Introduction to Transverse Beam Dynamics

Andrea Latina (CERN)

andrea.latina@cern.ch

JUAS, January 2017

Part 1.

Basics, single-particle dynamics

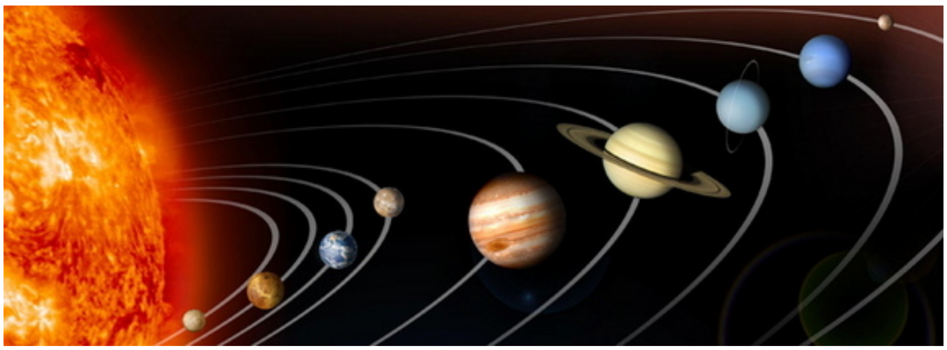
Luminosity run of a typical storage ring

In a storage ring: the protons are accelerated and stored for ~ 12 hours

The distance traveled by particles running at *nearly* the speed of light, $v \approx c$, for 12 hours is

$$\textbf{distance} \approx \mathbf{12 \times 10^{11}\ km}$$

$\rightarrow$ this is about 100 times the distance from Sun to Pluto and back !



Introduction and basic ideas

It's a circular machine: we need a transverse deflecting force$\rightarrow$the Lorentz force

$$\vec{F} = q \cdot \left(\vec{E} + \vec{v} \wedge \vec{B}\right)$$

where, in high energy machines, $|\vec{v}| \approx c \approx 3 \cdot 10^8$ m/s. Usually there is no electric field, and the transverse deflection is given by a magnetic field only.

Example

$$B = 1\ \text{T} \rightarrow \quad \begin{aligned} F &= q \cdot 3 \cdot 10^8\ \frac{m}{s} \cdot 1\ \text{T} \\ &= q \cdot 3 \cdot 10^8\ \frac{m}{s} \cdot 1\ \frac{Vs}{m^2} \\ &= q \cdot 300\ \frac{MV}{m} \end{aligned}$$

Notice that there is a technical limit for an electric field:

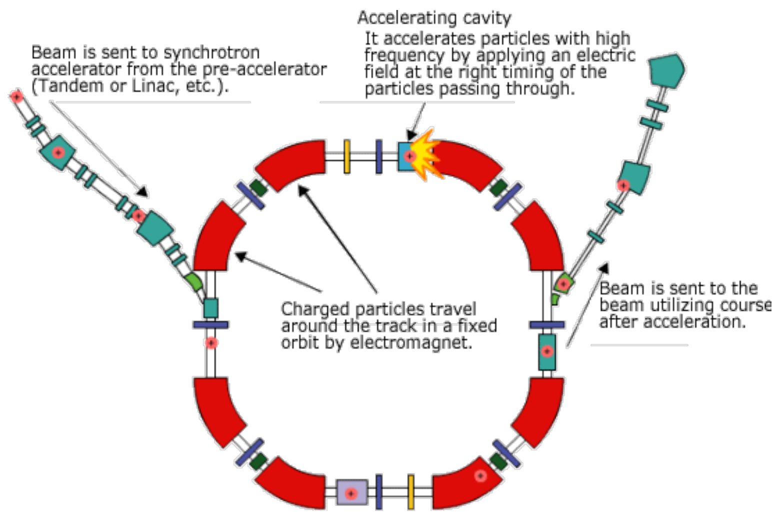
$$E \lesssim 1\ \frac{MV}{m}$$

Therefore | in an accelerator, use magnetic fields wherever it's possible |

$$\left.\begin{aligned} \text{Lorentz force} \quad F_L &= qvB \\ \text{Centripetal force} \quad F_{\text{centr}} &= \frac{\gamma m_0 v^2}{\rho} \\ \frac{\gamma m_0 v^{\cancel{2}}}{\rho} &= q\cancel{v}B \end{aligned}\right\}$$

