD / ELENA LINAC / LEIR***
16:00–16:15	**Coffee Break**	**Coffee Break**	**Coffee Break**	**Coffee Break**	
16:15–17:15	**Introduction to CERN practical days** *RF, Vacuum*	**RF Engineering** lecture *F. Caspers*	**Accelerator driven system** Seminar *D. Vandeplassche*	**RF Engineering** lecture *F. Caspers*	***Bus leaves at 18:00 from CERN***
17:15					

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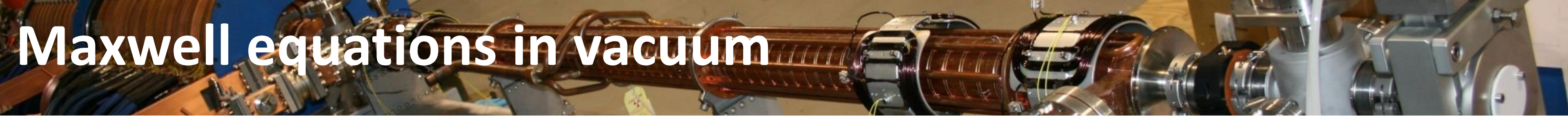


Maxwell equations in vacuum

$$\nabla \cdot \vec{E} = \rho/\epsilon_0$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} + \mu_0 \vec{J}$$

$\vec{E}$	**Electric Field**	(V/m)	**fields**
$\vec{B}$	**Magnetic Flux Density**	(Wb/m^2)	
ρ	**Electric Charge Density**	(C/m^3)	**sources**
$\vec{J}$	**Electric Current Density**	(A/m^2)	

$\mu_0 = 4\pi\ 10^{-7}\ (H/m)$

Magnetic constant

$\epsilon_0 = 1/c^2\mu_0 = 8.8542\ 10^{-12}\ (F/m)$

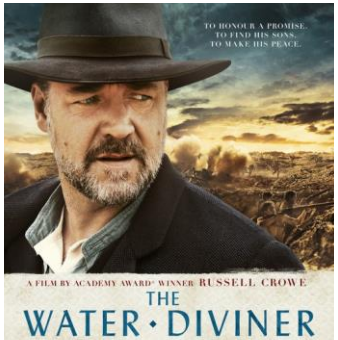
Electric constant

$c = 1/\sqrt{\mu_0\epsilon_0} = 299792458\ (m/s)$

Speed of light

Divergence operator

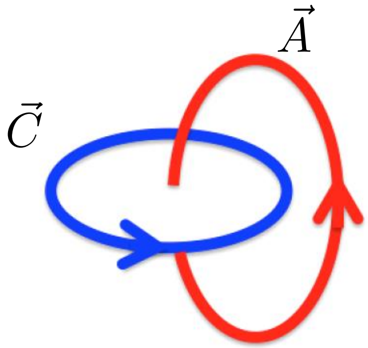
$$\nabla \cdot \vec{A} = \ldots$$



The source of $\vec{A}$ is ...

Curl operator

$$\nabla \times \vec{A} = \vec{C}$$



$\vec{A}$ is chained to $\vec{C}$

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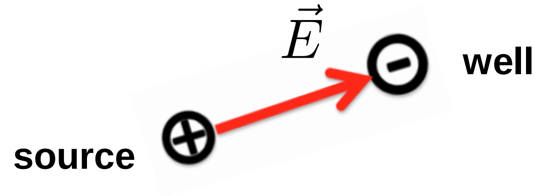
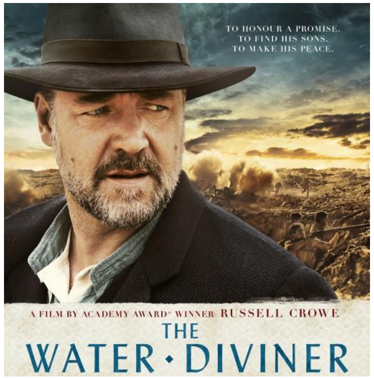
Maxwell equations logo

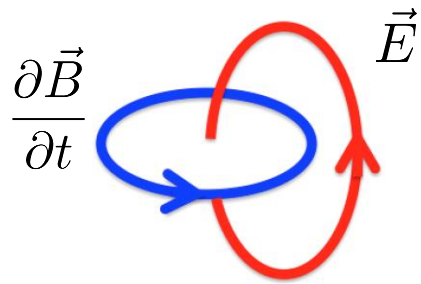
$$\nabla \cdot \vec{E} = \rho / \epsilon_0$$



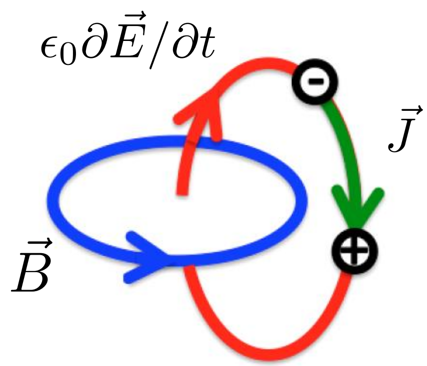


$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$



$$\nabla \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} + \mu_0 \vec{J}$$



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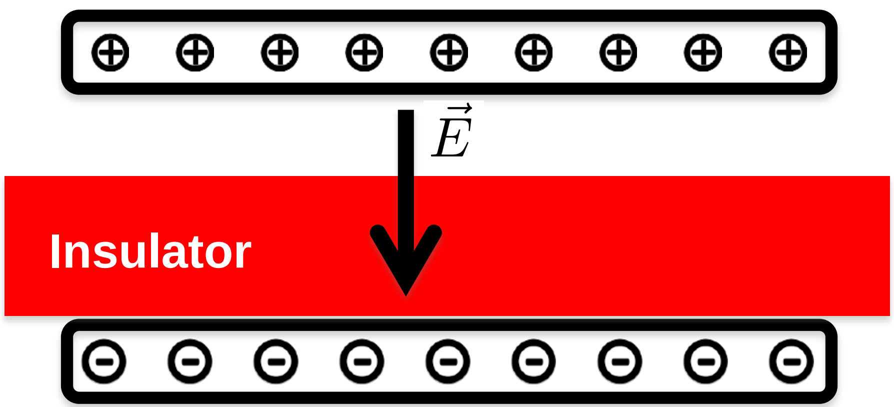
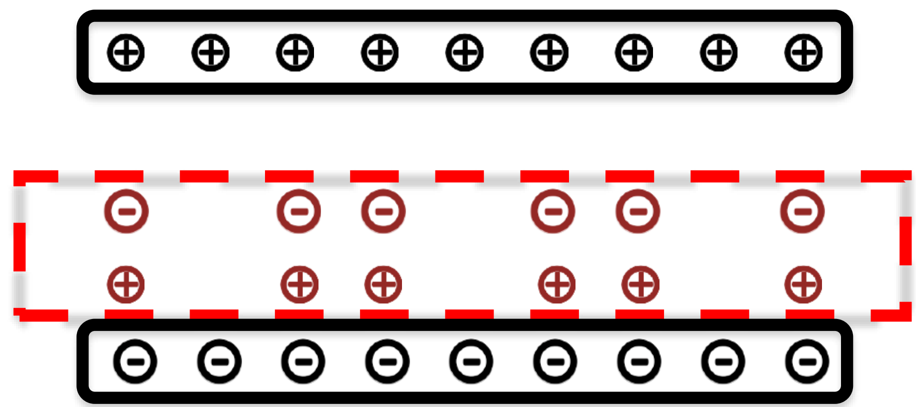
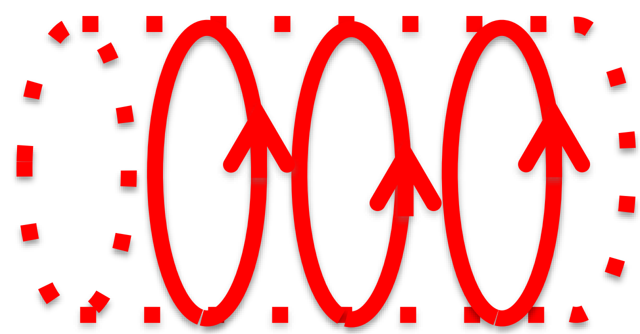
Maxwell equations in matter: the physical approach

The reality … | **… the model**





charges and currents IN VACUUM



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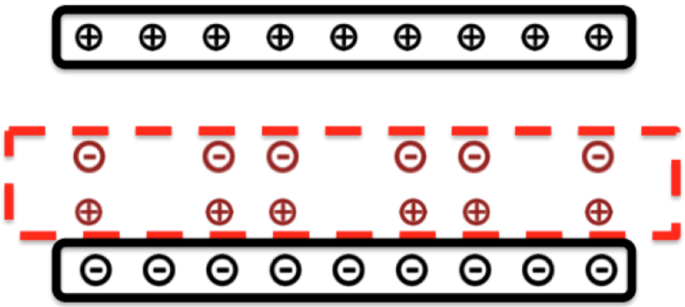
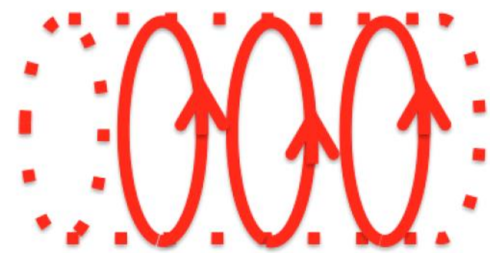
Maxwell equations in matter: the mathematics



Electric insulators (dielectric)



Magnetic materials
(ferrite, superconductor)

Polarization charges

Magnetization currents

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P}$$

Constitutive relations

$$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}$$

$\vec{D}$ **Electric Flux Density** (C/m^2)

$\vec{H}$ **Magnetic Field** (A/m)

$\vec{P}$ **Electric Polarization** (C/m^2)

$\vec{M}$ **Magnetization** (A/m)

Equivalence Principles in Electromagnetics Theory

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Maxwell equations: general expression

$$\nabla \cdot \vec{D} = \rho$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

fields

$\vec{E}$	**Electric Field**	(V/m)
$\vec{H}$	**Magnetic Field**	(A/m)
$\vec{B}$	**Magnetic Flux Density**	(Wb/m^2)
$\vec{D}$	**Electric Flux Density**	(C/m^2)

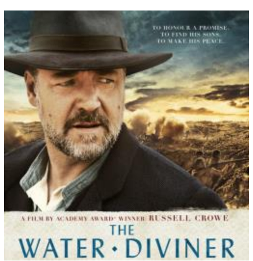
$$\vec{H} = \frac{\vec{B}}{\mu_0}$$

in vacuum

sources

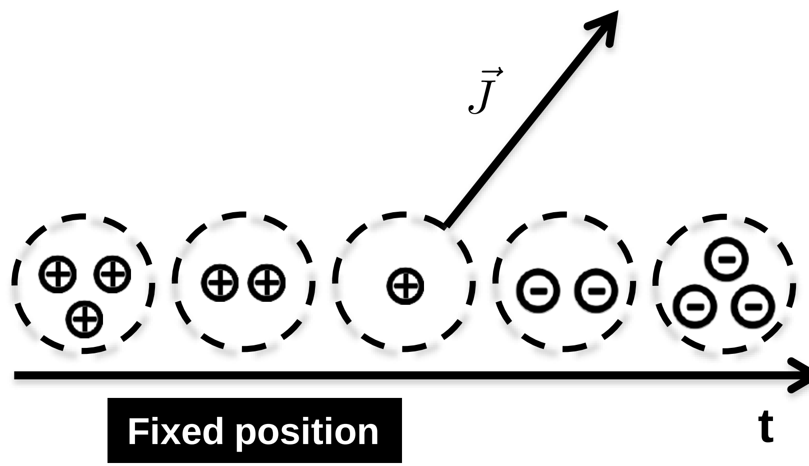
ρ	**Electric Charge Density**	(C/m^3)
$\vec{J}$	**Electric Current Density**	(A/m^2)

Continuity equation is included



$$\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$$

At a given position, the source of $\vec{J}$ is the decrease of charge in time



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Maxwell equations: the static limit

$$\frac{\partial}{\partial t} = 0$$

$$\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t} = 0$$

Ohm Law

→ **Kirchhoff Laws**

Lumped elements (electric networks)

$$\frac{\partial}{\partial t} \approx 0$$

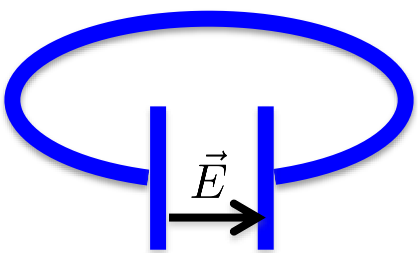
The lumped elements model for electric networks is used also when the field variation is negligible over the size of the network.

$$\frac{\partial}{\partial t} = 0$$

$$\nabla \times \vec{E} = 0$$

The E field is conservative.

The energy gain of a charge in closed circuit is zero.



No static, circular accelerators (RF

Electrostatics

$$\nabla \times \vec{E} = 0 \longrightarrow \vec{E} = -\nabla V \xrightarrow[\text{free space}]{\nabla \cdot \vec{E} = 0} \nabla^2 V = 0$$

Laplace equation

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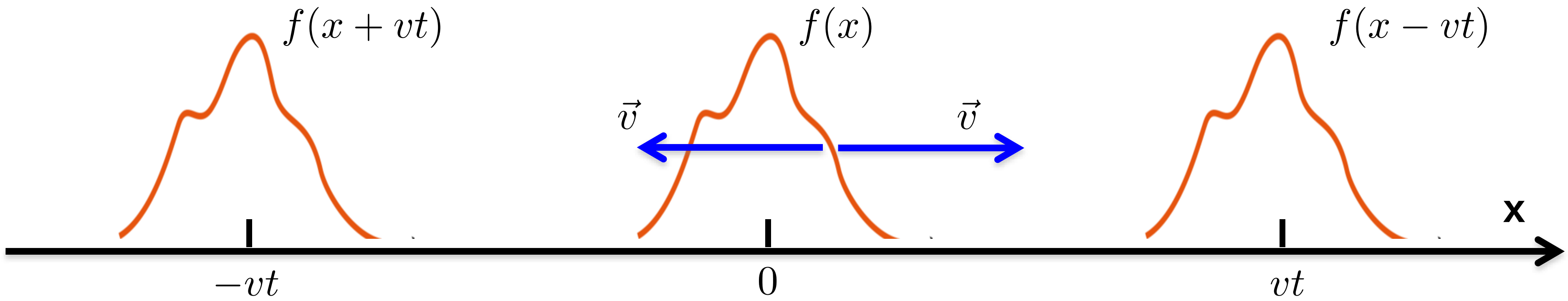
Solution of Maxwell Equations: the EM waves



$$f(x \pm vt) \iff \frac{\partial^2 f}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$$

Wave equation (1D)

Maxwell Equations: free space, no sources

$$\nabla \cdot \vec{E} = 0 \qquad \nabla \cdot \vec{H} = 0$$

$$\nabla \times \vec{E} = -\mu_0 \frac{\partial \vec{H}}{\partial t}$$

$$\nabla \times \vec{H} = \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E} = -\nabla^2 \vec{E}$$

$$||$$

$$\nabla \times \nabla \times \vec{E}$$

$$||$$

$$-\mu_0 \frac{\partial}{\partial t}(\nabla \times \vec{H}) = -\mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t}$$

$$\nabla^2 \vec{E} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\nabla^2 \vec{H} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{H}}{\partial t^2}$$

$$\frac{1}{v^2} = \mu_0 \epsilon_0 \Longrightarrow v = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = c$$

Wave equation (3D)

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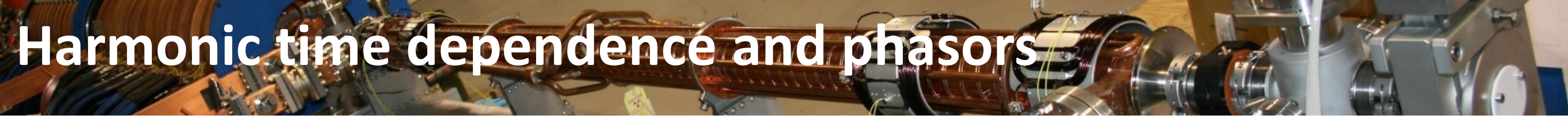

Harmonic time dependence and phasors

Assuming sinusoidal electric field (Fourier)

Time dependence $\longrightarrow$ $e^{j\omega t} = e^{j2\pi f\ t}$ $\longrightarrow$ $\frac{\partial}{\partial t} \cdots = j\omega \ldots$

$$\vec{E}(\vec{r},t) = Re\left\{\vec{E}(\vec{r},\omega)e^{j\omega t}\right\}$$

Phasors are complex vectors

Power/Energy depend on time average of quadratic quantities

$$\left|\vec{E}(\vec{r},t)\right|_{average} = \frac{1}{T}\int_0^T \vec{E}(\vec{r},t)\cdot\vec{E}(\vec{r},t)dt = \cdots = \frac{1}{2}\vec{E}(\vec{r},\omega)\cdot\vec{E}^*(\vec{r},\omega) = \left|\vec{E}_{RMS}(\vec{r},\omega)\right|^2$$

$$\left|\vec{E}_{RMS}\right| = \left|\vec{E}\right|/\sqrt{2}$$

In the following we will use the same symbol for

Real vectors	Complex vectors
$\vec{E}(\vec{r},t), \vec{H}(\vec{r},t), \ldots$	$\vec{E}(\vec{r},t\omega), \vec{H}(\vec{r},\omega), \ldots$

Note that, with phasors, a time animation is identical to phase rotation.

