Joint University Accelerator School

Examination of Transverse Beam Dynamics (solutions)

JUAS 2017

9th February 2017

1 Exercise: FODO cell [5 points]

We want to design a storage ring based on a periodic FODO lattice, for electrons with nominal momentum 1 GeV/c. The ring is composed by 32 cells, and its circumference is 160 m long. Each cell features 2 dipole magnets, whose total filling factor is 60%.

1. Characterise the dipole magnet. What are its: length? bending angle? bending radius? magnetic field? [1 pt]

   SOLUTION: Each bending magnets has $L_{\text{dipole}} = 1.5$ m; $\theta = 5.625$ degrees $= \pi/32 = 0.0982$ rad; $\rho = 15.28$ m; $B = 0.218$ T.

2. What values of the phase advance are permitted in order to have $\beta_{\max} \leq 10$ m? [1 pt]

   SOLUTION: There are at least two ways to answer this question. Let's write the formula for $\beta_{\max} \leq 10$:

$$\beta_{\max} = \frac{L\left(1 + \sin\frac{\mu}{2}\right)}{\sin\mu} \leq 10 \text{ [m]},$$

   with $L = 5$ m, length of each cell. Then:

$$\beta_{\max} = \frac{2 \text{ [m]} \left(1 + \sin\frac{\mu}{2}\right)}{\sin\mu} \leq 10 \text{ [m]},$$

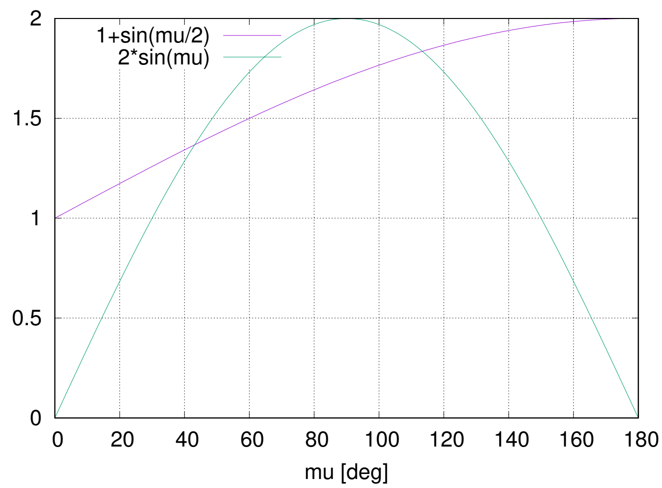
   that is, $\beta_{\max} \leq 10$ m when

$$\frac{1 + \sin\frac{\mu}{2}}{\sin\mu} \leq 2.$$

   In a FODO cell μ is always $0 \leq \mu \leq \pi$. So, we can rewrite this inequality as:

$$1 + \sin\frac{\mu}{2} \leq 2\sin\mu. \tag{1}$$

   Let's draw the two curves corresponding to the LHS and to the RHS of Eqn. (1):



The interval where μsatisfies the inequality in Eqn. (1) is roughly:

$$\sim 47 \text{ deg } (\sim 0.82 \text{ rad}) < \mu_x <\sim 112 \text{ deg } (\sim 1.92 \text{ rad})$$

which is the answer to the question. Alternatively, one can identify the same interval by going to slide 72 of the lectures, and picking the range where $\beta_{\max}/L \leq 2$.

3. The quadrupoles are 0.5 m long. What is the quadrupole gradient required to achieve the maximum μ of point 2? [0.5 pt]

   SOLUTION: $G_{\text{quad}} = k_{\text{quad}} \cdot \frac{P}{q} = \frac{1}{f_{\text{quad}} L_{\text{quad}}} \cdot \frac{P}{q} = 4.461$ T/m. With $f_{\text{quad}} = \frac{L_{\text{cell}}}{4 \sin \frac{\mu}{2}}$.

4. For the gradient computed in the previous point, what are the values of the maximum and of the minimum dispersion? [0.5 pt]

   SOLUTION: From $D^{\pm} = \frac{L\theta_{\text{cell}}\left(1 \pm \frac{1}{2} \sin \frac{\mu}{2}\right)}{4 \sin^2 \frac{\mu}{2}}$ one obtains: $D_{\max} = 0.58$ m; $D_{\min} = 0.26$ m.