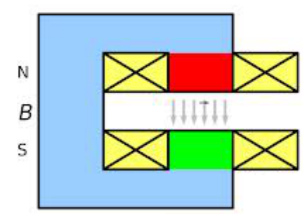
$P = m_0 \gamma v = mv$ "momentum"
$B\rho =$ "beam ridigity"

$$\boxed{\frac{P}{q} = B\rho}$$



Dipole magnets: the magnetic guide

- Dipole magnets:
  - define the ideal orbit
  - in a homogeneous field created by two flat pole shoes, $B = \frac{\mu_0 n I}{h}$



- Normalise magnetic field to momentum:

$$\boxed{\frac{P}{q} = B\rho \;\Rightarrow\; \frac{1}{\rho} = \frac{qB}{P} \quad [\text{m}^{-1}]} \qquad B = [T]; \quad \text{P}=\left[\frac{GeV}{c}\right]; \quad 1 \text{ T}=\frac{1\ V \cdot 1\ s}{1\ m^2}$$

- Example: the LHC, accelerating protons (q=1 e)

$$\left.\begin{aligned} B &= 8.3 \text{ T} \\ p &= 7000\ \frac{GeV}{c} \end{aligned}\right\} \quad \frac{1}{\rho} = \text{e}\frac{8.3\ \frac{Vs}{m^2}}{7000 \cdot 10^9\ \frac{eV}{c}} = \frac{8.3\text{s}\ \cdot 3 \cdot 10^8\ \frac{m}{s}}{7000 \cdot 10^9\ \text{m\^{}2}} =$$

$$= 0.333 \cdot \frac{8.3}{7000}\ \frac{1}{m} = \frac{1}{2.53}\ \frac{1}{km}$$

Dipole magnets: the magnetic guide

Very important rule of thumb:

$$\frac{1}{\rho\ [m]} \approx 0.3 \frac{B\ [T]}{P\ [GeV/c]}$$

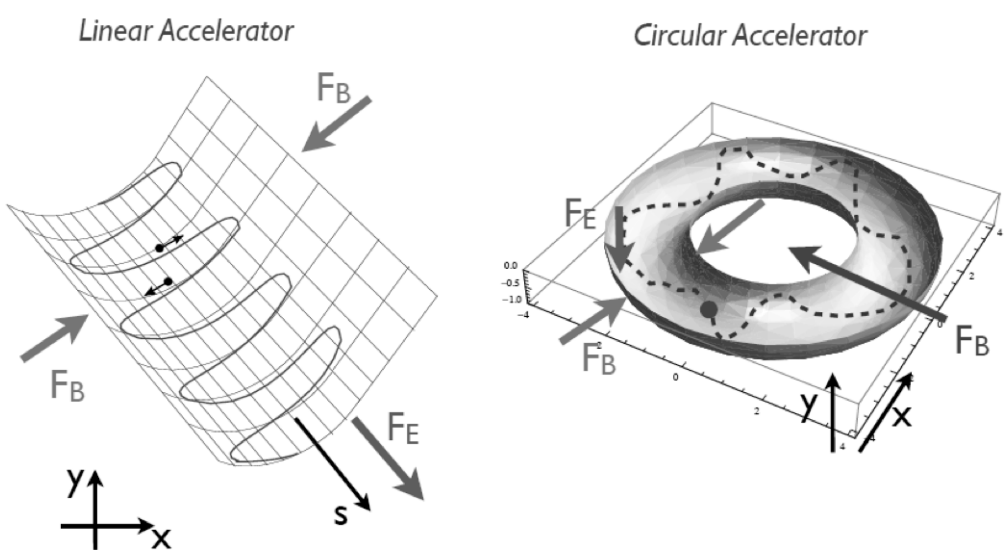
In the LHC, $\rho = 2.53$ km. The circumference $2\pi\rho = 15.9$ km $\approx 60\%$ of the entire LHC.