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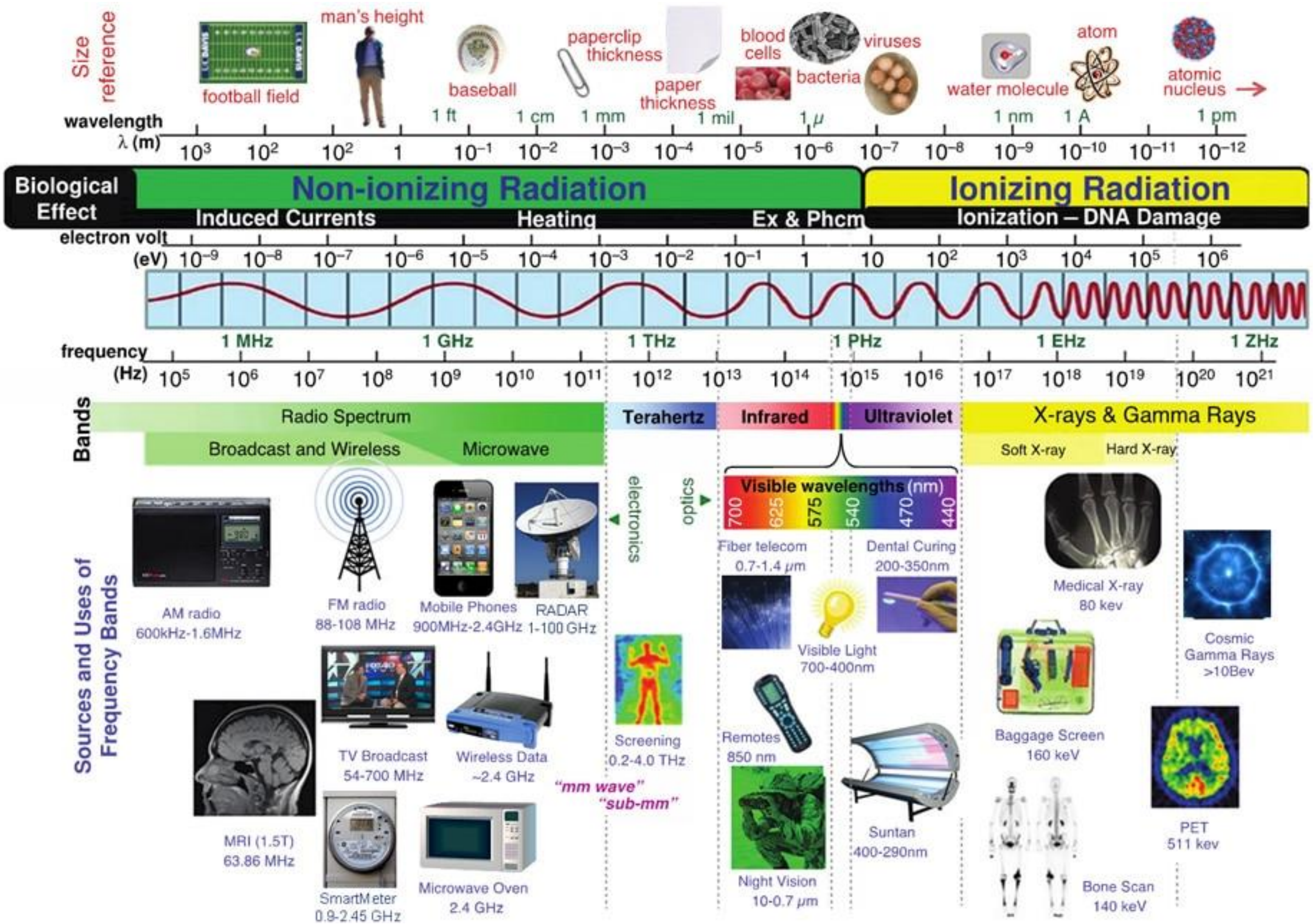
Electromagnetic radiation spectrum



Source: Common knowledge (Wikipedia)

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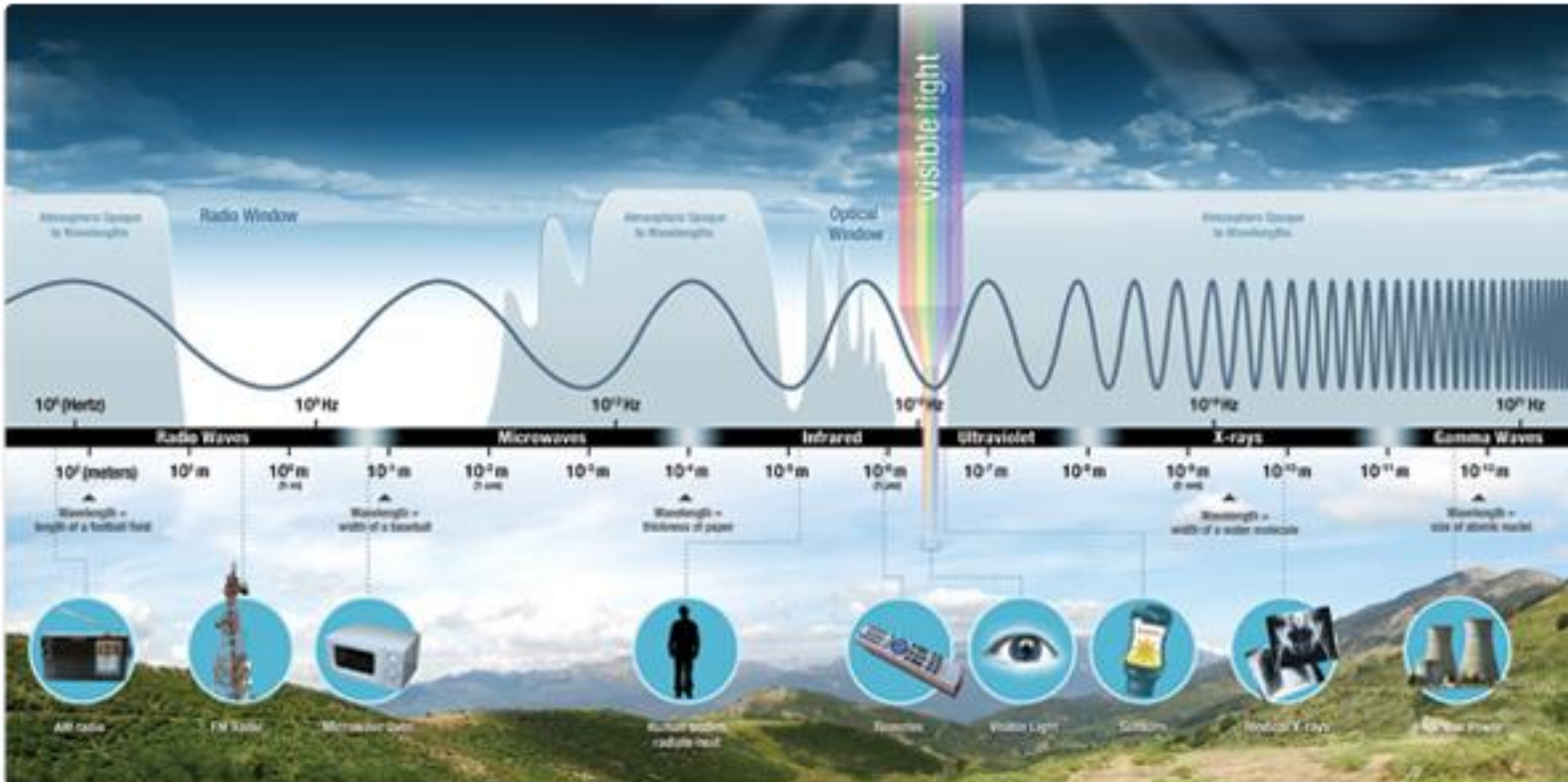
Electromagnetic radiation spectrum: users point of view



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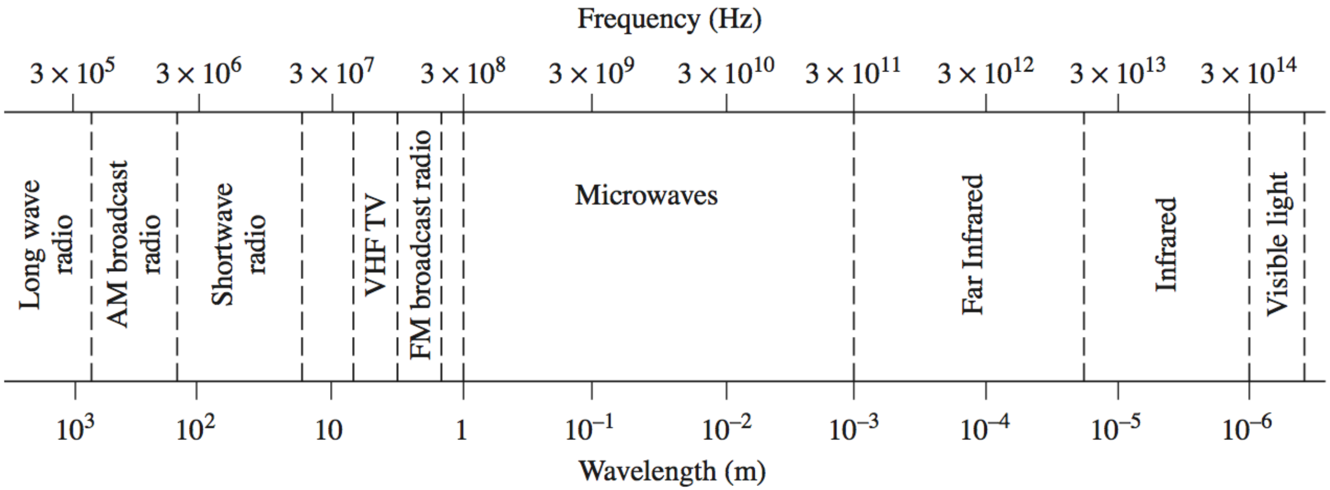
The electromagnetic spectrum for RF engineers



Typical Frequencies	
AM broadcast band	535–1605 kHz
Short wave radio band	3–30 MHz
FM broadcast band	88–108 MHz
VHF TV (2–4)	54–72 MHz
VHF TV (5–6)	76–88 MHz
UHF TV (7–13)	174–216 MHz
UHF TV (14–83)	470–890 MHz
US cellular telephone	824–849 MHz
	869–894 MHz
European GSM cellular	880–915 MHz
	925–960 MHz
GPS	1575.42 MHz
	1227.60 MHz
Microwave ovens	2.45 GHz
US DBS	11.7–12.5 GHz
US ISM bands	902–928 MHz
	2.400–2.484 GHz
	5.725–5.850 GHz
US UWB radio	3.1–10.6 GHz

Approximate Band Designations	
Medium frequency	300 kHz–3 MHz
High frequency (HF)	3 MHz–30 MHz
Very high frequency (VHF)	30 MHz–300 MHz
Ultra high frequency (UHF)	300 MHz–3 GHz
L band	1–2 GHz
S band	2–4 GHz
C band	4–8 GHz
X band	8–12 GHz
Ku band	12–18 GHz
K band	18–26 GHz
Ka band	26–40 GHz
U band	40–60 GHz
V band	50–75 GHz
E band	60–90 GHz
W band	75–110 GHz
F band	90–140 GHz

Source: Pozar, Microwave Engineering 4ed, 2012

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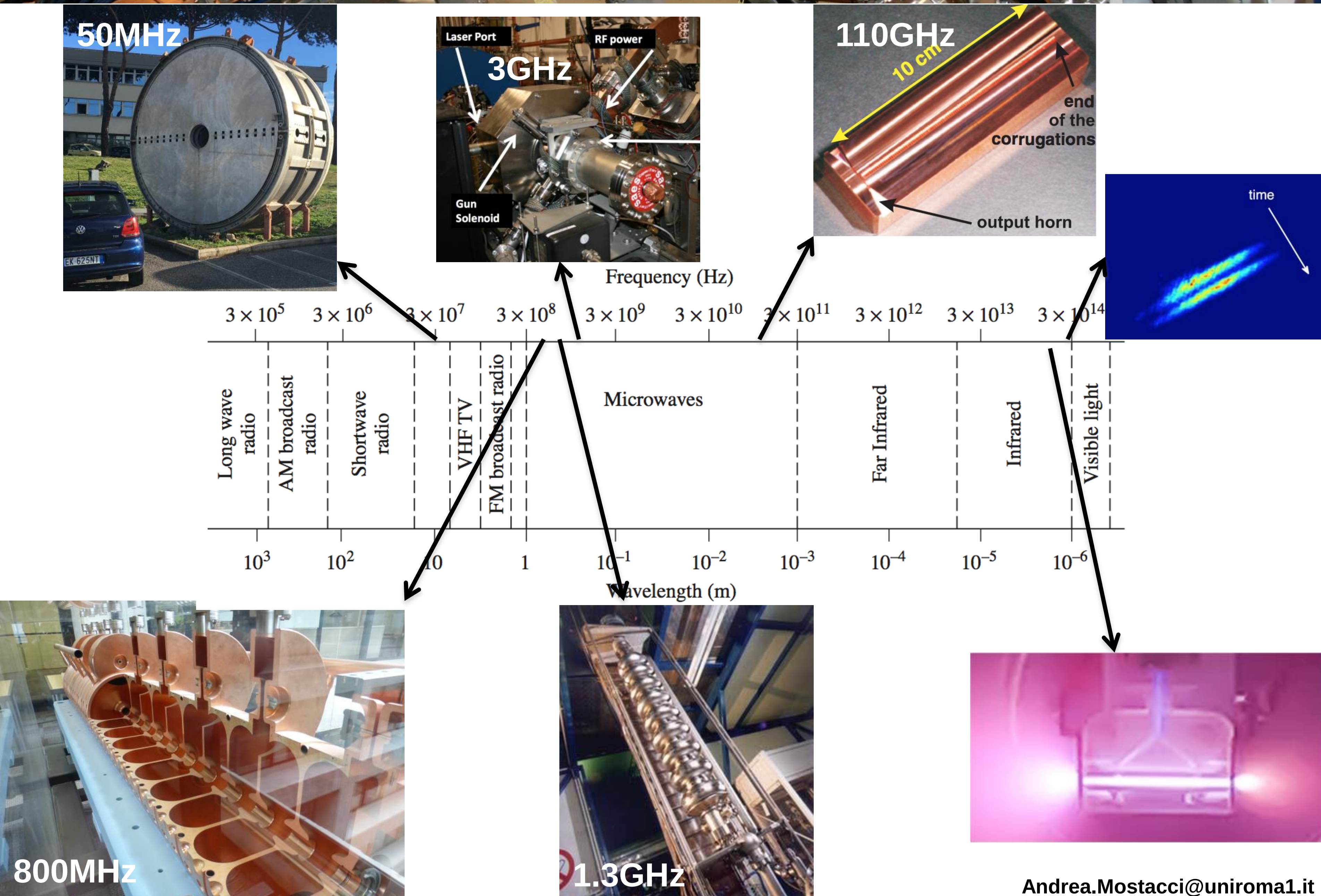
The RF spectrum and particle accelerators



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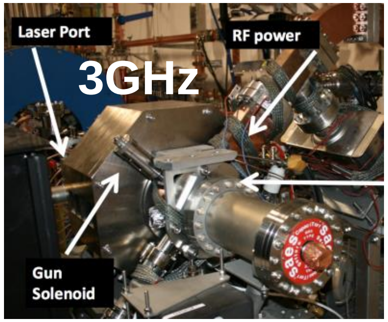
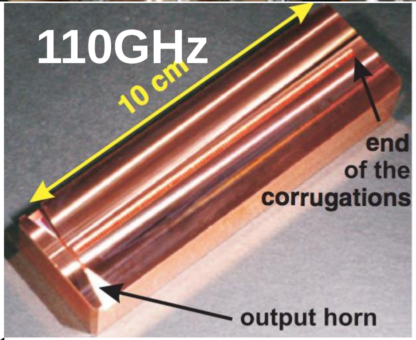
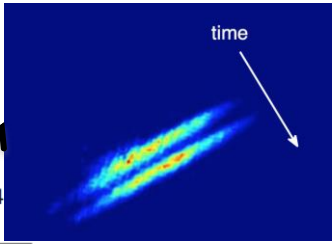
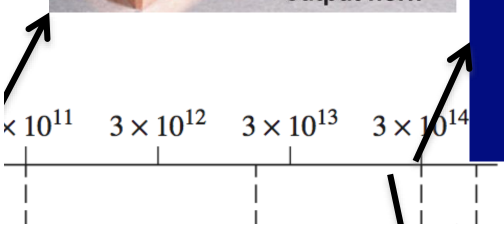
The RF spectrum and particle accelerators