5. Estimate the momentum compaction, α, and the length variation of the closed orbit for a particle with $\frac{\Delta P}{P} = -1\%$. [2 pt]

   SOLUTION: Recall the definition of momentum compaction factor, α, as the coefficient of proportionality between the momentum error and the associated circumference change:

$$\frac{\Delta C}{C} = \alpha \frac{\Delta P}{P},$$

   where C is the circumference for the on-energy particle, and ΔC is the circumference change for a particle off-momentum by $\frac{\Delta P}{P}$. The momentum compaction α can be computed using the following formula:

$$\alpha = \frac{1}{C} \oint_C \frac{D\left(s\right)}{\rho\left(s\right)} \mathrm{d}s.$$

   We compute:

$$D_{\text{average}} = \frac{D_{\max} + D_{\min}}{2} = 0.1756 \text{ m}.$$

   The momentum compaction, α, is then:

$$\alpha = \frac{1}{C} \oint_C \frac{D\left(s\right)}{\rho\left(s\right)} \mathrm{d}s = \frac{1}{C} \sum_{\text{all cells}} \frac{2 \cdot D_{\text{average}}}{\rho} L_{\text{dipole}} = \frac{1}{C} 32 \cdot 2 \cdot D_{\text{average}} \theta = 2\pi \frac{D_{\text{average}}}{C} = 0.00689.$$

   Then for a particle with $\frac{\Delta P}{P} = -1\%$,

$$\Delta C = \alpha C \frac{\Delta P}{P} = -1.103 \text{ cm}.$$

2 Exercise: Dispersion in a transfer line [3 points]

A misaligned quadrupole introduces dispersion. Consider a quadrupole, with gradient $G = 10$ T/m and length 1 m, installed in a region where the nominal momentum is 7 GeV/c. If the quadrupole is installed with a horizontal offset of 1 mm, what are the values of dispersion and dispersion prime 10 m downstream?

SOLUTION: The average kick given by a misaligned quadrupole is:

$$\Delta x' = K L_{\text{quad}} \Delta x$$

where

$$K = \frac{G}{P/q} = 0.428 \text{ m}^{-2}.$$

Then

$$\Delta x' = 0.000428 \text{ radians.}$$

The value of the dispersion 10 m downstream can be computed representing the misaligned quadrupole with a thin dipole magnet, bending by an angle $\theta = \Delta x'$:

$$M_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix}$$

The values of dispersion and dispersion prime 10 m downstream are the matrix elements m_{13} and m_{23} of the total transfer matrix dipole kick + drift:

$$M = \begin{pmatrix} 1 & L & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & L & \boxed{\text{D}\equiv\text{L}\theta} \\ 0 & 1 & \boxed{D' \equiv \theta} \\ 0 & 0 & 1 \end{pmatrix}$$

3 Exercise: CLIC lattice [3 points]

The main linac of CLIC consists of 12 sectors filled with FODO cells with increasing length and fixed phase advance, $\mu = 72°$ per cell, in both axes. In each sector the cell length scales approximatively with the square root of the energy: $L_{\text{cell}} \propto \sqrt{E}$. If one wants to keep fixed the quadrupole gradient G in all sectors, how must the length of the quadrupole magnets scale? Justify your answer.

SOLUTION. The focal length of a FODO cell is

$$f = \pm \frac{L_{\text{cell}}}{4 \sin \frac{\mu}{2}}.$$

As the phase advance is fixed to 72°, the focal length is proportional to the cell length. Since the cell length scales with $\sqrt{E}$ by design, the focal length will also do the same:

$$f \propto \sqrt{E}.$$

Now, from

$$f = \frac{1}{k_{\text{quad}} L_{\text{quad}}}$$

and

$$k_{\text{quad}} = \frac{G}{P/e} = \frac{Gc}{E/e} = \frac{1}{f L_{\text{quad}}},$$

it follows that

$$G = \frac{E/e}{c\, f\, L_{\text{quad}}}.$$

In conclusion, since $f \propto \sqrt{E}$, in order to keep constant the gradient G, the quadrupoles must have an increasing length, L_{quad} , which scales with the energy just like the FODO cell length:

$$L_{\text{quad}} = \frac{E/e}{Gc\sqrt{E}} \propto \sqrt{E}.$$

4 Exercise: the Achromatic Telescopic Squeeze (ATS) [9 points]

The ATS is a proposed alternative optics for the LHC and the baseline optics for its upgrade: the HL-LHC. The ATS allows to further squeeze the beam at the IP overcoming the limitation imposed by the sextupole strengths for the chromatic correction.