The field B is $\approx 1 \ldots 8$ T

which is a sort of "normalised bending strength", normalised to the momentum of the particles.

The focusing force

$$\vec{F} = q \cdot \left(\vec{E} + \vec{v} \wedge \vec{B}\right)$$



Remember the 1d harmonic oscillator: $F = -k\,x$

Reminder: the 1d Harmonic oscillator

Restoring force

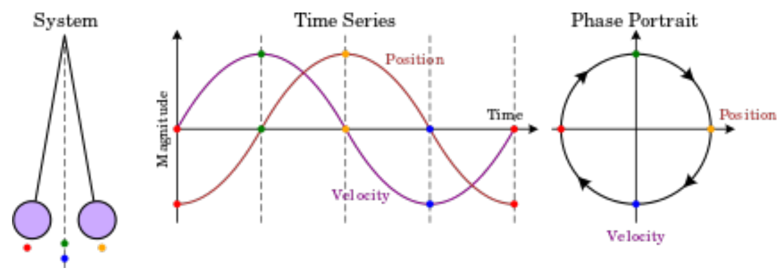
$$F = -k\,x$$

Equation of motion:

$$x'' = -\frac{k}{m}x$$

which has solution:

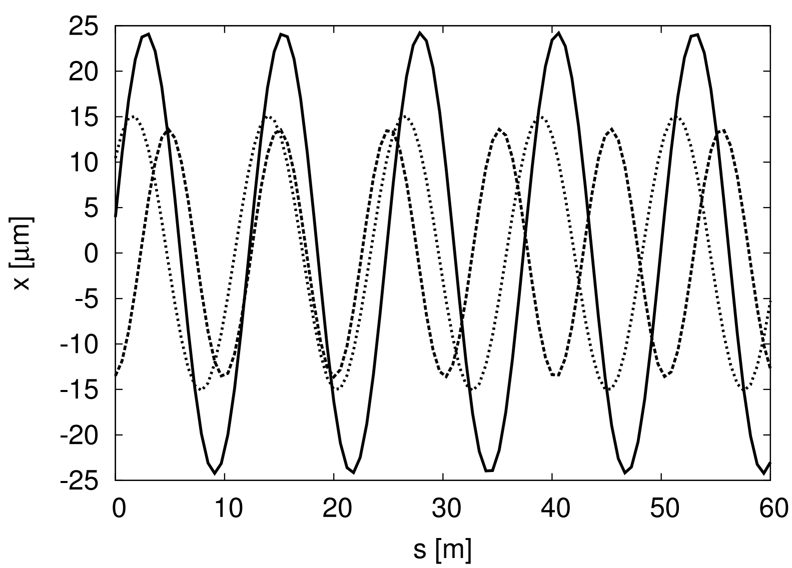
$$x(t) = A\cos(\omega t + \phi) \underset{\text{or}}{=} a_1\cos(\omega t) + a_2\sin(\omega t)$$



- F , restoring force, N or MeV/m
- k, spring constant or *focusing strength*, N/m or MeV/m^2
- $\omega = \sqrt{\frac{k}{m}} = 2\pi f$, angular velocity, rad/s
- ϕ, initial phase, rad
- f, rotation frequency, 1/s or Hz
- $m = m_0\gamma$, particle's mass, MeV/c^2
- m_0, particle's rest mass, MeV/c^2
- A, oscillation amplitude, m

Exercise

The following plot represents the trajectories of three particles traveling in a transfer line with constant focusing strength.



Among the three particles, one is significantly off-momentum. Which one is it (full, small-dot or large-dot line)? Is its rigidity higher or lower than the on-momentum particles?