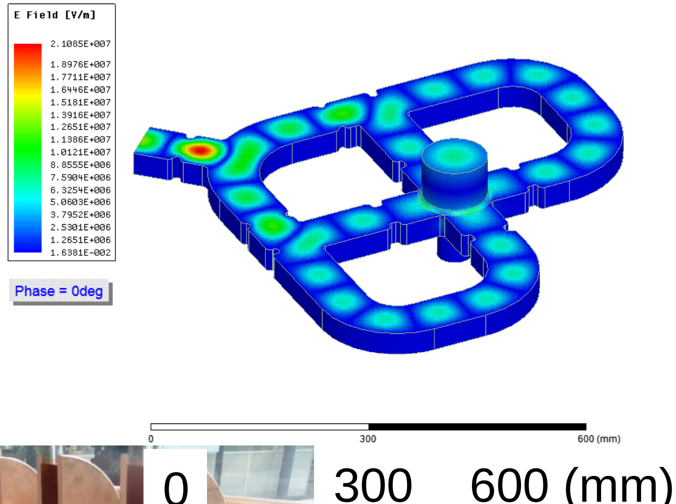


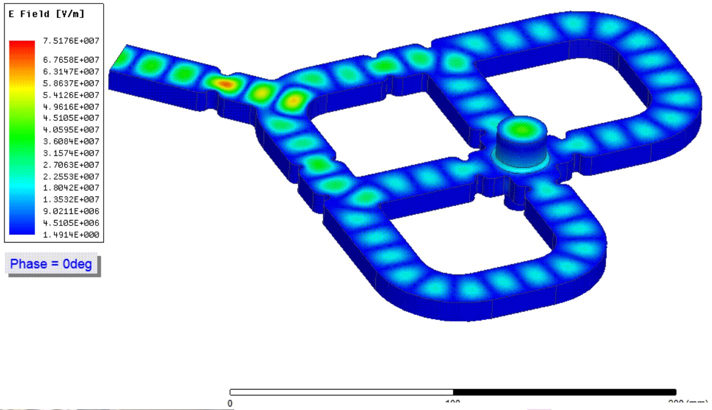


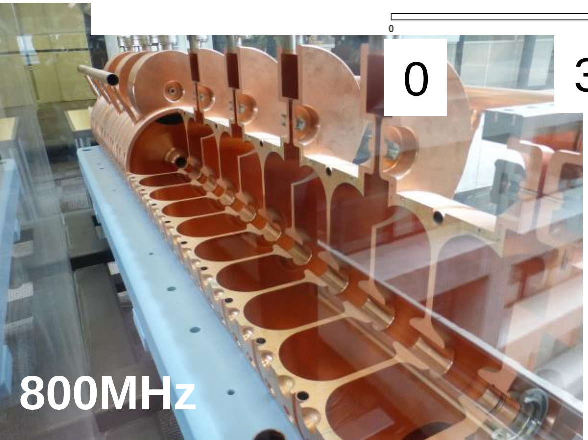


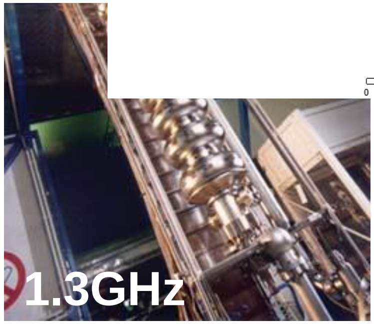


Animations by G. Castorina









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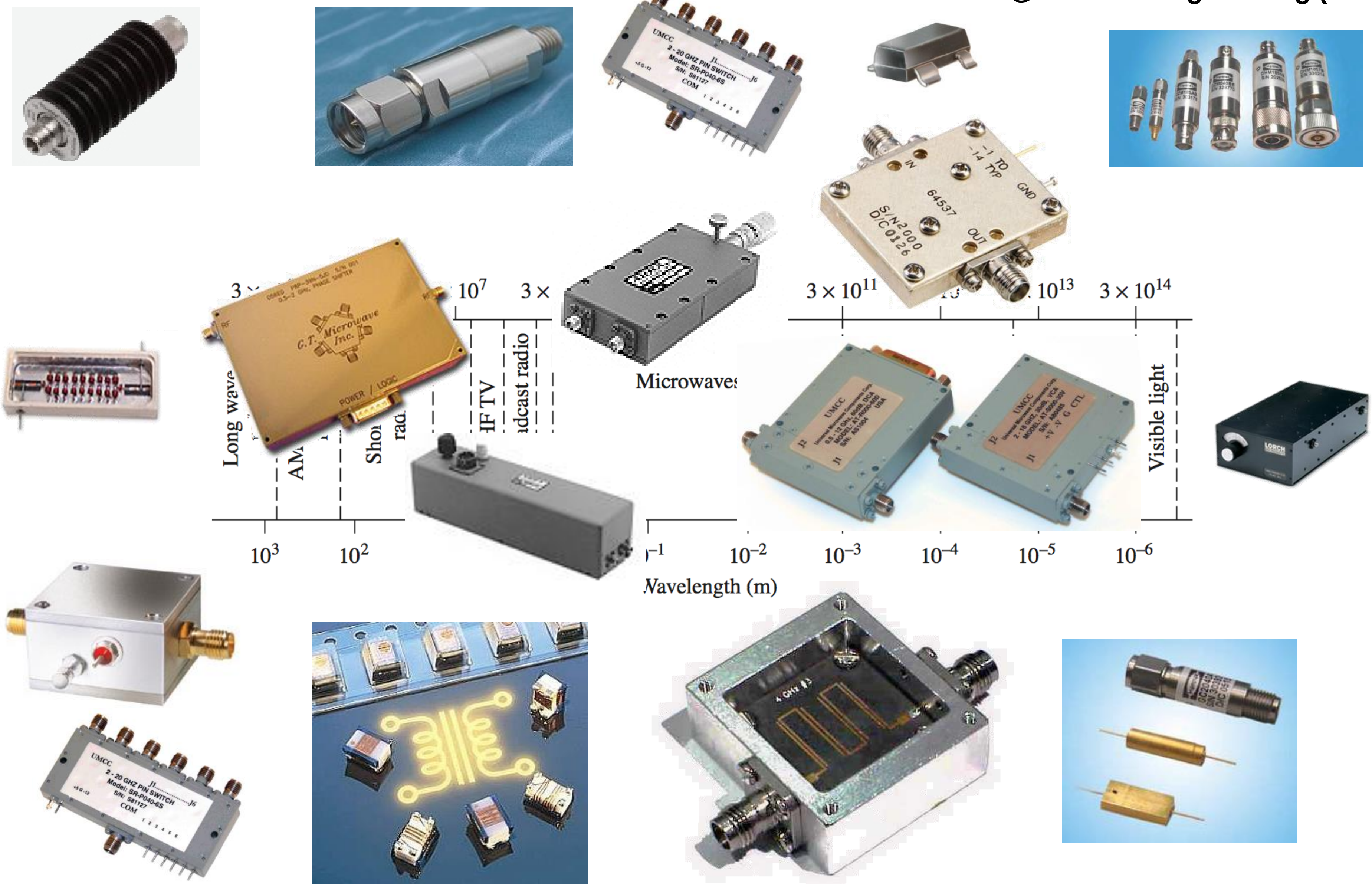
The RF spectrum and particle accelerators

A. Gallo Lecture @ CAS RF engineering (2010)



Harmonic fields in media: constitutive relations

Hyp: Linear, Homogeneous, Isotropic and non Dispersive media

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P} \qquad \vec{D} = \epsilon_c \vec{E} \qquad \epsilon_c = \epsilon' - j\epsilon''$$ **complex permittivity**

Losses (heat) due to damping of vibrating dipoles

$$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M} \qquad \vec{B} = \mu \vec{H} \qquad \mu = \mu' - j\mu''$$ **complex permeability**

Ohm Law $\vec{J}_c = \sigma \vec{E}$ $\quad \sigma$ **conductivity** (S/m)

Losses (heat) due to moving charges colliding with lattice

Material	Conductivity S/m (20°C)	Material	Conductivity S/m (20°C)
Aluminum	3.816×10^7	Nichrome	1.0×10^6
Brass	2.564×10^7	Nickel	1.449×10^7
Bronze	1.00×10^7	Platinum	9.52×10^6
Chromium	3.846×10^7	Sea water	3–5
Copper	5.813×10^7	Silicon	4.4×10^{-4}
Distilled water	2×10^{-4}	Silver	6.173×10^7
Germanium	2.2×10^6	Steel (silicon)	2×10^6
Gold	4.098×10^7	Steel (stainless)	1.1×10^6
Graphite	7.0×10^4	Solder	7.0×10^6
Iron	1.03×10^7	Tungsten	1.825×10^7
Mercury	1.04×10^6	Zinc	1.67×10^7
Lead	4.56×10^6		

Source: Pozar, Microwave Engineering 4ed, 2012

Harmonic fields in media: Maxwell Equations

Hyp: **Linear, Homogeneous, Isotropic and non Dispersive media**

$\vec{D} = \epsilon_0 \vec{E} + \vec{P}$ $\vec{D} = \epsilon_c \vec{E}$ $\epsilon_c = \epsilon' - j\epsilon''$ **complex permittivity**

Losses (heat) due to damping of vibrating dipoles

$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}$ $\vec{B} = \mu \vec{H}$ $\mu = \mu' - j\mu''$ **complex permeability**

Ohm Law $\vec{J}_c = \sigma \vec{E}$ σ **conductivity** (S/m)

Losses (heat) due to moving charges colliding with lattice

$\frac{\partial}{\partial t}\cdots = j\omega\cdots$

$$\nabla \cdot \vec{D} = \rho \qquad \nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -j\omega\mu\vec{H}$$

$$\nabla \times \vec{H} = j\omega\vec{D} + \vec{J}_c + \vec{J} = \cdots = j\omega\epsilon\vec{E} + \vec{J} \qquad \epsilon = \epsilon' - j\epsilon'' - j\frac{\sigma}{\omega}$$

$$\tan\delta = \frac{\omega\epsilon'' + \sigma}{\omega\epsilon'} = \frac{\text{Losses}}{\text{Displacement current}}$$

$$\epsilon' = \epsilon_r \epsilon_0$$

Loss tangent

$$\epsilon = \epsilon_r \epsilon_0 \left(1 - j\tan\delta\right)$$

Dielectric constant

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Harmonic fields in media: Maxwell Equations

DIELECTRIC CONSTANTS AND LOSS TANGENTS FOR SOME MATERIALS

Material	Frequency	ϵ_r	$\tan\delta$ (25°C)
Alumina (99.5%)	10 GHz	9.5–10.	0.0003
Barium tetratitanate	6 GHz	$37 \pm 5\%$	0.0005
Beeswax	10 GHz	2.35	0.005
Beryllia	10 GHz	6.4	0.0003
Ceramic (A-35)	3 GHz	5.60	0.0041
Fused quartz	10 GHz	3.78	0.0001
Gallium arsenide	10 GHz	13.0	0.006
Glass (pyrex)	3 GHz	4.82	0.0054
Glazed ceramic	10 GHz	7.2	0.008
Lucite	10 GHz	2.56	0.005
Nylon (610)	3 GHz	2.84	0.012
Parafin	10 GHz	2.24	0.0002
Plexiglass	3 GHz	2.60	0.0057
Polyethylene	10 GHz	2.25	0.0004
Polystyrene	10 GHz	2.54	0.00033
Porcelain (dry process)	100 MHz	5.04	0.0078
Rexolite (1422)	3 GHz	2.54	0.00048
Silicon	10 GHz	11.9	0.004
Styrofoam (103.7)	3 GHz	1.03	0.0001
Teflon	10 GHz	2.08	0.0004
Titania (D-100)	6 GHz	$96 \pm 5\%$	0.001
Vaseline	10 GHz	2.16	0.001
Water (distilled)	3 GHz	76.7	0.157

Source: Pozar, Microwave Engineering 4ed, 2012

ı Dispersive media

$''$ **complex permittivity**

ι'' **complex permeability**

ıctivity (S/m)

Losses (heat) due to moving charges colliding with lattice

$\vec{E} + \vec{J}$ $\qquad \epsilon = \epsilon' - j\epsilon'' - j\dfrac{\sigma}{\omega}$

Loss tangent

$$\epsilon = \epsilon_r \epsilon_0 \left(1 - j \tan\delta\right)$$

Dielectric constant

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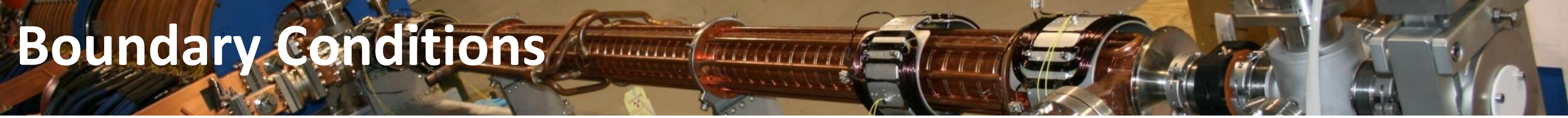
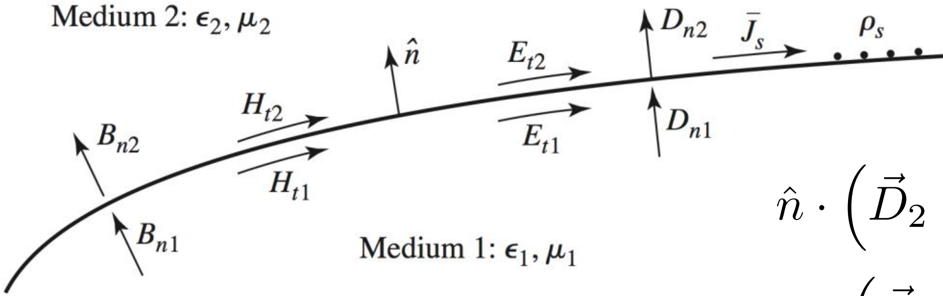


Boundary Conditions



ρ_s **Surface Charge Density** (C/m^2)

$\vec{J}_s$ **Surface Current Density** (A/m)

$$\hat{n}\cdot\left(\vec{D}_2-\vec{D}_1\right)=\rho_s \qquad \hat{n}\cdot\left(\vec{B}_2-\vec{B}_1\right)=0$$

$$\hat{n}\times\left(\vec{E}_2-\vec{E}_1\right)=0 \qquad \hat{n}\times\left(\vec{H}_2-\vec{H}_1\right)=\vec{J}_s$$

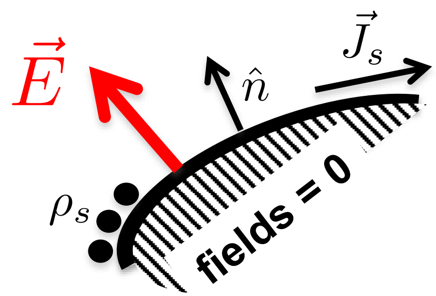
Fields at a lossless dielectric interface

$$\rho_s=0 \qquad \hat{n}\cdot\vec{D}_1=\hat{n}\cdot\vec{D}_2 \qquad \hat{n}\cdot\vec{B}_1=\hat{n}\cdot\vec{B}_2$$

$$\vec{J}_s=0 \qquad \hat{n}\times\vec{E}_1=\hat{n}\times\vec{E}_2 \qquad \hat{n}\times\vec{H}_1=\hat{n}\times\vec{H}_2$$

Perfect conductor (electric wall)

$$\sigma\to\infty$$

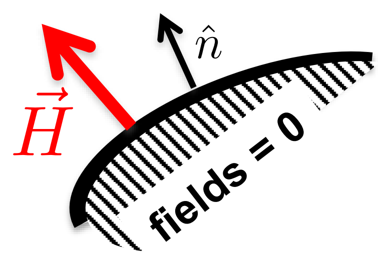


$$\hat{n}\cdot\vec{D}=\rho_s \qquad \hat{n}\cdot\vec{B}=0$$

$$\hat{n}\times\vec{E}=0 \qquad \hat{n}\times\vec{H}=\vec{J}_s$$

Magnetic Wall (dual of the E-wall)

approx.



$$\hat{n}\cdot\vec{D}=0 \qquad \hat{n}\cdot\vec{B}=0$$

$$\hat{n}\times\vec{H}=0 \qquad \hat{n}\times\vec{E}\neq 0$$

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Helmotz equation and its simplest solution

$$\nabla^2 \vec{E} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$e^{j\omega t} \rightarrow \frac{\partial^2}{\partial t^2} \cdots = -\omega^2 \ldots$$

Helmotz equation

$$\nabla^2 \vec{E} + \omega^2 \mu\epsilon \vec{E} = 0$$

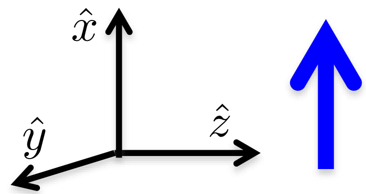
$$\nabla^2 \vec{H} + \omega^2 \mu\epsilon \vec{H} = 0$$

$$k = \omega\sqrt{\mu\epsilon} \qquad (1/m)$$

Propagation/phase constant

Wave number

The simples solution: the plane wave



$$\vec{E} = E_x \hat{x}$$

Uniform in x, y

Lossless medium

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial y} = 0 \qquad \frac{d^2 E_x}{dz^2} + k^2 E_x = 0 \qquad E_x(z) = E^+ e^{-jkz} + E^- e^{jkz}$$

$$E_x(z,t) = Re\left\{E(x,\omega)e^{j\omega t}\right\} = E^+ \cos(\omega t - kz) + E^- \cos(\omega t + kz)$$

It is a wave, moving in the +z direction or –z direction

Phase velocity **Velocity at which a fixed phase point on the wave travels**

$$\omega t \mp kz = \text{const} \qquad v_p = \frac{dz}{dt} = \frac{d}{dt}\left(\frac{\omega t \mp \text{const}}{k}\right) = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\epsilon}}$$

Speed of light

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Wave length **Distance between two consecutive maxima (or minima or …)**

$$(\omega t - kz) - [\omega t - k(z+\lambda)] = 2\pi$$

$$\lambda = \frac{2\pi}{k} = \frac{2\pi v_p}{\omega} = \frac{v_p}{f}$$

$$\nabla \times \vec{E} = -j\omega\mu\vec{H}$$

Compute H …

$$E_x(z) = E^+ e^{-jkz} + E^- e^{jkz}$$

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Plane waves and Transverse Electro-Magnetic (TEM) waves

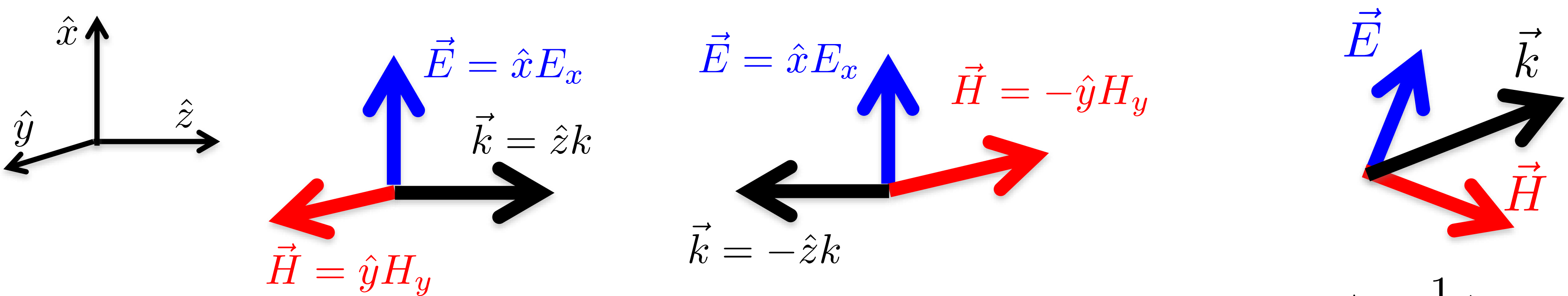
Wave length **Distance between two consecutive maxima (or minima or …)**

$$(\omega t - kz) - [\omega t - k(z+\lambda)] = 2\pi \qquad \lambda = \frac{2\pi}{k} = \frac{2\pi v_p}{\omega} = \frac{v_p}{f}$$

$$\nabla \times \vec{E} = -j\omega\mu\vec{H} \qquad H_x = H_z = 0 \qquad H_y = \frac{j}{\omega\mu}\frac{\partial E_x}{\partial z} = \frac{1}{\eta}\left(E^+ e^{-jkz} - E^- e^{jkz}\right)$$

$$\eta = \frac{\omega\mu}{k} = \sqrt{\frac{\mu}{\epsilon}}$$ **Intrinsic impedance of the medium** (Ω) $$\eta_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} = 377\ \Omega$$