In the traditional squeeze the reduction of beta at the IP is entirely produced within the straight section and the dynamics in the arcs is not changed.

1. Write the expression for $\beta(s)$ starting from a waist ($\beta_w = \beta^*$, $\alpha_W = 0$) and show that as β^* is reduced, the chromaticity of the ring generated by the squeezing magnets (placed at a distance L^* from the IP) increases. [0.5 pt]

   The ATS scheme adopts a FODO cell matched to 90° to transport the beam in the arcs, connected by means of dispersion suppressors to the straight section, where the Interaction Point (IP) and the detector are placed. The sextupoles, which perform the chromatic correction, are placed next to each quadrupole in the arc.

   SOLUTION:

$$\beta(s) = \beta^* + \frac{s^2}{\beta^*} \approx \frac{s^2}{\beta^*}$$

   The last approximation being valid for $\beta^* \ll s$.

   We see that $\beta(L^*)$ is inversely proportional to β^*, therefore as we push down β^*, $\beta(L^*)$ increases and so does the natural chromaticity generated by the last quadrupoles:

$$\xi_N = -\frac{1}{4\pi}\oint \beta(s)k(s)ds \Rightarrow \Delta\xi_N = -\frac{1}{4\pi}\Delta\beta(L^*)k_Q L_Q$$

2. Briefly explain why it is useless to place sextupoles for the chromatic correction in the straight section. [0.5 pt]

   In order to get additional sextupolar strength we can mismatch the beta function in the arcs. We shall see how to do this in a controlled way. Use the thin lens approximation and neglect the dipole magnets.

   SOLUTION: A fundamental ingredient for the chromatic correction is the dispersion function, which is typically null in the straight sections (remember that we want to keep the beam size at the IP as small as possible and therefore we remove the dispersion with dispersion suppressors).

3. Express the focal length of the quadrupoles in term of the distance L between them, so that we achieve a 90° FODO. [0.5 pt]

   SOLUTION:

$$f = \frac{L}{2\sin(90^\circ/2)} = \frac{L}{\sqrt{2}}$$

4. Compute the transfer matrix of a cell, starting from the centre of a focussing quadrupole (use thin-lens approximation). [1 pt]

   SOLUTION: This simply requires to multiply the matrices together starting from a focussing quad with twice focal length:

$$M = \begin{pmatrix} 1 & 0 \\ -\frac{1}{2f} & 1 \end{pmatrix}\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix}\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\frac{1}{2f} & 1 \end{pmatrix}$$

   where f is the expression computed at the previous step. The result is

$$M = \begin{pmatrix} 0 & (2+\sqrt{2})L \\ \left(\frac{1}{\sqrt{2}} - 1\right)\frac{1}{L} & 0 \end{pmatrix}$$

5. What are M^2 and M^4? Briefly comment the result. [0.5 pt]

   SOLUTION:

$$M^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$M^4 = \left(M^2\right)^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

   The particles go back to the exact positions in the phase space after four cells, that is a phase advance of 2π.

6. Write down T, the 3x3 matrix that transport the Twiss and show that $T^2 = I$, the identity matrix. [1 pt]

SOLUTION:

$$T = \begin{pmatrix} C^2 & -2SC & S^2 \\ -CC' & SC' + S'C & -SS' \\ C'^2 & -2S'C' & S'^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & (2+\sqrt{2})^2 L^2 \\ 0 & -1 & 0 \\ \left(\frac{1}{\sqrt{2}} - 1\right)^2 \frac{1}{L^2} & 0 & 0 \end{pmatrix}$$

7. If we enter in a cell with: $\beta = \beta_0$, $\alpha = 0$ what are their values β_1, α_1 after one cell? And after two cells? [1 pt]

SOLUTION: Since $\alpha_0 = 0$, $\gamma_0 = 1/\beta_0$, therefore:

$$\beta_1 = \frac{(2+\sqrt{2})^2 L^2}{\beta_0}$$
$$\alpha_1 = -\alpha_0 = 0$$

After two cells we recover the initial values, since $T^2 = I$. This is important because it allows us to mismatch the cell, ensuring that the periodicity is maintained for a system made by two cells.