Quadrupole magnets: the focusing force

Quadrupole magnets are required to keep the trajectories in vicinity of the ideal orbit

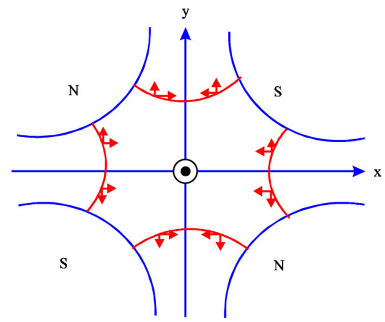
They exert a linearly-increasing Lorentz force, thru a linearly-increasing magnetic field:

$$\begin{matrix} B_x = gy \\ B_y = gx \end{matrix} \quad \Rightarrow \quad \begin{matrix} F_x = -qv_z gx \\ F_y = qv_z gy \end{matrix}$$

Gradient of a quadrupole magnet:

$$g = \frac{2\mu_0 nI}{r^2_{\text{aperture}}} \left[\frac{T}{m}\right] = \frac{B_{\text{poles}}}{r_{\text{aperture}}} \left[\frac{T}{m}\right]$$

- LHC main quadrupole magnets: $g \approx 25 \ldots 235$ T/m



the arrows show the force exerted on a particle

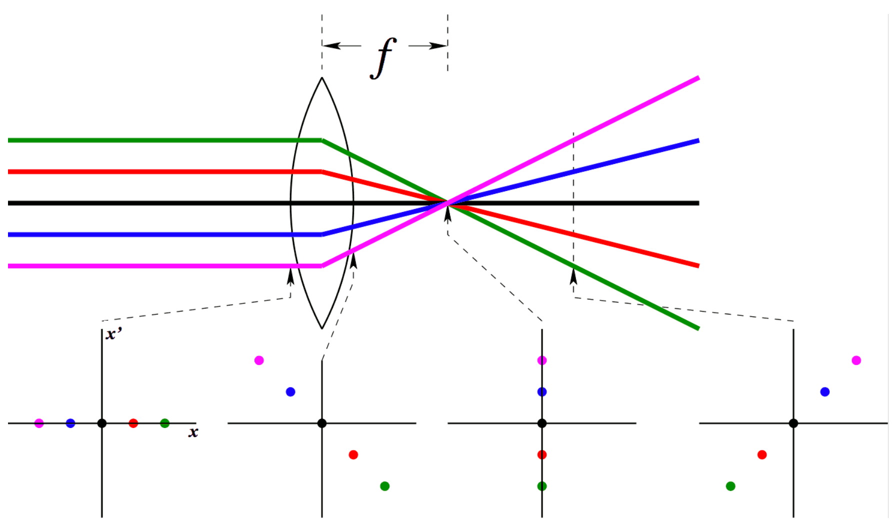
Divide by p/q to find the *normalised focusing strength*, k:

$$\boxed{k = \frac{g}{P/q} \left[m^{-2}\right]} \quad \Rightarrow \quad g = \left[\frac{T}{m}\right]; \quad q = [e]; \quad \frac{P}{q} = \left[\frac{\text{GeV}}{\text{c} \cdot e}\right] = \left[\frac{GV}{c}\right] = [T\ m]$$

A simple rule: $k \left[m^{-2}\right] \approx 0.3 \dfrac{g\ [T/m]}{P/q\ [GeV/c/e]}$.

Focal length of a quadrupole

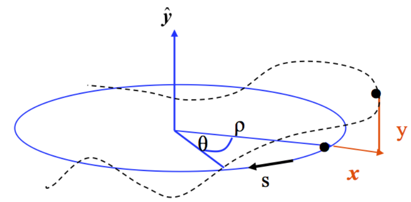
The focal length of a quadrupole is $f = \frac{1}{k \cdot L}$ [m], where L is the quadrupole length:



Towards the equation of motion

Linear approximation:

- the ideal particle $\Rightarrow$ stays on the **design orbit** (i.e. $x, y, P_x, P_y = 0;\ P = P_0$)
- any other particle $\Rightarrow$ has coordinates x, y
  - which are small quantities: $x, y \ll \rho$
  - $P_x,\ P_y$ are small, and $P \neq P_0$
- only linear terms in x and y of B are taken into account



Let's recall some useful relativistic formulæ and definitions:

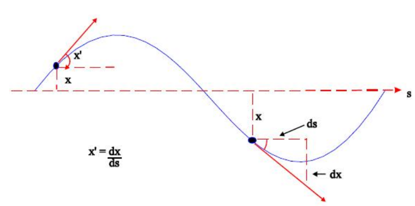
$P_0 = m_0\,\gamma\,v_0$	reference momentum
$P = P_0\,(1+\delta)$	total momentum
$\delta = (P - P_0)\,/\,P_0$	relative momentum offset
$E = \sqrt{P^2c^2 + m_0^2c^4} = m_0\,\gamma\,c^2 = K + m_0\,c^2$	total energy
$K = E - m_0\,c^2$	kinetic energy
$\beta = \frac{v}{c} = \frac{Pc}{E};\qquad \gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{E}{m_0\,c^2}$	relativistic beta and gamma

Phase-space coordinates

The state of a particle is represented with a 6-dimensional phase-space vector:

$$(x,\ x',\ y,\ y',\ z,\ \delta)$$

where x' and y' are the transverse angles:



with

$$x \qquad [\text{m}]$$

$$x' = \frac{dx}{ds} = \frac{dx}{dt}\frac{dt}{ds} = \frac{V_x}{V_z} = \frac{P_x}{P_z} \approx \frac{P_x}{P_0} \qquad [\text{rad}]$$

$$y \qquad [\text{m}]$$

$$y' = \frac{dy}{ds} = \frac{dy}{dt}\frac{dt}{ds} = \frac{V_y}{V_z} = \frac{P_y}{P_z} \approx \frac{P_y}{P_0} \qquad [\text{rad}]$$

$$z \qquad [\text{m}]$$

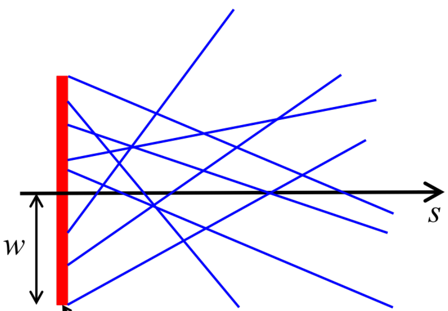
$$\delta = \frac{\Delta P}{P_0} = \frac{P - P_0}{P_0} \qquad [\#]$$

where P_0 is the momentum of the reference particle (reference momentum), and $P = P_0\,(1+\delta)$

Exercise: Phase space representations

1. Consider a cathode, located at position s_0 with radius w, emitting particles. What does the phase space look like for the particles just created? Which portion of the phase space is occupied by the emitted particles?

   Hint: the particle source in the configuration space



Towards the equation of motion

Taylor expansion of the B_y field:

$$B_y(x) = B_{y0} + \frac{\partial B_y}{\partial x}x + \frac{1}{2}\frac{\partial^2 B_y}{\partial x^2}x^2 + \frac{1}{3!}\frac{\partial^3 B_y}{\partial x^3}x^3 + \dots$$

Now we drop the suffix 'y' and normalise to the magnetic rigidity $p/q = B\rho$

$$\frac{B(x)}{P/q} = \frac{B_0}{B_0\rho} + \frac{g}{P/q}x + \frac{1}{2}\frac{g'}{P/q}x^2 + \frac{1}{3!}\frac{g''}{P/q}x^3 + \dots$$

$$= \frac{1}{\rho} + kx + \frac{1}{2}mx^2 + \frac{1}{3!}nx^3 + \dots$$

In the linear approximation, only the terms linear in x and y are taken into account:

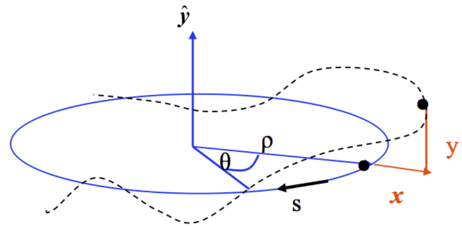
- dipole fields, $1/\rho$
- quadrupole fields, k

It is more practical to use "separate function" magnets, rather than combined ones:

- split the magnets and optimise them regarding their function
  - bending
  - focusing, etc.