The ration of E and E component is an impedance called wave impedance



$$\vec{H} = \frac{1}{\eta}\hat{k} \times \vec{E}$$

TEM wave **E and H field are transverse to the direction of propagation.**

$$Z_{TEM} = \eta$$

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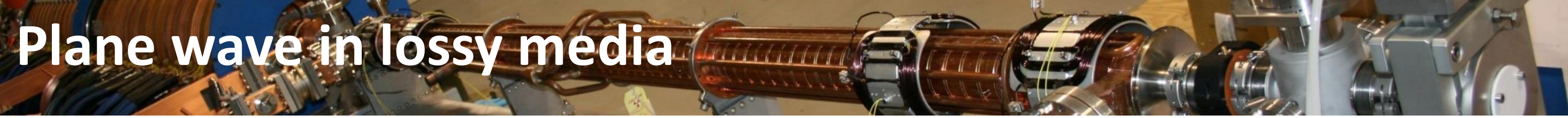
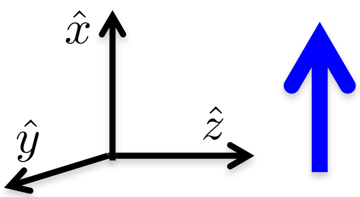


Plane wave in lossy media

$$\nabla^2 \vec{E} + \omega^2 \mu\epsilon \vec{E} = 0 \qquad \epsilon = \epsilon_r \epsilon_0 \left(1 - j\tan\delta\right) \qquad \tan\delta = \frac{\omega\epsilon'' + \sigma}{\omega\epsilon'}$$

Definition: $\gamma = \alpha + j\beta = j\omega\sqrt{\mu\epsilon} = j\omega\sqrt{\mu\epsilon_0\epsilon_r(1 - j\tan\delta)}$

Attenuation constant (α) **Phase constant** (β)



$\vec{E} = E_x \hat{x}$

Uniform in x, y

$$\frac{d^2 E_x}{dz^2} - \gamma^2 E_x = 0 \qquad E_x(z) = E^+ e^{-\gamma z} + E^- e^{\gamma z}$$

Positive z direction

$$e^{-\gamma z} = e^{-\alpha z} e^{-j\beta z}$$

— time →

$$e^{-\alpha z}\cos\left(\omega t - \beta z\right)$$

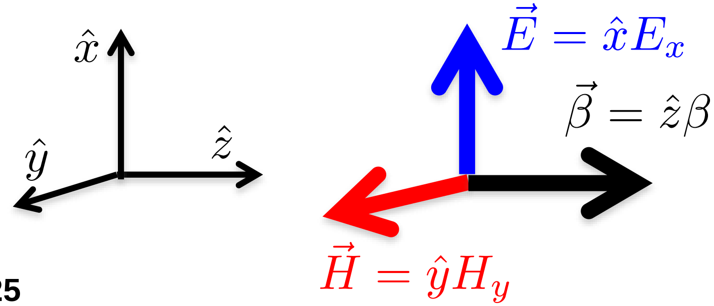
$$v_p = \frac{\omega}{\beta} \qquad \lambda = \frac{2\pi}{\beta}$$

$$H_y = \frac{j}{\omega\mu}\frac{\partial E_x}{\partial z} = -\frac{j\gamma}{\omega\mu}\left(E^+ e^{-\gamma z} - E^- e^{\gamma z}\right) = \frac{1}{\eta}\left(E^+ e^{-\gamma z} - E^- e^{\gamma z}\right)$$

$$\eta = \frac{j\omega\mu}{\gamma} \rightarrow \sqrt{\frac{\mu}{\epsilon}}$$

complex

$$Z_{TEM} = \eta$$



$$\vec{H} = \frac{1}{\eta}\hat{\beta} \times \vec{E}$$

Attenuating TEM "wave" …

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Plane waves in good conductors

Good conductor

Conduction current >> displacement current

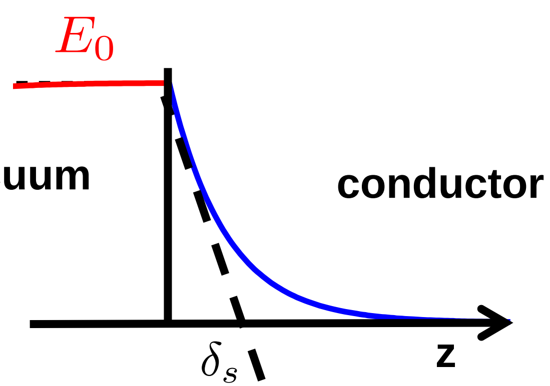
$$\sigma E \gg \omega\epsilon_c E$$

$$\tan\delta = \frac{\omega\epsilon'' + \sigma}{\omega\epsilon'} \approx \frac{\sigma}{\omega\epsilon_0\epsilon_r}$$

$$\gamma = \alpha + j\beta = j\omega\sqrt{\mu\epsilon} \simeq (1+j)\sqrt{\frac{\omega\mu\sigma}{2}}$$

Characteristic depth of penetration: skin depth

$$\delta_s = \frac{1}{\alpha} = \sqrt{\frac{2}{\omega\mu\sigma}}$$



Al	$\delta_s = 8.14\ 10^{-7}\ m$
Cu	$\delta_s = 6.60\ 10^{-7}\ m$
Au	$\delta_s = 7.86\ 10^{-7}\ m$
Ag	$\delta_s = 6.40\ 10^{-7}\ m$

@ 10 GHz

impedance of the medium

$$\eta = \frac{j\omega\mu}{\gamma} \simeq (1+j)\sqrt{\frac{\omega\mu}{2\sigma}} = (1+j)\frac{1}{\sigma\delta_s}$$

? Copper @ 100 MHz

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Maxwell equations and boundary value problem

Maxwell equation with sources + boundary conditions = boundary value problem

Homogeneous medium

Sources

$\vec{J}, \ \rho$

$$\nabla \cdot \vec{E} = \rho/\epsilon \qquad \nabla \cdot \vec{H} = 0$$

$$\nabla \times \vec{E} = -j\omega\mu\vec{H} \qquad \nabla \times \vec{H} = +j\omega\epsilon\vec{E} + \vec{J}$$

Do you see asymmetries?

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Maxwell equations and boundary value problem

Maxwell equation with sources + boundary conditions = boundary value problem

Homogeneous medium

Sources

$\nabla \cdot \vec{E} = \rho/\epsilon$ $\quad$ $\nabla \cdot \vec{H} = \rho_m/\mu$ $\quad$ $\vec{J},\ \rho$ $\quad$ **Actual or equivalent**

$\nabla \times \vec{E} = -j\omega\mu\vec{H} - \vec{J}_m$ $\quad$ $\nabla \times \vec{H} = +j\omega\epsilon\vec{E} + \vec{J}$ $\quad$ $\vec{J}_m,\ \rho_m$ $\quad$ **equivalent**

Vector Helmotz Equation

$$\nabla^2\vec{E} + k^2\vec{E} = \nabla \times \vec{J}_m + j\omega\mu\vec{J} + \frac{1}{\epsilon}\nabla\rho$$

$$\nabla^2\vec{H} + k^2\vec{H} = -\nabla \times \vec{J} + j\omega\epsilon\vec{J}_m + \frac{1}{\mu}\nabla\rho_m$$

$$k^2 = \omega^2\mu\epsilon$$

Solution

Step 1 **Source free region** $\vec{J} = \vec{J}_m = \rho_m = \rho = 0$ **Homogeneous problem**

Step 2 $\text{Solution} = \sum_k C_k\left(\vec{J}, \vec{J}_m, \rho_m, \rho\right) \text{Solution-Homogeneous-Problem}_k$

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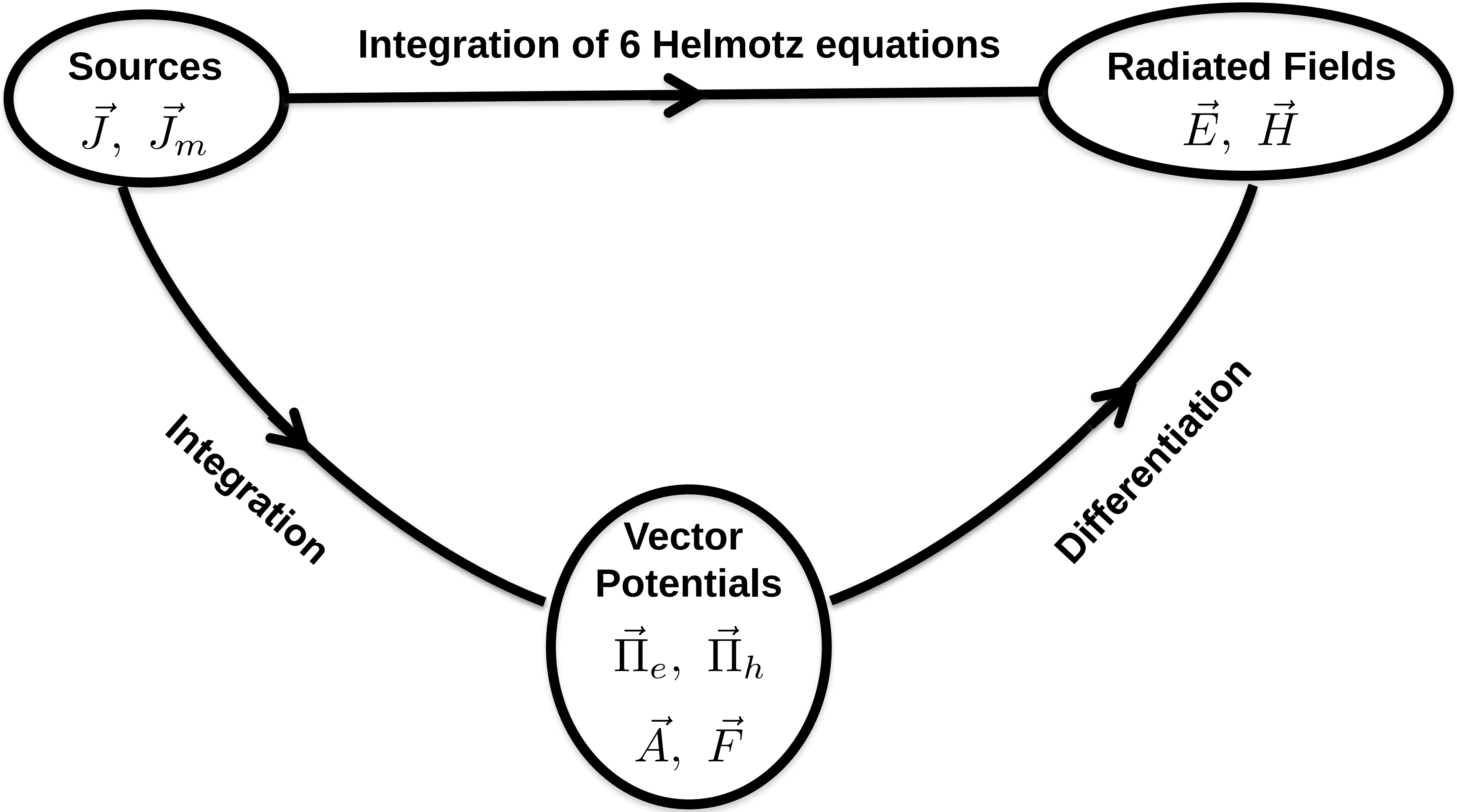
Method of solution of Helmotz equations



Solution of the homogeneous equation → **Shape of radiated field** → **MODES**

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Solution of Helmotz equations using potentials

Sources

$\vec{J},\ \vec{J}_m$

Integration of 6 Helmotz equations →

Radiated Fields

$\vec{E},\ \vec{H}$

Integration of Helmotz equations

if $\nabla \cdot \vec{H} = 0$

$$\nabla^2 \vec{A} + k^2 \vec{A} = -\mu \vec{J}$$

if $\nabla \cdot \vec{E} = 0$

$$\nabla^2 \vec{F} + k^2 \vec{F} = -\mu \vec{J}_m$$

$$k^2 = \omega^2 \mu \epsilon$$

Vector Potentials

$\vec{A},\ \vec{F}$

Differentiation

$$\vec{H}_A = \frac{1}{\mu} \nabla \times \vec{A}$$

$$\vec{E}_A = -j\omega \vec{A} - \frac{j}{\omega \mu \epsilon} \nabla \left(\nabla \cdot \vec{A} \right)$$

$$\vec{E}_F = -\frac{1}{\epsilon} \nabla \times \vec{F}$$

$$\vec{H}_F = -j\omega \vec{F} - \frac{j}{\omega \mu \epsilon} \nabla \left(\nabla \cdot \vec{F} \right)$$

$$\vec{E} = \vec{E}_A + \vec{E}_F$$

$$\vec{H} = \vec{H}_A + \vec{H}_F$$

Why/when is it convenient?

MODES → **Solution of the homogeneous equations** →

$$\nabla^2 \vec{A} + k^2 \vec{A} = 0$$

$$\nabla^2 \vec{F} + k^2 \vec{F} = 0$$

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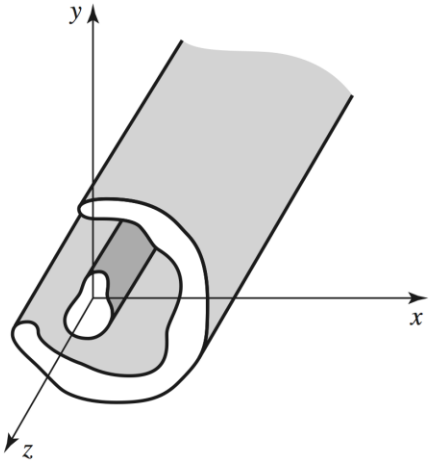
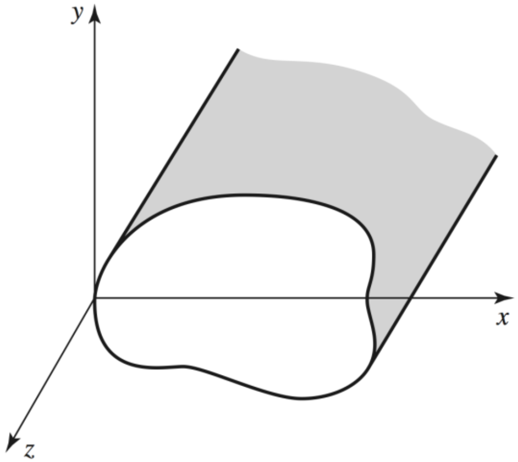
Modes of cylindrical waveguides: propagating field





Field propagating in the positive z direction

$$\vec{A} = \hat{z}\ A_z(x,y)\ e^{-j\beta z} = \hat{z}\ A$$

$$\vec{F} = \hat{z}\ F_z(x,y)\ e^{-j\beta z} = \hat{z}\ F$$

$$\nabla^2 = \nabla_t^2 + \frac{\partial^2}{\partial z^2} \longrightarrow \begin{array}{l} \nabla_t^2 A_z + \left(k^2 - \beta^2\right) A_z = 0 \\ \nabla_t^2 F_z + \left(k^2 - \beta^2\right) F_z = 0 \end{array}$$

2 Helmotz equations
(transverse coordinates)

$$\vec{H}_A = \frac{1}{\mu}\nabla \times (\hat{z}A) \longrightarrow \vec{H}_A = \vec{h}_t\ e^{-j\beta z}$$

$$\vec{E}_A = -j\omega A\hat{z} - \frac{\beta}{\omega\mu\epsilon}\nabla A \longrightarrow \vec{E}_A = [\vec{e}_t + \hat{z}\ e_z]\, e^{-j\beta z}$$

Only E field along z
E-mode
Transverse Magnetic (TM)

$$\vec{E}_F = -\frac{1}{\epsilon}\nabla \times (\hat{z}F) \longrightarrow \vec{E}_F = \vec{e}_t\ e^{-j\beta z}$$

$$\vec{H}_F = -j\omega F\hat{z} - \frac{\beta}{\omega\mu\epsilon}\nabla F \longrightarrow \vec{H}_F = \left[\vec{h}_t + \hat{z}\ h_z\right] e^{-j\beta z}$$

Only H field along z
H-mode
Transverse Electric (TE)

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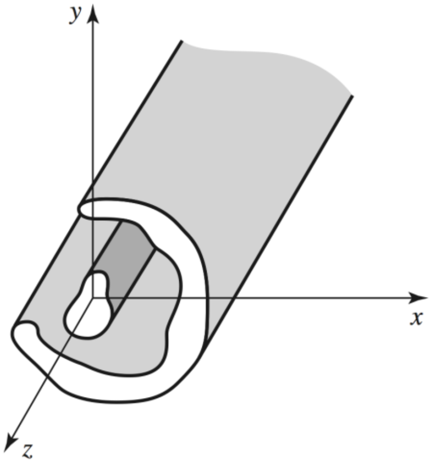
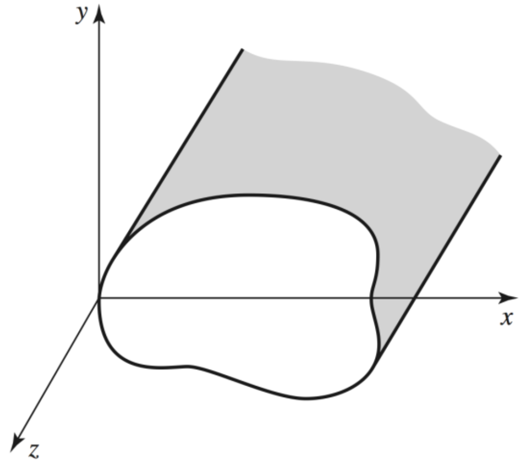
Modes of cylindrical waveguides: propagating field





Field propagating in the positive z direction

$$\vec{A} = \hat{z}\ A_z(x,y)\ e^{-j\beta z} = \hat{z}\ A$$

$$\vec{F} = \hat{z}\ F_z(x,y)\ e^{-j\beta z} = \hat{z}\ F$$

$$\vec{H}_A = \frac{1}{\mu}\nabla \times (\hat{z}A) \longrightarrow \vec{H}_A = \vec{h}_t\ e^{-j\beta z}$$

$$\vec{E}_A = -j\omega A\hat{z} - \frac{\beta}{\omega\mu\epsilon}\nabla A \longrightarrow \vec{E}_A = [\vec{e}_t + \hat{z}\ e_z]\, e^{-j\beta z}$$