8. Which value of β_0 is such that $\beta_0 = \beta_1 \equiv \beta_{\text{Match}}$? Is this result in agreement with the formula for the matched FODO? [1 pt]

SOLUTION: By setting $\beta_0 = \beta_1$ in the previous point we find:

$$\beta_0 = (2+\sqrt{2})L$$

as we are starting in a focusing quad, we expect to have the maximum β of the FODO here, therefore, from the lectures:

$$\beta^+ = \frac{2L\left(1 + \sin\frac{\mu}{2}\right)}{\sin\mu} = \left(2+\sqrt{2}\right) L$$

for $\mu = 90°$ (remember that here we took L to be the distance between quads, so half of the cell length).

9. Now consider a system made of two cells starting at the centre of the focussing quadrupole. Remember: this system is periodic for any initial Twiss parameters as $T^2 = I$. We enter into this lattice with $\alpha_0 = 0$, $\beta_0 = r\beta_{\text{Match}}$, where $r \in \mathbb{R}^+$. Write the expressions of β at the focussing quadrupoles as a function of the enhancing parameter: r, assuming $L = 29.3$ m. [1 pt]

SOLUTION: The given L results in $\beta_{\text{Match}} = (2+\sqrt{2})L = 100\,\text{m}$

At the first focussing quadrupole we simply have:

$$\beta_{Qf1} = \beta_0 = r\beta_{\text{Match}} = r\ 100\,\text{m}$$

At the second focussing quadrupole we can use the expression for $\beta_1$derived before:

$$\beta_{Qf2} = \beta_1 = \frac{(2+\sqrt{2})^2 L^2}{r\beta_{\text{Match}}} = \frac{\left(\beta_{\text{Match}}\right)^2}{r\beta_{\text{Match}}} = \frac{\beta_{\text{Match}}}{r} = \frac{100\,\text{m}}{r}$$

10. In the arc, sextupole magnets are installed next to each quadrupole and tuned to compensate for the natural chromaticity. Show that if the chromaticity of the arc is corrected to zero, it remains null for each value of r (you can consider only the focussing quadrupoles). [1 pt]

SOLUTION:

$$\begin{aligned} \xi^{\text{ARC}} = \xi_N + \xi_{\text{sext}} &= -\frac{1}{4\pi}\oint \beta(s)K(s)\mathrm{d}s + \frac{1}{4\pi}\oint K_2(s)\,D(s)\beta(s)\,\mathrm{d}s \\ &= -\frac{1}{4\pi f}\left(r\beta_{\text{Match}} + \frac{\beta_{\text{Match}}}{r}\right) + \frac{1}{4\pi}K_2 L_{\text{sext}} D\left(r\beta_{\text{Match}} + \frac{\beta_{\text{Match}}}{r}\right) \\ &= \left(r + \frac{1}{r}\right)\left(-\frac{1}{4\pi f}\beta_{\text{Match}} + \frac{1}{4\pi}K_2 L_{\text{sext}} D\beta_{\text{Match}}\right) \\ &= 0 \end{aligned}$$

the factor $r + 1/r$ cancels, therefore the correction holds for each r.