The equation of motion in radial coordinates

Let's consider a local segment of one particle's trajectory:



and recall the radial centrifugal acceleration: $a_r = \frac{\mathrm{d}^2\rho}{\mathrm{d}t^2} - \rho\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2 = \frac{\mathrm{d}^2\rho}{\mathrm{d}t^2} - \rho\omega^2.$

- For an ideal orbit: $\rho = \text{const} \Rightarrow \frac{\mathrm{d}\rho}{\mathrm{d}t} = 0$

  $\Rightarrow$the force is
  $$\begin{aligned} F_{\text{centrifugal}} &= -m\rho\omega^2 = -mv^2/\rho \\ F_{\text{Lorentz}} &= qB_y v = -F_{\text{centrifugal}} \end{aligned} \quad \Rightarrow \quad \frac{P}{q} = B_y\rho$$

- For a general trajectory: $\rho \to \rho + x$:

  $$F_{\text{centrifugal}} = m\,a_r = -F_{\text{Lorentz}} \quad \Rightarrow \quad m\left[\frac{\mathrm{d}^2}{\mathrm{d}t^2}(\rho + x) - \frac{v^2}{\rho + x}\right] = -qB_y v$$

$$F = \underbrace{m\frac{\mathrm{d}^2}{\mathrm{d}t^2}(\rho + x)}_{\text{term 1}} - \frac{mv^2}{\underbrace{\rho + x}_{\text{term 2}}} = -qB_y v$$

- Term 1: As ρ =const...

$$m\frac{\mathrm{d}^2}{\mathrm{d}t^2}(\rho + x) = m\frac{\mathrm{d}^2}{\mathrm{d}t^2}x$$

- Term 2: Remember: $x \approx$ mm whereas $\rho \approx$ m $\to$ we develop for small x

remember

$$\frac{1}{\rho + x} \approx \frac{1}{\rho}\left(1 - \frac{x}{\rho}\right)$$

Taylor expansion:

$$f(x) = f(x_0) + \\ + (x - x_0) f'(x_0) + \frac{(x - x_0)^2}{2!} f''(x_0) + \cdots$$

$$m\frac{d^2x}{\mathrm{d}t^2} - \frac{mv^2}{\rho}\left(1 - \frac{x}{\rho}\right) = -qB_y v$$

The guide field in linear approximation $B_y = B_0 + x\frac{\partial B_y}{\partial x}$

$$m\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} - \frac{mv^2}{\rho}\left(1-\frac{x}{\rho}\right) = -qv\left\{B_0 + x\frac{\partial B_y}{\partial x}\right\} \qquad \text{let's divide by } m$$

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} - \frac{v^2}{\rho}\left(1-\frac{x}{\rho}\right) = -\frac{qvB_0}{m} - x\frac{qvg}{m}$$

Independent variable: $t \to s$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}t} = x'v$$

$$\frac{d^2x}{\mathrm{d}t^2} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\underbrace{\frac{\mathrm{d}x}{\mathrm{d}s}}_{x'}\underbrace{\frac{\mathrm{d}s}{\mathrm{d}t}}_{v}\right) = \frac{\mathrm{d}}{\mathrm{d}t}(x'v) =$$

$$= \frac{\mathrm{d}}{\mathrm{d}s}\underbrace{\frac{\mathrm{d}s}{\mathrm{d}t}}_{v}(x'v) = \frac{\mathrm{d}}{\mathrm{d}s}\left(x'v^2\right) = x''v^2 + \cancel{x'2v\frac{\mathrm{d}v}{\mathrm{d}s}}$$

$$x''v^2 - \frac{v^2}{\rho}\left(1-\frac{x}{\rho}\right) = -\frac{qvB_0}{m} - x\frac{vg}{m} \qquad \text{let's divide by } v^2$$

$$x'' - \frac{1}{\rho}\left(1-\frac{x}{\rho}\right) = -\frac{qB_0}{mv} - x\frac{qg}{mv}$$

$$x'' - \frac{1}{\rho} + \frac{x}{\rho^2} = -\frac{B_0}{P/q} - \frac{xg}{P/q}$$

$$x'' \cancel{-\frac{1}{\rho}} + \frac{x}{\rho^2} = \cancel{-\frac{1}{\rho}} - kx$$