Only E field along z
E-mode
Transverse Magnetic (TM)

$$\vec{E}_F = -\frac{1}{\epsilon}\nabla \times (\hat{z}F) \longrightarrow \vec{E}_F = \vec{e}_t\ e^{-j\beta z}$$

$$\vec{H}_F = -j\omega F\hat{z} - \frac{\beta}{\omega\mu\epsilon}\nabla F \longrightarrow \vec{H}_F = \left[\vec{h}_t + \hat{z}\ h_z\right] e^{-j\beta z}$$

Only H field along z
H-mode
Transverse Electric (TE)

$$\vec{E} = \vec{E}_A + \vec{E}_F \qquad \vec{H} = \vec{H}_A + \vec{H}_F$$

$\longrightarrow$ **TM modes** + **TE modes**

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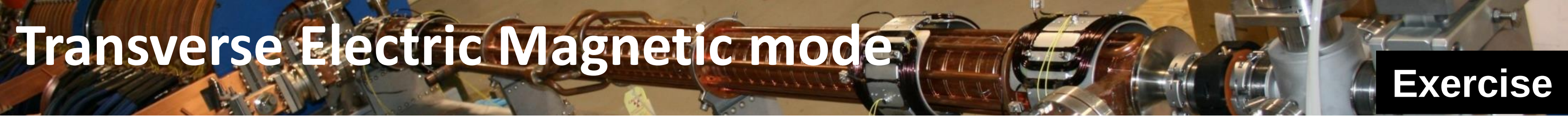

Transverse Electric Magnetic mode

Exercise

Look for a Transverse Electric Magnetic mode $E_z = H_z = 0$

$$\vec{E},\ \vec{H},\ v_p?$$

Hint 1 **Start from a TM mode (vector potential A)** $H_z = 0$

$$\nabla = \nabla_t + \hat{z}\frac{\partial}{\partial z} \qquad \nabla \cdot \vec{A} = \cdots$$

Hint 2

$$\vec{E}_A = \cdots$$

Solution

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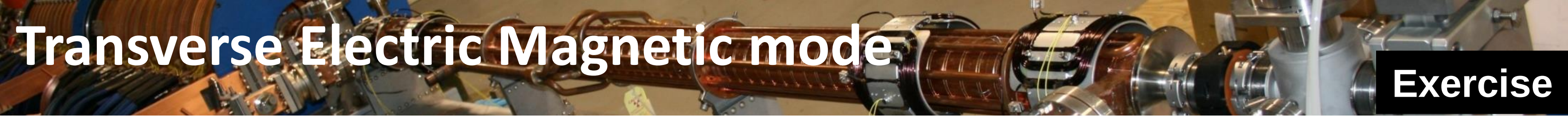

Transverse Electric Magnetic mode

Exercise

Look for a Transverse Electric Magnetic mode $E_z = H_z = 0$

$\vec{E}, \ \vec{H}, \ v_p?$

Hint 1 **Start from a TM mode (vector potential A)** $H_z = 0$

$$\nabla = \nabla_t + \hat{z}\frac{\partial}{\partial z} \qquad \nabla \cdot \vec{A} = \cdots = -j\beta A_z e^{-j\beta z}$$

Hint 2

$$\vec{E}_A = \cdots = -j\omega \hat{z} A_z e^{-j\beta z} - \frac{j}{\omega\mu\epsilon}\left[\nabla_t + \hat{z}\frac{\partial}{\partial z}\right](-j\beta)A_z e^{-j\beta z} =$$

$$= -\frac{j}{\omega\mu\epsilon}\left[\omega^2\mu\epsilon - \beta\right] A_z e^{-j\beta z}\hat{z} - \frac{\beta}{\omega\mu\epsilon}\nabla_t A_z \ e^{-j\beta z}$$

$$\text{if } \beta^2 = \omega^2\mu\epsilon = k^2 \quad \Longrightarrow \quad e_z = 0$$

Solution **For a given** A_z

$$\vec{H} = \frac{1}{\mu}\nabla_t \times (\hat{z}A_z)\, e^{-j\omega\sqrt{\mu\epsilon}z} \qquad \vec{E} = -\frac{1}{\sqrt{\mu\epsilon}} \nabla_t A_z \ e^{-j\omega\sqrt{\mu\epsilon}z}$$

1. $\nabla_t^2 A_z = -\left(k^2 - \beta^2\right) A_z = 0$ **The transverse E field is “electrostatic”**

2. **As plane waves:** $\ldots e^{-j\omega\sqrt{\mu\epsilon}z} \quad \Longrightarrow \quad v_p = 1/\sqrt{\mu\epsilon}$

$$\vec{h}_t = \sqrt{\frac{\epsilon}{\mu}}\ \hat{z} \times \vec{e}_t = \frac{1}{Z_{TEM}}\hat{z} \times \vec{e}_t$$

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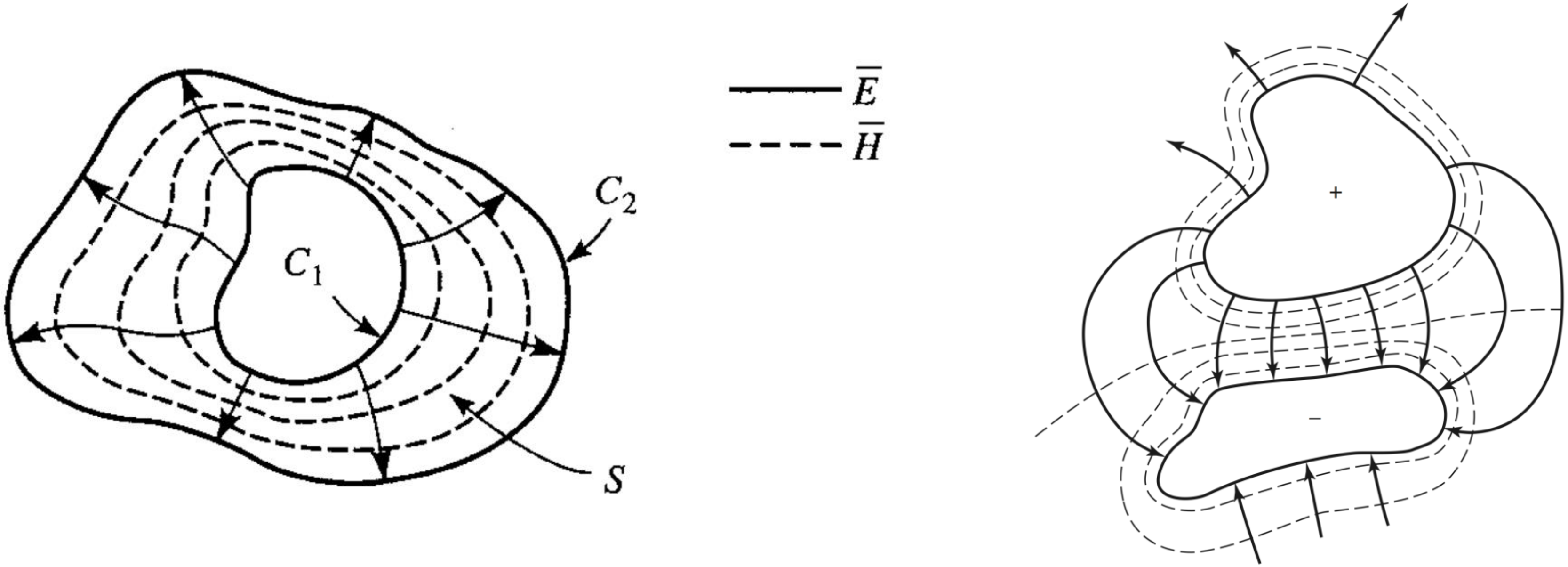
Transverse Electric Magnetic mode in waveguides

Solution **For a given** A_z $$\vec{H} = \frac{1}{\mu}\nabla_t \times (\hat{z}A_z)\, e^{-j\omega\sqrt{\mu\epsilon}z} \qquad \vec{E} = -\frac{1}{\sqrt{\mu\epsilon}}\,\nabla_t A_z\; e^{-j\omega\sqrt{\mu\epsilon}z}$$

3. **TEM waves are possible only if there are at least two conductors.**



4. **The plane wave is a TEM wave of two infinitely large plates separated to infinity**

5. **Electrostatic problem with boundary conditions**

$$\vec{e}_t \;\longrightarrow\; \vec{h}_t = \frac{1}{Z_{TEM}}\hat{z}\times\vec{e}_t \;\longrightarrow\; \begin{aligned}\vec{E} &= \vec{e}_t\; e^{-j\omega\sqrt{\mu\epsilon}z}\\ \vec{H} &= \vec{h}_t\; e^{-j\omega\sqrt{\mu\epsilon}z}\end{aligned}$$

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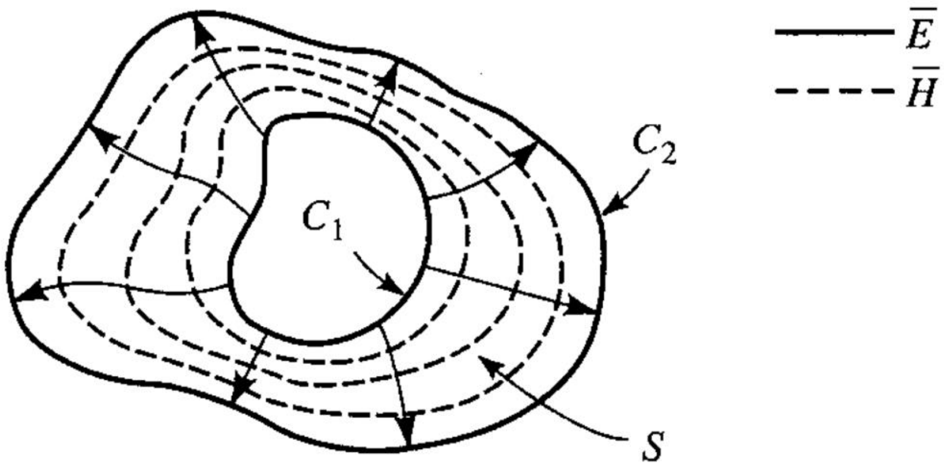
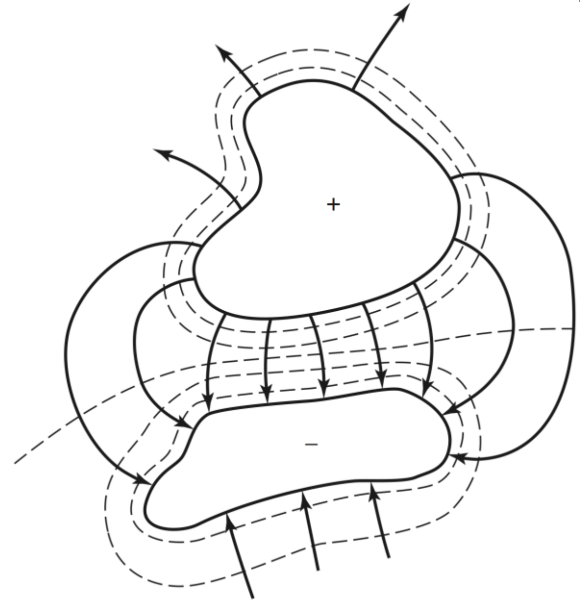
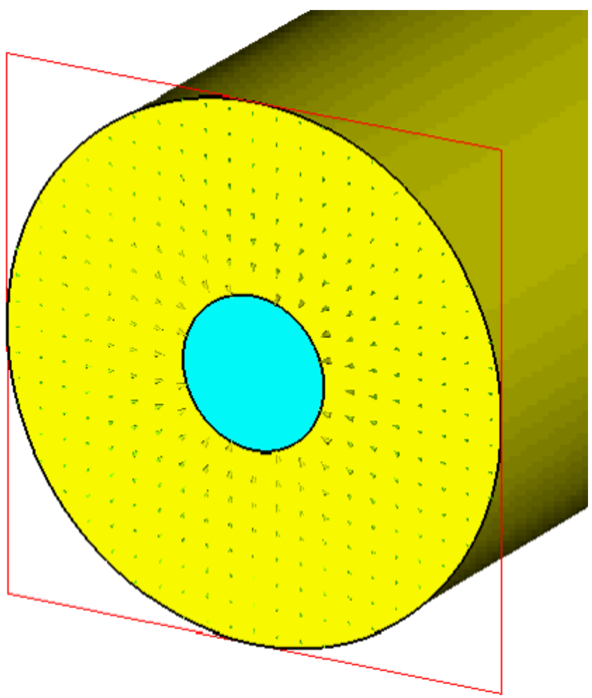
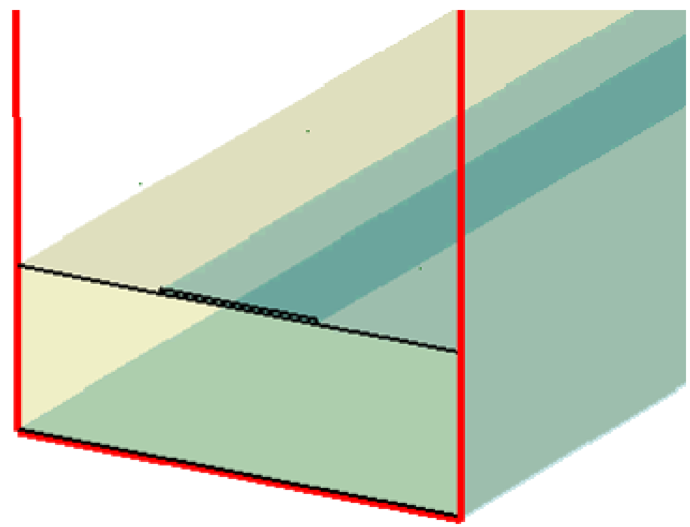
Common TEM waveguides









Animations by S. Pisa

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General solution for fields in cylindrical waveguide

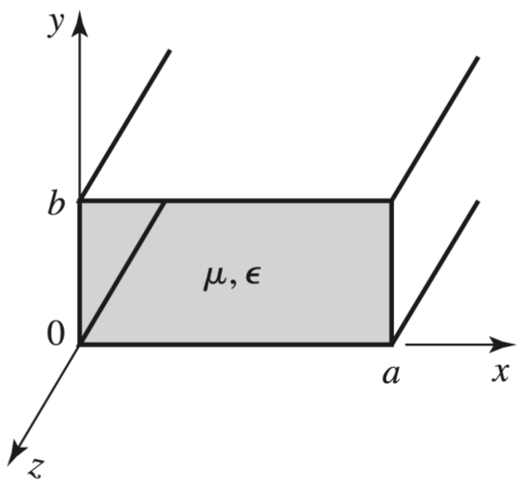
1. **Write the Helmotz equations for potentials**

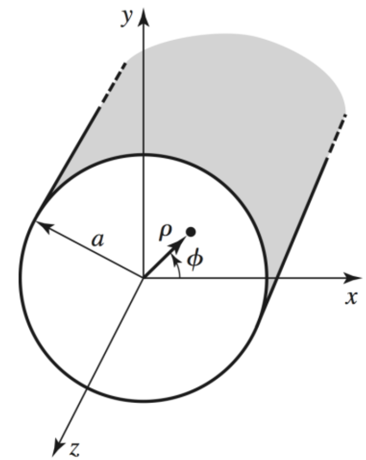
TM waves $\nabla_t^2 A_z + k_t^2 A_z = 0$

TE waves $\nabla_t^2 F_z + k_t^2 F_z = 0$

$$k_t^2 = k^2 - \beta^2 = \omega^2 \mu \epsilon - \beta^2$$

$$\epsilon = \epsilon_r \epsilon_0 \left(1 - j \tan \delta\right)$$





Cartesian coordinates

$$\nabla_t^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

Cylindrical coordinates

$$\nabla_t^2 = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2}$$

2.

$$A_z(x, y) = X(x) Y(y)$$

$$A_z(\rho, \phi) = R(\rho) \Phi(\phi)$$

Separation of variables

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General solution for fields in cylindrical waveguide

3. Eigenvalue problem: Eigenvalues + Eigen-function

TM $\nabla_t^2 A_z + k_t^2 A_z = 0$ $\qquad k_t \qquad A_z,\ F_z$

TE $\nabla_t^2 F_z + k_t^2 F_z = 0$

4. Compute the fields and apply the boundary conditions

$$\vec{e} = \vec{e}_t + \hat{z}\ e_z$$

$$\vec{h} = \vec{h}_t + \hat{z}\ h_z$$

$\longrightarrow$

$$\vec{e}_{m,n} \qquad \vec{h}_{m,n}$$

$$\beta_{m,n} = \sqrt{\omega^2\mu\epsilon - k_t^2{}_{(m,n)}}$$

Mode (m,n)

5.