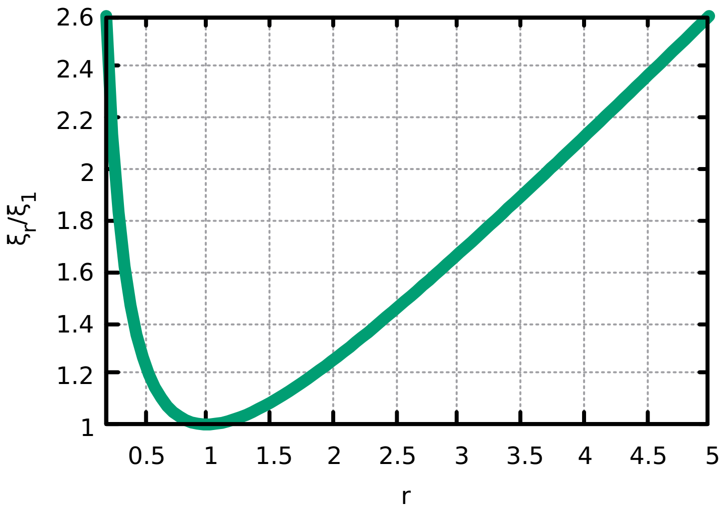
11. If we want to correct for the natural chromaticity produced in the straight section, we need to generate some additional positive chromaticity in the arc so that: $\xi^{\mathrm{ARC}} = -\xi_N^{\mathrm{IP}}$. Suppose that for $r = 1$ the total chromaticity of the arc is some positive value: ξ_1^{ARC}. Write an expression and sketch the plot for $\frac{\xi_r^{\mathrm{ARC}}}{\xi_1^{\mathrm{ARC}}}$. Briefly comment on how this can help in squeezing β_{IP} down to lower values. [1 pt]

SOLUTION: From the previous point we see that we can write:

$$\xi_1^{\mathrm{ARC}} = 2\left(-\frac{1}{4\pi f}\beta_{\mathrm{Match}} + \frac{1}{4\pi}K_2 L_{\mathrm{sext}} D \beta_{\mathrm{Match}}\right)$$

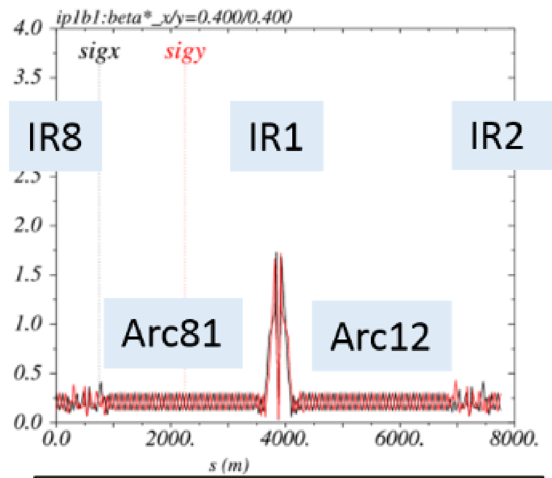
ξ_r^{ARC} scales with $\left(r + \frac{1}{r}\right)$ therefore we can write:

$$\frac{\xi_r^{\mathrm{ARC}}}{\xi_1^{\mathrm{ARC}}} = \frac{1}{2}\left(r + \frac{1}{r}\right)$$

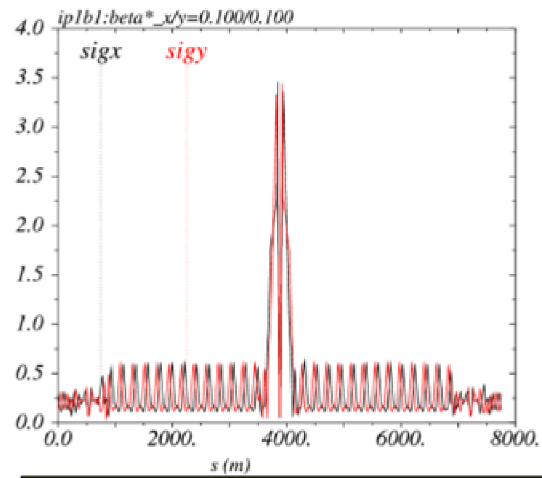


we see that for any r, the chromaticity produced in the arc is enhanced.

If properly tuned this can be used to compensate for the additional chromaticity produced while squeezing the beam, from here the name Achromatic. The name Telescopic follows from the fact that the mismatch is generated by matching quadrupoles before the arcs, when it is propagated to the IP, it results in a smaller β^*. The following plots show the betas in the real machine for $r = 1$ and $r = 4$. Note that in this process the dispersion remains unchanged as we only adjust quadrupoles where the dispersion is null (IR8, IR2).



Pre-squeeze to β*= 40 cm



Tele-squeeze to β*= 10 cm