It can be complex

$$\vec{E} = \sum_{m,n} a_{m,n}\ \vec{e}_{m,n}\ e^{-j\beta_{m,n} z} \qquad \vec{H} = \sum_{m,n} b_{m,n}\ \vec{h}_{m,n}\ e^{-j\beta_{m,n} z}$$

It depends on the sources

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Rectangular waveguides

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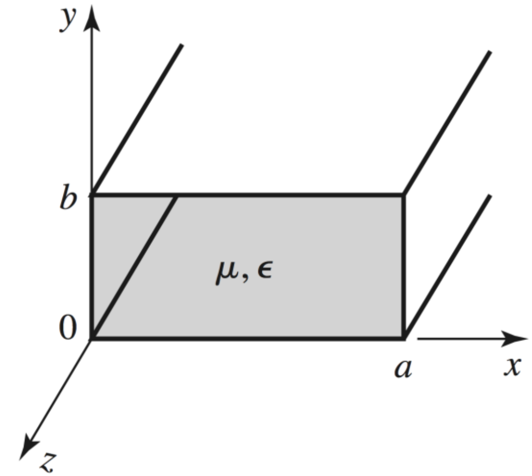
Rectangular waveguides: TE mode

Example

$$F_z = X(x)Y(y)$$

Write the Helmotz equation



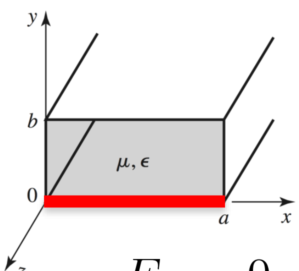
$$-k_x^2 - k_y^2 + k_t^2 = 0$$

constraint condition

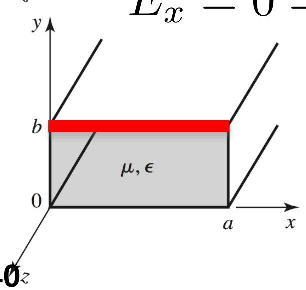
$$X(x) =$$

$$Y(y) =$$

$$e_x = -\frac{1}{\epsilon}\frac{\partial F_z}{\partial y} = -\frac{1}{\epsilon} X Y'$$



$$E_x = 0 \Longrightarrow e_x = 0$$



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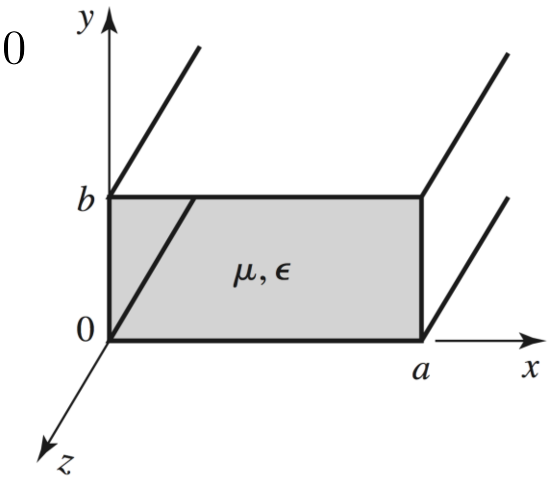
Rectangular waveguides: TE mode

Example



$$F_z = X(x)Y(y) \qquad \nabla_t^2 F_z + k_t^2 F_z = YX'' + XY'' + k_t^2 XY = 0$$

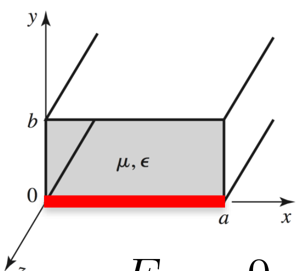
$$\frac{X''}{X} + \frac{Y''}{Y} + k_t^2 = 0 \qquad -k_x^2 - k_y^2 + k_t^2 = 0$$

constraint condition

$$\frac{X''}{X} = -k_x^2 \longrightarrow X(x) = C_1 \cos(k_x x) + D_1 \sin(k_x x)$$

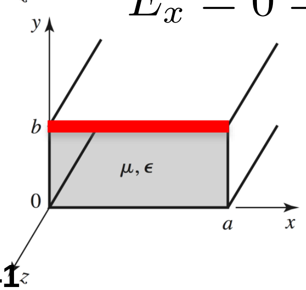
$$\frac{Y''}{Y} = -k_y^2 \longrightarrow Y(y) = C_2 \cos(k_y y) + D_2 \sin(k_y y)$$

$$e_x = -\frac{1}{\epsilon}\frac{\partial F_z}{\partial y} = -\frac{1}{\epsilon} XY' = -\frac{k_y}{\epsilon}\left[C_1 \cos(k_x x) + D_1 \sin(k_x x)\right]\left[-C_2 \sin(k_y y) + D_2 \cos(k_y y)\right]$$



$$e_x(0 \le x \le a, y = 0) = \ldots [-C_2 \cdot 0 + D_2 \cdot 1] = 0 \iff D_2 = 0$$

$$E_x = 0 \Longrightarrow e_x = 0$$



$$e_x(0 \le x \le a, y = b) = \ldots [-C_2 \sin(k_y b)] = 0 \iff k_y b = n\pi$$

$$n = 0, 1, 2, \ldots$$

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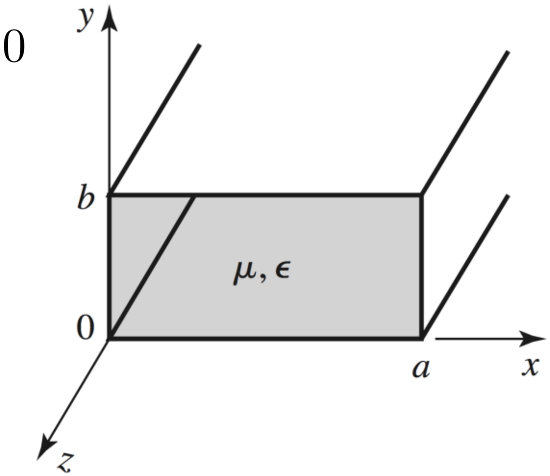
Rectangular waveguides: TE mode

Example

$$F_z = X(x)Y(y) \qquad \nabla_t^2 F_z + k_t^2 F_z = YX'' + XY'' + k_t^2 XY = 0$$



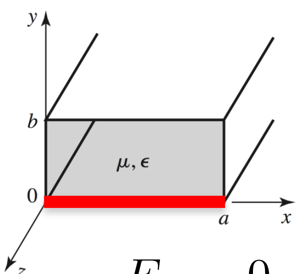
$$\frac{X''}{X} + \frac{Y''}{Y} + k_t^2 = 0 \qquad -k_x^2 - k_y^2 + k_t^2 = 0$$

constraint condition

$$\frac{X''}{X} = -k_x^2 \longrightarrow X(x) = C_1 \cos(k_x x) + D_1 \sin(k_x x)$$

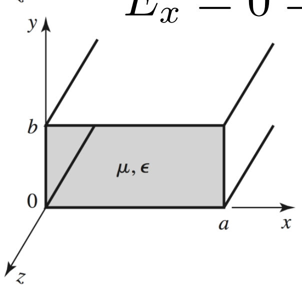
$$\frac{Y''}{Y} = -k_y^2 \longrightarrow Y(y) = C_2 \cos(k_y y) + D_2 \sin(k_y y)$$

$$e_x = -\frac{1}{\epsilon}\frac{\partial F_z}{\partial y} = -\frac{1}{\epsilon} XY' = -\frac{k_y}{\epsilon}\left[C_1 \cos(k_x x) + D_1 \sin(k_x x)\right]\left[-C_2 \sin(k_y y) + D_2 \cos(k_y y)\right]$$



$$e_x(0 \le x \le a, y = 0) = \ldots\left[-C_2 \cdot 0 + D_2 \cdot 1\right] = 0 \iff D_2 = 0$$

$$E_x = 0 \Longrightarrow e_x = 0$$



$$e_x(0 \le x \le a, y = b) = \ldots\left[-C_2 \sin(k_y b)\right] = 0 \iff \begin{array}{c} k_y b = n\pi \\ n = 0, 1, 2, \ldots \end{array}$$

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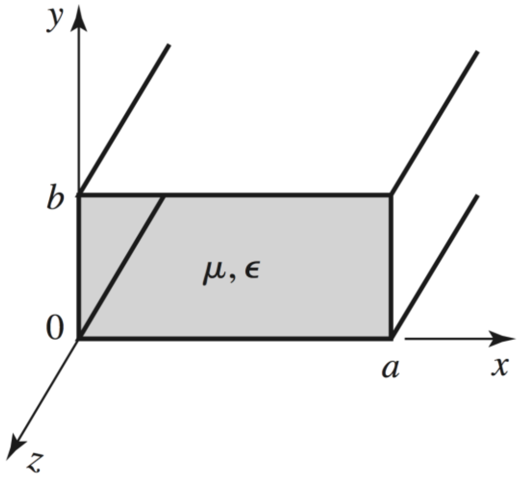
Eigenvalues and cut-off frequencies (TE mode, rect. WG)

$$k_t^2 = k_x^2 + k_y^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 = \omega^2\mu\epsilon - \beta^2$$

constraint condition



$$\vec{H} = \sum_{m,n} b_{m,n}\, \vec{h}_{m,n}\, e^{-j\beta_{m,n} z}$$

$$\beta_{m,n} = \sqrt{\omega^2\mu\epsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$

$$\vec{E} = \sum_{m,n} a_{m,n}\, \vec{e}_{m,n}\, e^{-j\beta_{m,n} z}$$

Cut-off frequencies f_c such that $\beta_{m,n} = 0$

$$(f_c)_{m,n} = \frac{1}{2\pi\sqrt{\mu\epsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} \qquad \begin{array}{c} m,\ n = 0, 1, 2, \ldots \\ m = n \neq 0 \end{array}$$

$f < (f_c)_{m,n}$ **mode m, n is attenuated exponentially (evanescent mode)**

$f > (f_c)_{m,n}$ **mode m, n is propagating with no attenuation**

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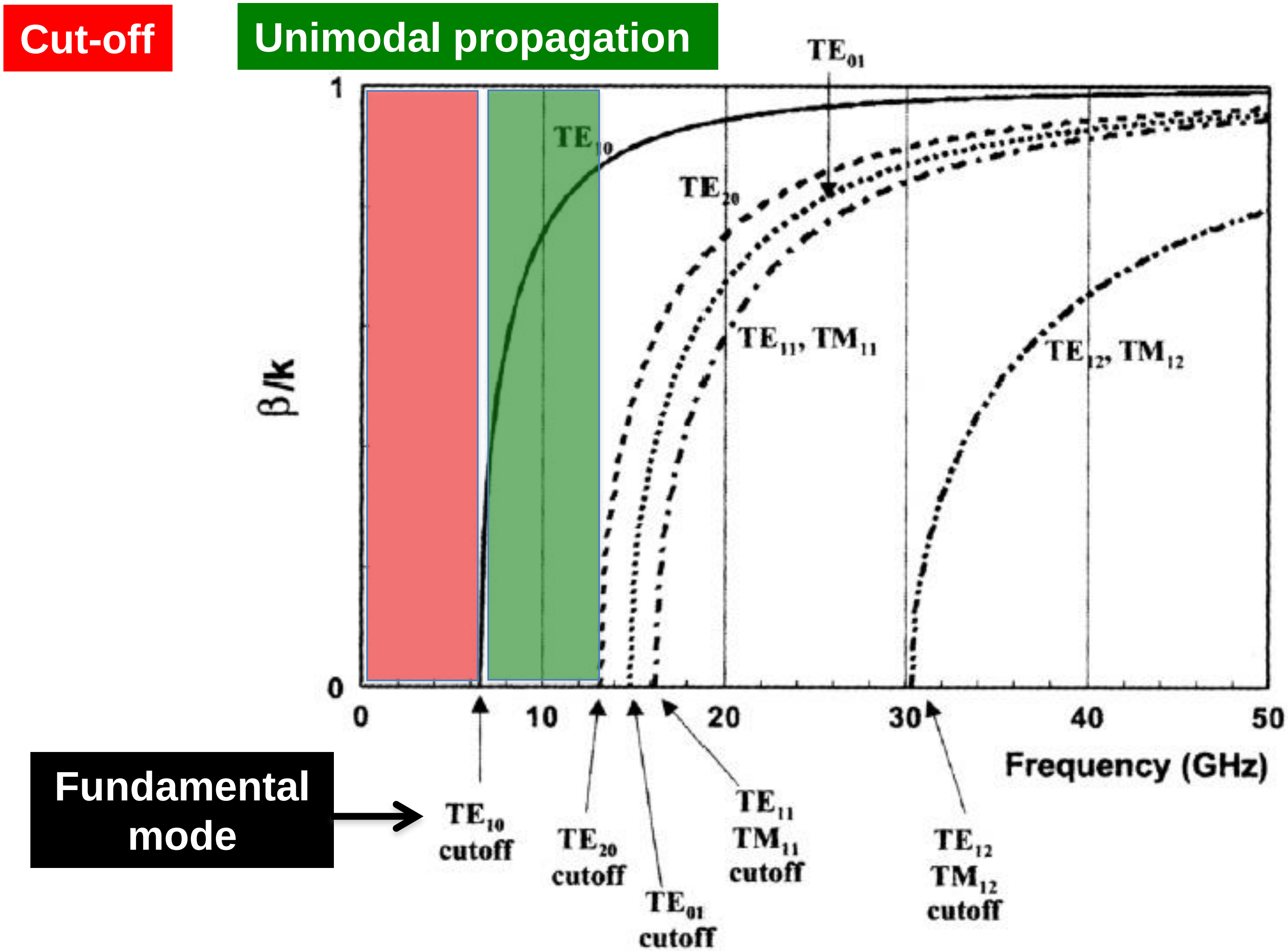
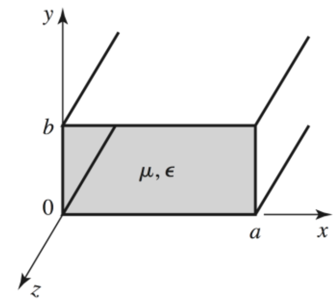
Waveguide dispersion curve

Courtesy of S. Pisa





Same curve for TE and TM mode, but n=0 or m=0 is possible only for TE modes.

In any metallic waveguide the fundamental mode is TE.

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Single mode operation of a rectangular waveguide

Exercise

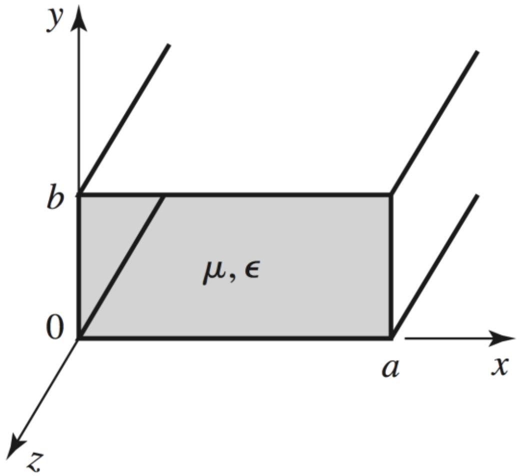
1. **Find the smallest ratio a/b allowing the largest bandwidth of single mode operation**

2. **State the largest bandwidth of single mode operation**

3. **Defining the single mode bandwidth as**

$$1.25\ (f_c)_1 < f < 0.95\ (f_c)_2$$

Find the single mode BW for WR-90 waveguide (a=22.86mm and b=10.16 mm)



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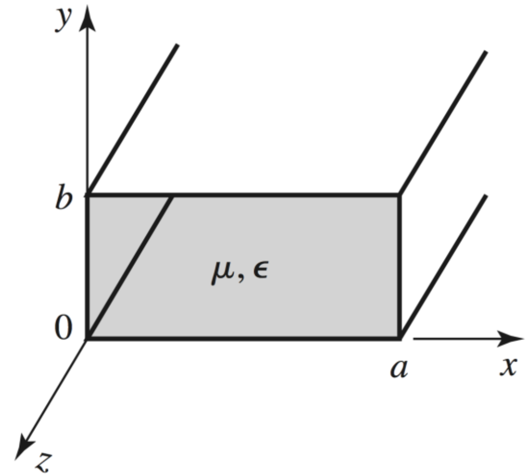
Single mode operation of a rectangular waveguide

Exercise



1. **Find the smallest ratio a/b allowing the largest bandwidth of single mode operation**

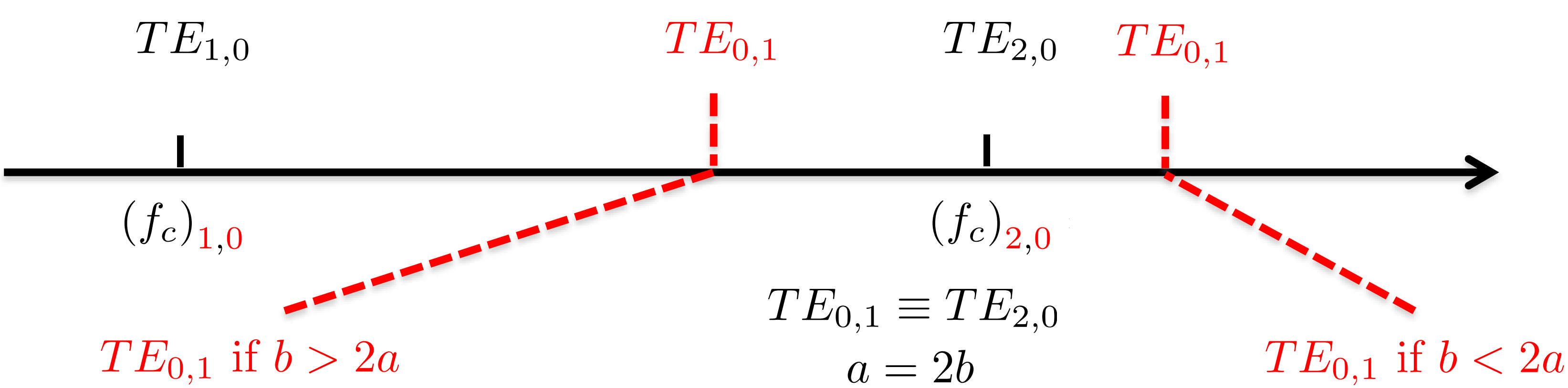
2. **State the largest bandwidth of single mode operation**

3. **Defining the single mode bandwidth as**

$$1.25\ (f_c)_1 < f < 0.95\ (f_c)_2$$

Find the single mode BW for WR-90 waveguide (a=22.86mm and b=10.16 mm)

$$(f_c)_{1,0} = \frac{1}{2\sqrt{\mu\epsilon}a} \qquad (f_c)_{2,0} = \frac{1}{\sqrt{\mu\epsilon}a} = 2\,(f_c)_{2,0} \qquad (f_c)_{0,1} = \frac{1}{\sqrt{\mu\epsilon}b}$$



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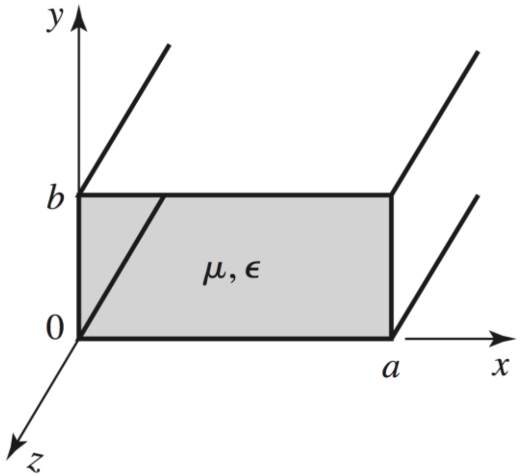
Single mode operation of a rectangular waveguide

Exercise

1. Find the smallest ratio a/b allowing the largest bandwidth of single mode operation

2. State the largest bandwidth of single mode operation

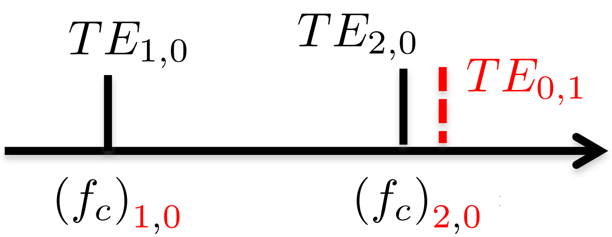


3. Defining the single mode bandwidth as

$$1.25\ (f_c)_1 < f < 0.95\ (f_c)_2$$

Find the single mode BW for WR-90 waveguide (a=22.86mm and b=10.16 mm)

a=0.9 inches b=0.4 inches



$$(f_c)_{1,0} = c/2a = 3\ 10^8/(2\ 22.86\ 10^{-3}) = 6.56\ \text{GHz}$$

$$(f_c)_{2,0} = c/a = 3\ 10^8/(22.86\ 10^{-3}) = 13.12\ \text{GHz}$$

Single mode BW $6.56\ 1.25 = 8.2\ \text{GHz} < f < 12.4\ \text{GHz} = 13.12\ 0.95$

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Eigenfunctions and mode pattern (TE mode, rect. WG)

$$E_x^{+,(m,n)} = a_{m,n}\frac{k_y}{\epsilon}\cos(k_x x)\sin(k_y y)e^{-j\beta z}$$

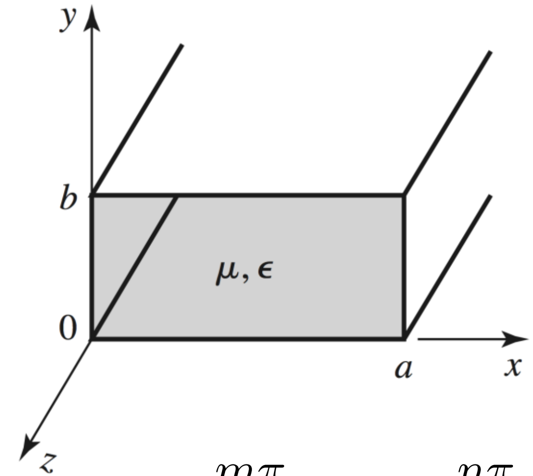
$$E_y^{+,(m,n)} = a_{m,n}\frac{k_y}{\epsilon}\sin(k_x x)\cos(k_y y)e^{-j\beta z}$$

$$E_z^{+,(m,n)} = 0$$

$$H_x^{+,(m,n)} = a_{m,n}\frac{k_x\beta}{\omega\mu\epsilon}\sin(k_x x)\cos(k_y y)e^{-j\beta z}$$

$$H_y^{+,(m,n)} = a_{m,n}\frac{k_y\beta}{\omega\mu\epsilon}\cos(k_x x)\sin(k_y y)e^{-j\beta z}$$

$$H_z^{+,(m,n)} = -ja_{m,n}\frac{k_t^2}{\omega\mu\epsilon}\cos(k_x x)\cos(k_y y)e^{-j\beta z}$$



$$k_x = \frac{m\pi}{a} \quad k_y = \frac{n\pi}{b}$$

$$\beta = \sqrt{\omega^2\mu\epsilon - k_x^2 - k_y^2}$$

$$TE_{m,n}^{+z}$$

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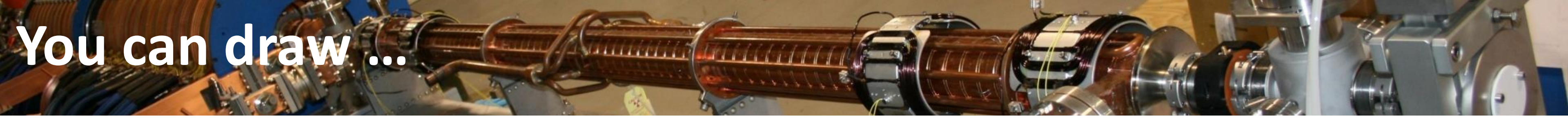
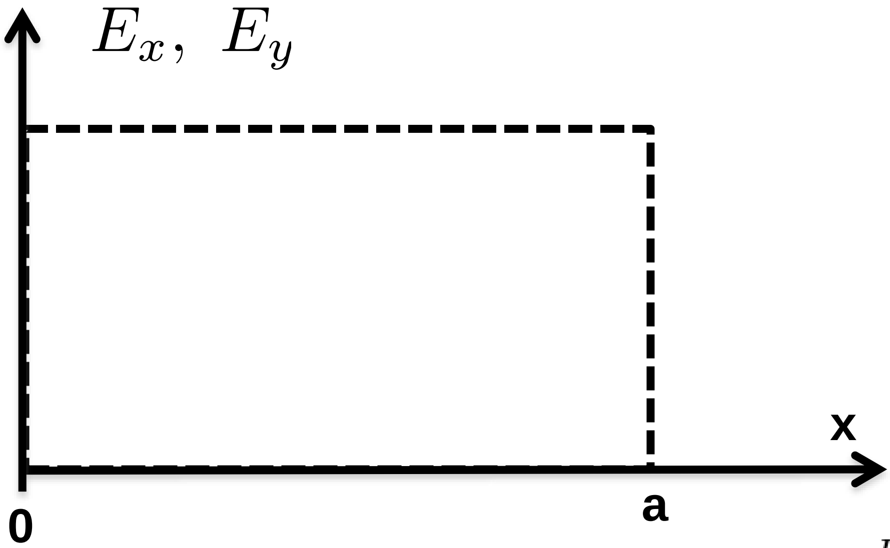
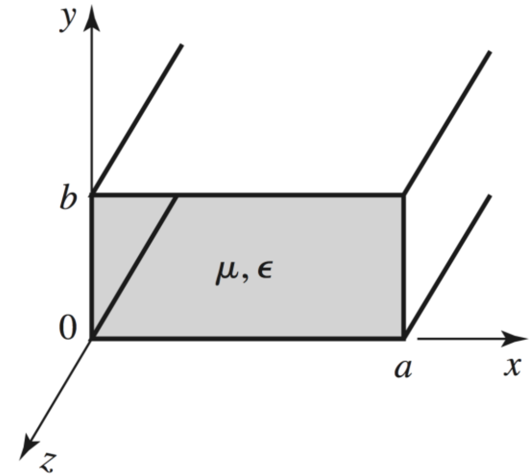
You can draw …





$$TE_{1,0}$$



$$E_x^{+,(m,n)} = a_{m,n}\frac{k_y}{\epsilon}\cos(k_x x)\sin(k_y y)e^{-j\beta z}$$

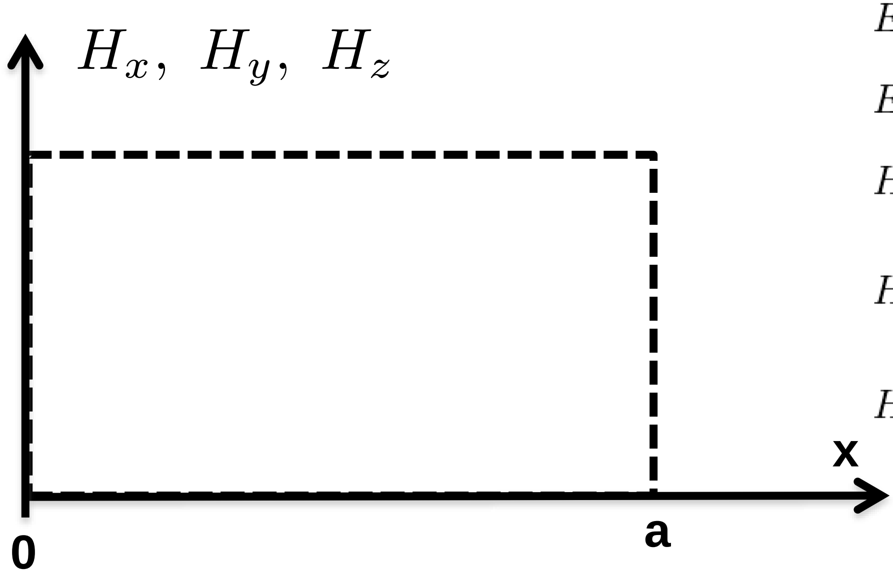
$$E_y^{+,(m,n)} = a_{m,n}\frac{k_y}{\epsilon}\sin(k_x x)\cos(k_y y)e^{-j\beta z}$$

$$E_z^{+,(m,n)} = 0$$

$$H_x^{+,(m,n)} = a_{m,n}\frac{k_x\beta}{\omega\mu\epsilon}\sin(k_x x)\cos(k_y y)e^{-j\beta z}$$

$$H_y^{+,(m,n)} = a_{m,n}\frac{k_y\beta}{\omega\mu\epsilon}\cos(k_x x)\sin(k_y y)e^{-j\beta z}$$

$$H_z^{+,(m,n)} = -ja_{m,n}\frac{k_t^2}{\omega\mu\epsilon}\cos(k_x x)\cos(k_y y)e^{-j\beta z}$$



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Eigenfunctions and mode pattern (TE mode, rect. WG)

$$E_x^{+,(m,n)} = a_{m,n} \frac{k_y}{\epsilon} \cos(k_x x) \sin(k_y y) e^{-j\beta z}$$

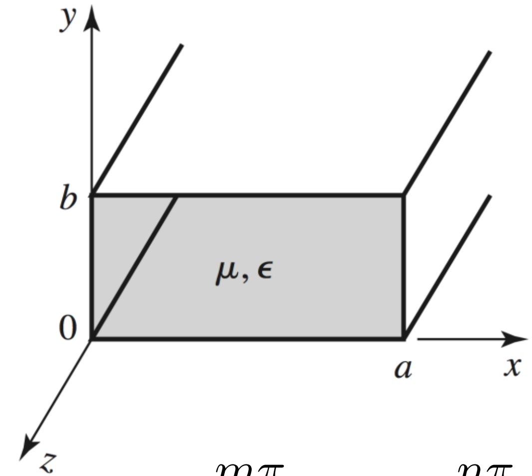
$$E_y^{+,(m,n)} = a_{m,n} \frac{k_y}{\epsilon} \sin(k_x x) \cos(k_y y) e^{-j\beta z}$$

$$E_z^{+,(m,n)} = 0$$

$$H_x^{+,(m,n)} = a_{m,n} \frac{k_x \beta}{\omega\mu\epsilon} \sin(k_x x) \cos(k_y y) e^{-j\beta z}$$

$$H_y^{+,(m,n)} = a_{m,n} \frac{k_y \beta}{\omega\mu\epsilon} \cos(k_x x) \sin(k_y y) e^{-j\beta z}$$

$$H_z^{+,(m,n)} = -j a_{m,n} \frac{k_t^2}{\omega\mu\epsilon} \cos(k_x x) \cos(k_y y) e^{-j\beta z}$$



$$k_x = \frac{m\pi}{a} \qquad k_y = \frac{n\pi}{b}$$

$$\beta = \sqrt{\omega^2\mu\epsilon - k_x^2 - k_y^2}$$

$$TE_{m,n}^{+z}$$

$$TE_{1,0}$$

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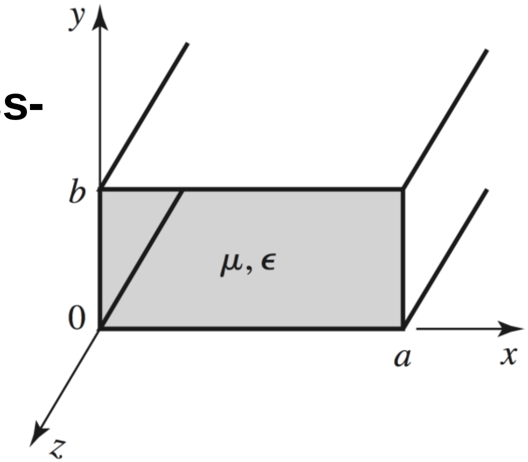
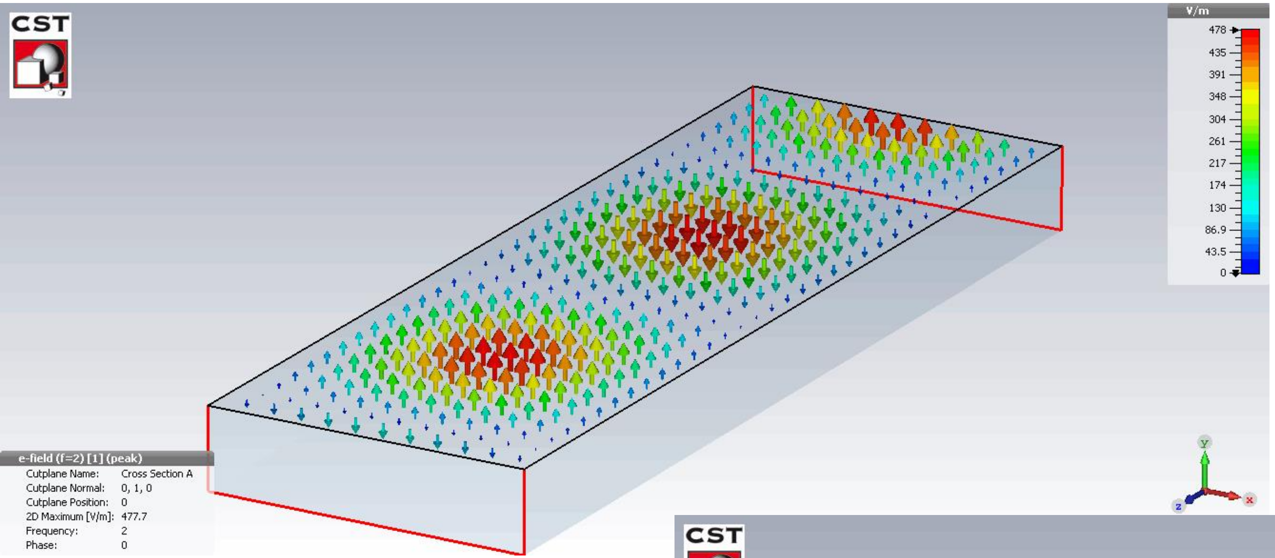
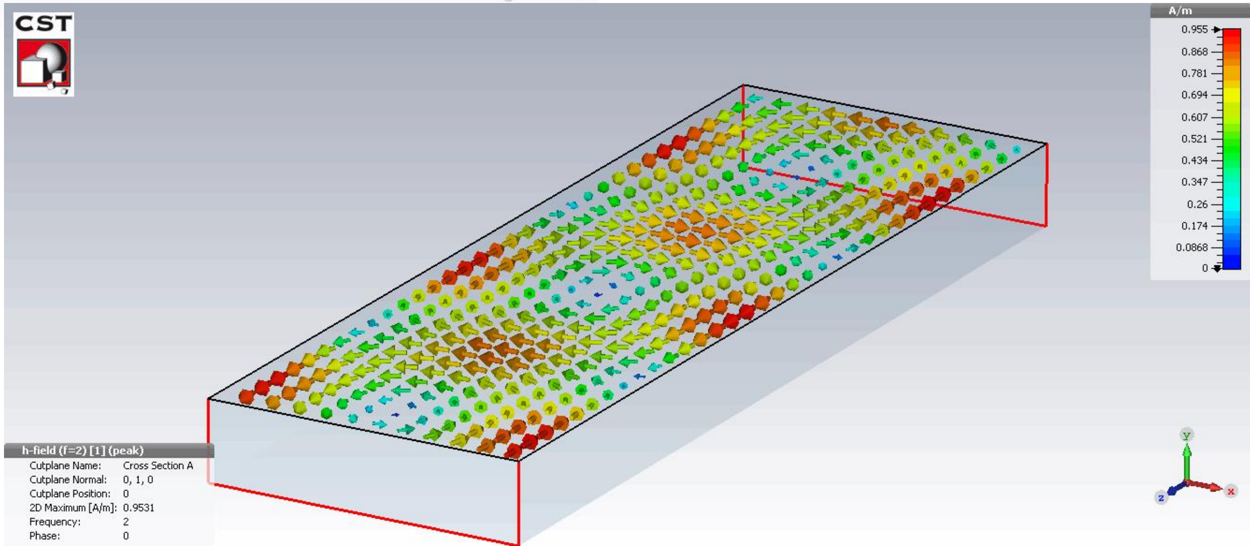
Field pattern (TE10 mode, rect. WG)

$TE_{m,n}^{+z}$ **m (n) is the number of half periods (or maxima/minima) along the x (y) axis in the cross-section.**



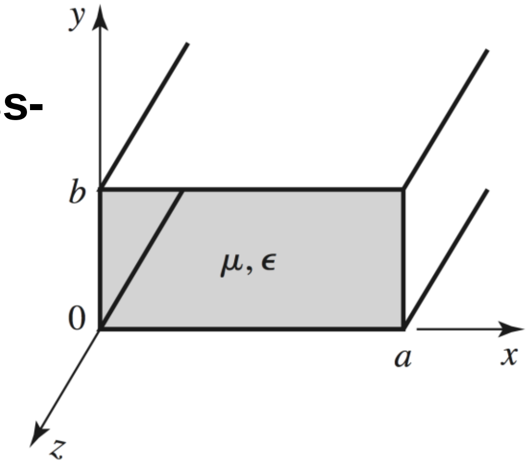


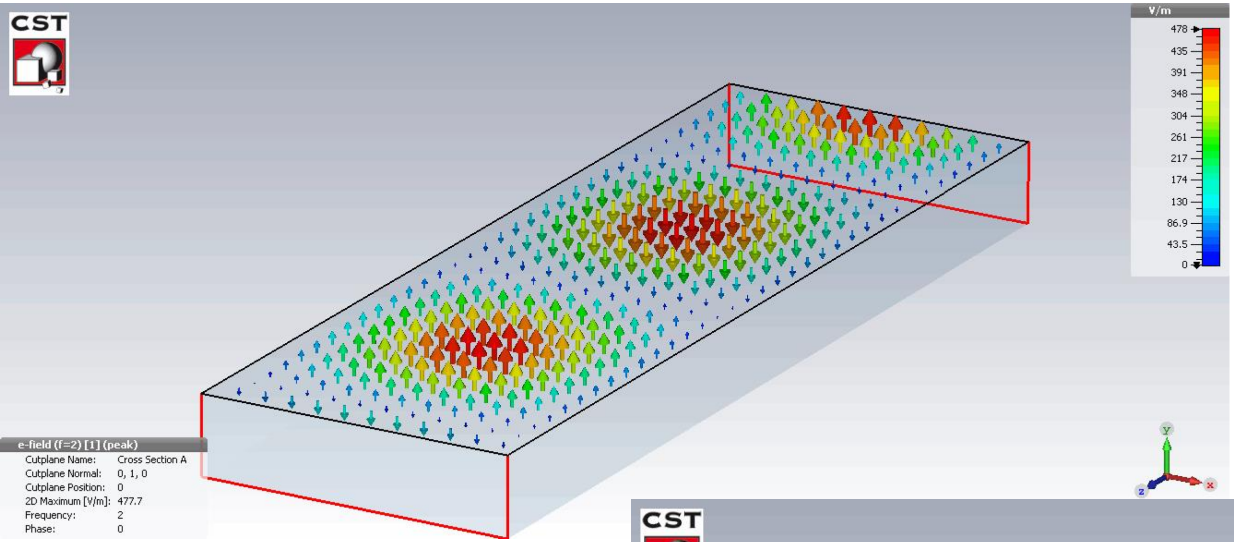


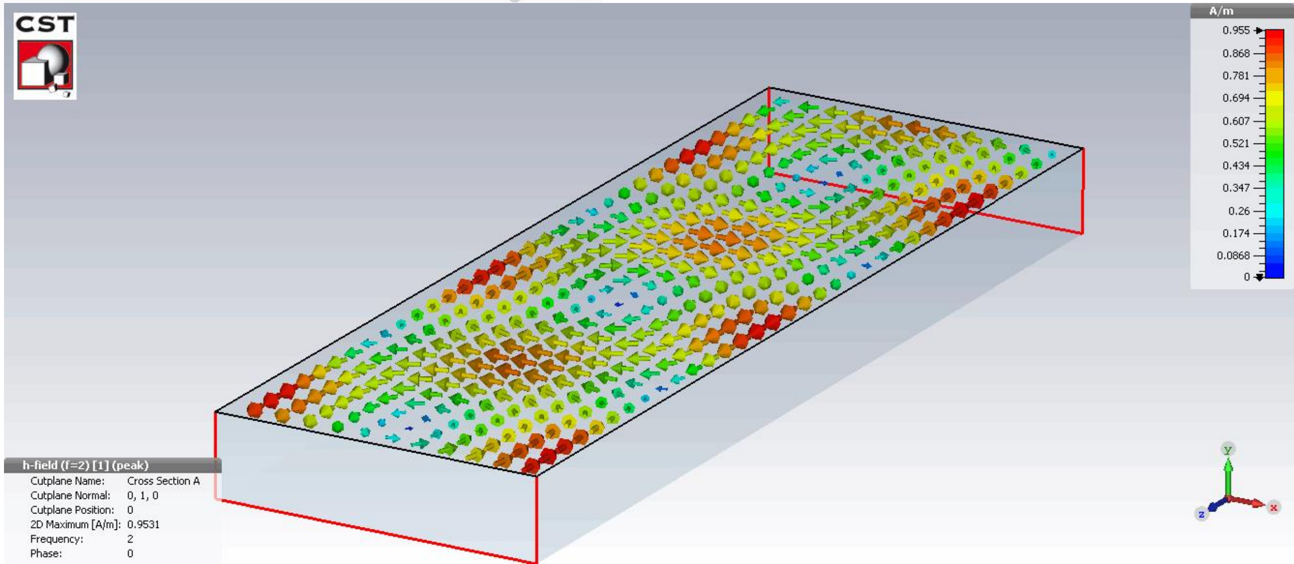
Simulations by L. Ficcadenti

Field pattern (TE10 mode, rect. WG)

$TE^{+z}_{m,n}$ **m (n) is the number of half periods (or maxima/minima) along the x (y) axis in the cross-section.**





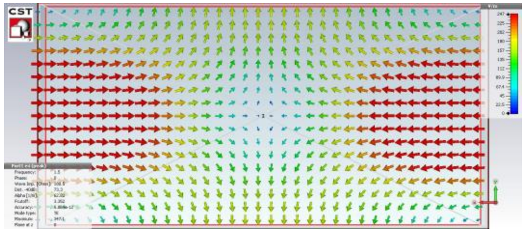


Animations by L. Ficcadenti

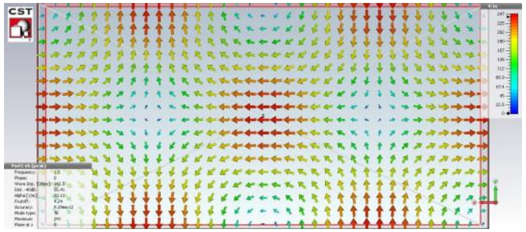
Field pattern at the cross section

$TE_{m,n}^{+z}$ **m (n) is the number of half periods (or maxima/minima) along the x (y) axis in the cross-section.**

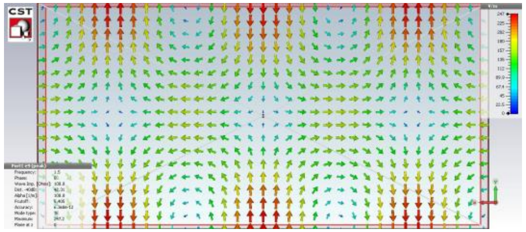
TE??



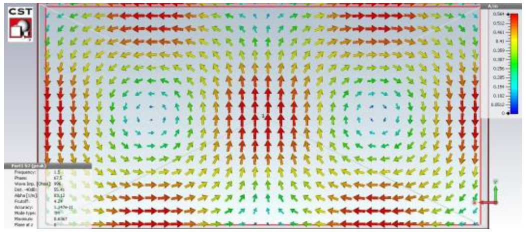
TE??

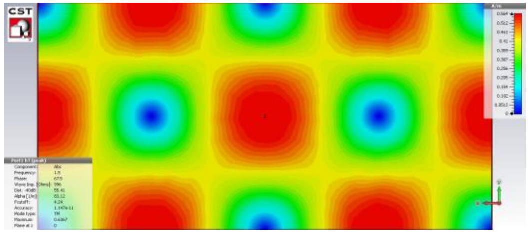


TE??

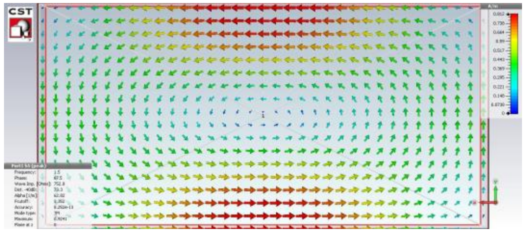


TM??

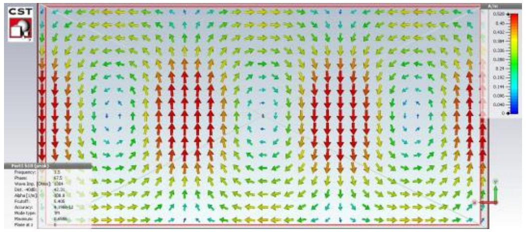




TM??



TM??



Simulations by L. Ficcadenti

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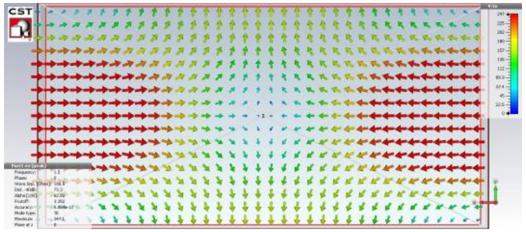
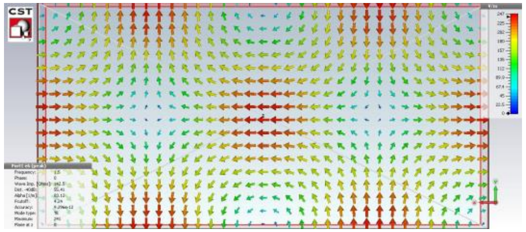
Field pattern at the cross section

$TE_{m,n}^{+z}$ **m (n) is the number of half periods (or maxima/minima) along the x (y) axis in the cross-section.**

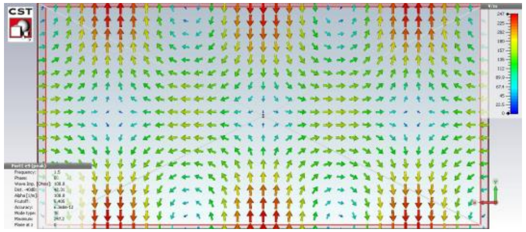
TE11



TE21



TE31



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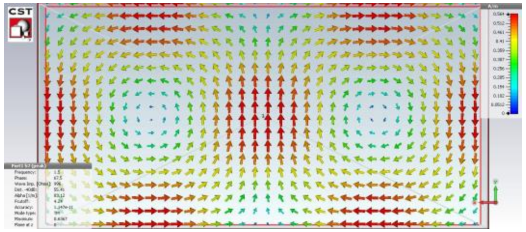

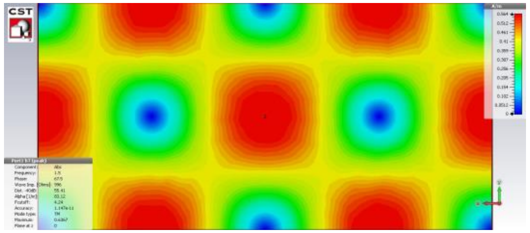

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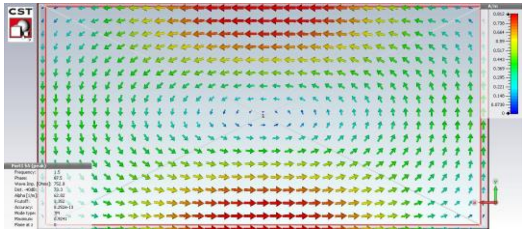

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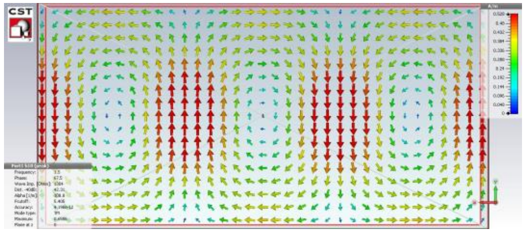

Simulations by L. Ficcadenti

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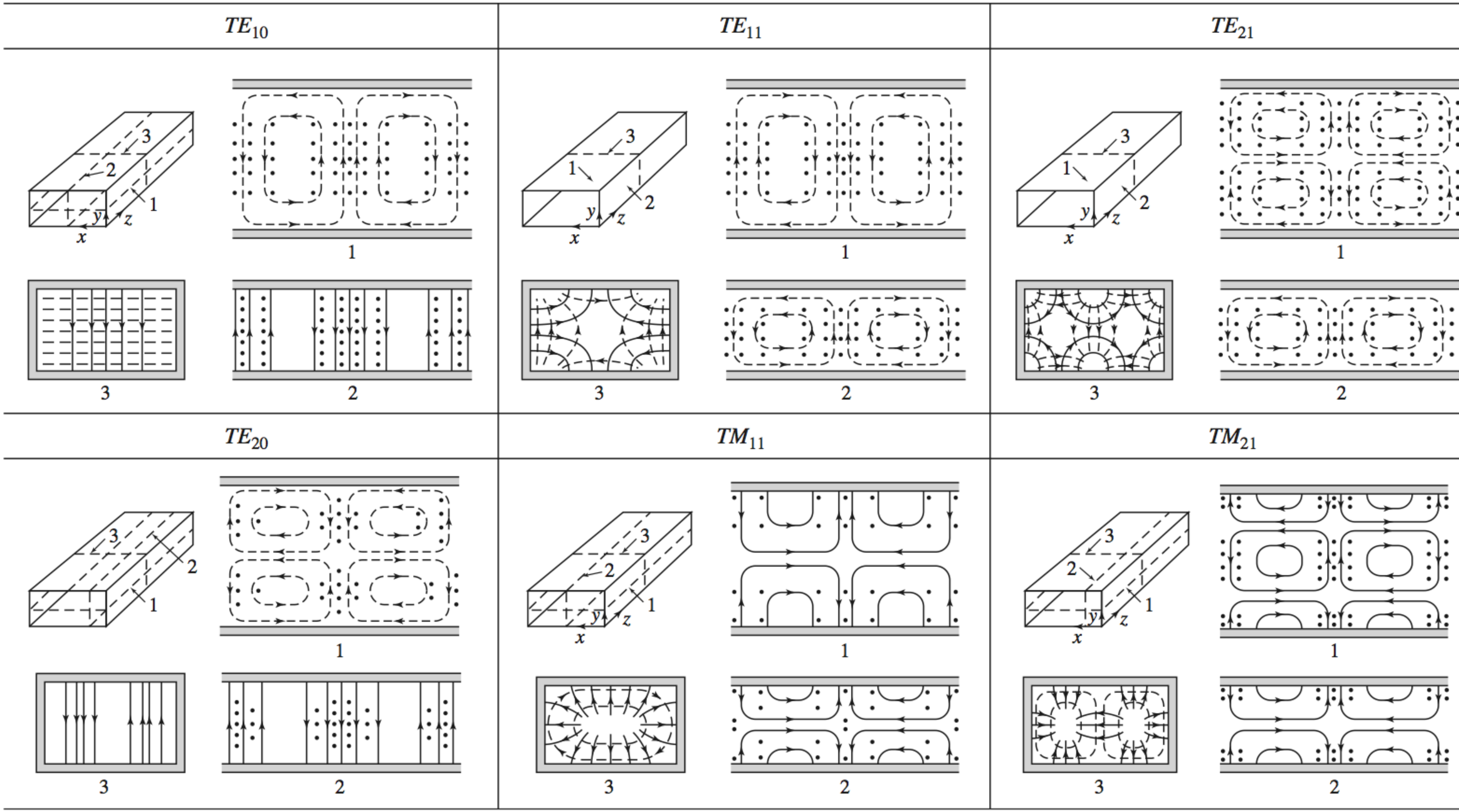
Field pattern (TE mode, rect. WG)

$TE_{m,n}^{+z}$ **m (n) is the number of half periods (or maxima/minima) along the x (y) axis in the cross-section.**



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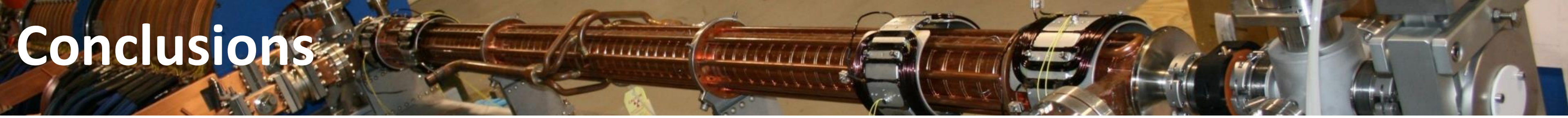
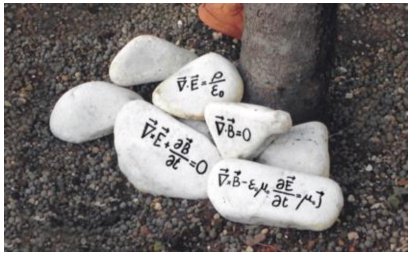


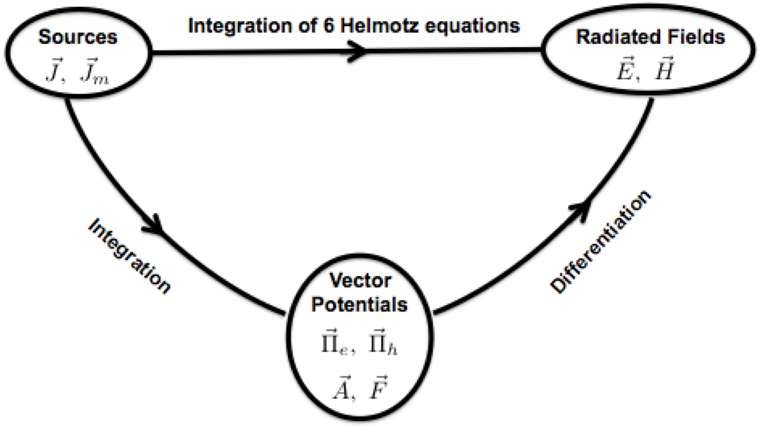
Conclusions

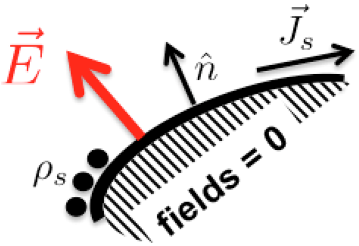
… The universe is written in the mathematical language and the letters are triangles, circles and other geometrical figures …

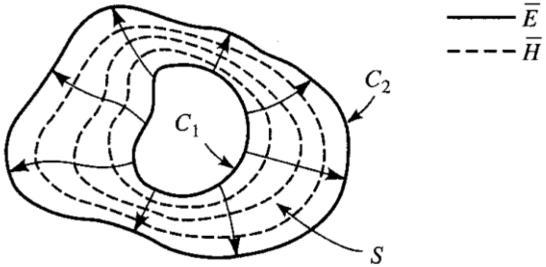
$\nabla \times$

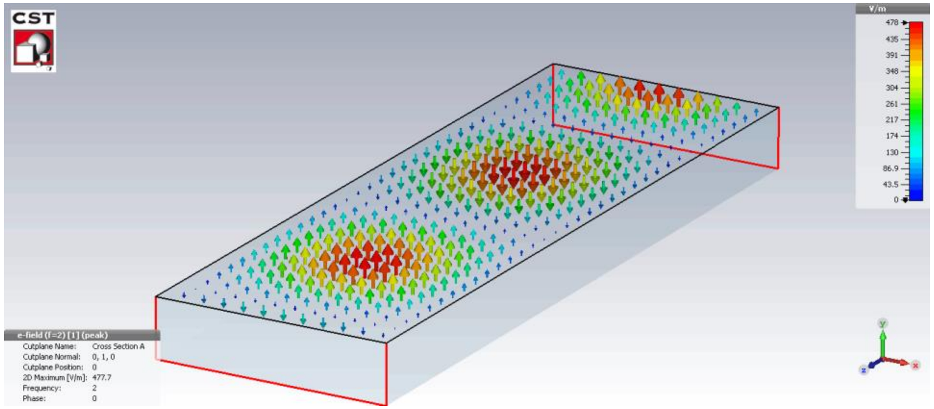
$\nabla \cdot$











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