Remember:

$$mv = p$$

Normalise to the momentum of the particle:

$$\frac{1}{\rho} = \frac{B_0}{P/q}\ [\mathrm{m}^{-1}]; \quad k = \frac{g}{P/q}\ [\mathrm{m}^{-2}]$$

$$x'' + x\left(\frac{1}{\rho^2} + k\right) = 0$$

Equation for the vertical motion

- $\frac{1}{\rho^2} = 0$ usually there are not vertical bends
- $k \longleftrightarrow -k$ quadrupole field changes sign

$$y'' - ky = 0$$

Weak focusing

- "Weak" focusing:

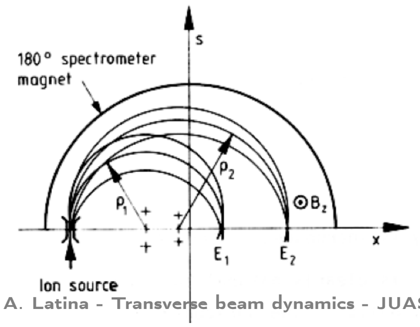
$$x''(s) + \underbrace{\left(\frac{1}{\rho^2} + k\right)}_{\text{focusing effect}} x(s) = 0$$

there is a focusing force, $\frac{1}{\rho^2}$, even without a quadrupole gradient,

$$k = 0 \quad \Rightarrow \quad x'' = -\frac{1}{\rho^2} x$$

even without quadrupoles there is retrieving force (focusing) in the bending plane of the dipole magnets

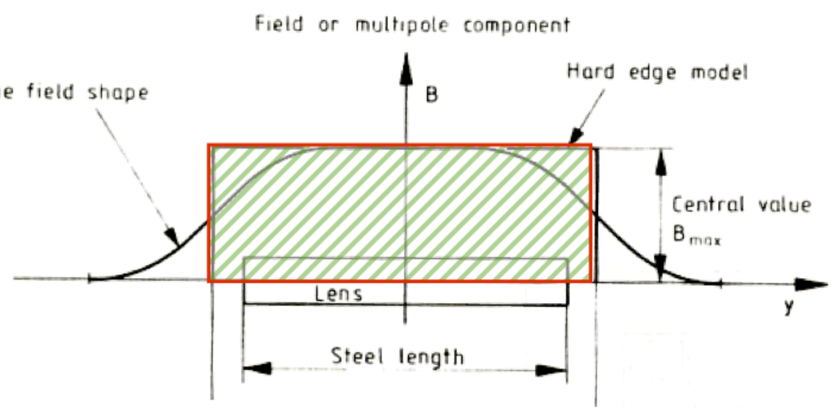
- In large machine this effect is very weak...



Mass spectrometer: particles are separated according to their energy and focused due to the $1/\rho$ effect of the dipole

Effective length

$$B_0 \cdot L_{eff} = \int_0^{l_{mag}} B(s)\, ds$$



Solution of the trajectory equations: focusing quadrupole

Definition:

$$\left.\begin{array}{lll}\text{horizontal plane} & K & = 1/\rho^2 + k \\ \text{vertical plane} & K & = -k\end{array}\right\} \quad x'' + Kx = 0$$

This is the differential equation of a harmonic oscillator ... with spring constant K. We make an ansatz:

$$x(s) = a_1 \cos(\omega s) + a_2 \sin(\omega s)$$

General solution: a linear combination of two independent solutions:

$$x'(s) = -a_1 \omega \sin(\omega s) + a_2 \omega \cos(\omega s)$$
$$x''(s) = -a_1 \omega^2 \cos(\omega s) + a_2 \omega^2 \sin(\omega s) = -\omega^2 x(s) \quad \rightarrow \quad \omega = \sqrt{K}$$

General solution, for $K > 0$:

$$x(s) = a_1 \cos\left(\sqrt{K} s\right) + a_2 \sin\left(\sqrt{K} s\right)$$

We determine a_1, a_2 by imposing the following boundary conditions:

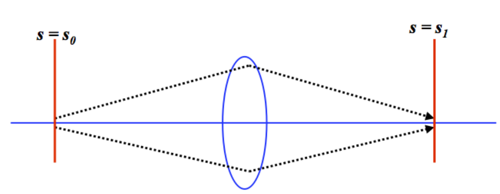
$$s = 0 \quad \rightarrow \quad \begin{cases} x(0) = x_0, & a_1 = x_0 \\ x'(0) = x'_0, & a_2 = \frac{x'_0}{\sqrt{K}} \end{cases}$$

Horizontal focusing quadrupole, $K > 0$:

$$x(s) = x_0 \cos\left(\sqrt{K} s\right) + x'_0 \frac{1}{\sqrt{K}} \sin\left(\sqrt{K} s\right)$$
$$x'(s) = -x_0 \sqrt{K} \sin\left(\sqrt{K} s\right) + x'_0 \cos\left(\sqrt{K} s\right)$$

For convenience we can use a matrix formalism:

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_1} = M_{foc} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}_{s_0}$$



For a quadrupole of length L:

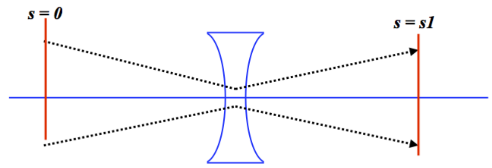
$$M_{\text{foc}} = \begin{pmatrix} \cos\left(\sqrt{K} L\right) & \frac{1}{\sqrt{K}} \sin\left(\sqrt{K} L\right) \\ -\sqrt{K} \sin\left(\sqrt{K} L\right) & \cos\left(\sqrt{K} L\right) \end{pmatrix}$$

Defocusing quadrupole

The equation of motion is

$$x'' + Kx = 0$$

with $K < 0$



Remember: $f(s) = \cosh(s)$, $f'(s) = \sinh(s)$

The solution is in the form:

$$x(s) = a_1 \cosh(\omega s) + a_2 \sinh(\omega s)$$

with $\omega = \sqrt{|K|}$. For a quadrupole of length L the transfer matrix reads:

$$M_{\text{defoc}} = \begin{pmatrix} \cosh\left(\sqrt{|K|}L\right) & \frac{1}{\sqrt{|K|}} \sinh\left(\sqrt{|K|}L\right) \\ \sqrt{|K|} \sinh\left(\sqrt{|K|}L\right) & \cosh\left(\sqrt{|K|}L\right) \end{pmatrix}$$

Notice that for a drift space, i.e. when $K = 0 \rightarrow M_{\text{drift}} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$

Summary of the transfer matrices

- Focusing quad, $K > 0$

$$M_{\text{foc}} = \begin{pmatrix} \cos\left(\sqrt{K}L\right) & \frac{1}{\sqrt{K}} \sin\left(\sqrt{K}L\right) \\ -\sqrt{K} \sin\left(\sqrt{K}L\right) & \cos\left(\sqrt{K}L\right) \end{pmatrix}$$

- Defocusing quad, $K < 0$

$$M_{\text{defoc}} = \begin{pmatrix} \cosh\left(\sqrt{|K|}L\right) & \frac{1}{\sqrt{|K|}} \sinh\left(\sqrt{|K|}L\right) \\ \sqrt{|K|} \sinh\left(\sqrt{|K|}L\right) & \cosh\left(\sqrt{|K|}L\right) \end{pmatrix}$$

- Drift space, $K = 0$

$$M_{\text{drift}} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$

With the assumptions we have made, the motion in the horizontal and vertical planes is independent: "... the particle motion in x and y is uncoupled"

Thin-lens approximation of a quadrupole magnet

When the focal length f of the quadrupolar lens is much bigger than the length of the magnet itself, L_Q

$$f = \frac{1}{k \cdot L} \qquad \gg L_Q$$

we can derive the limit for $L \to 0$ while keeping constant f, i.e. $k \cdot L_Q = \text{const}$.

The transfer matrices are

$$M_x = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \qquad M_y = \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix}$$

focusing, and defocusing respectively.

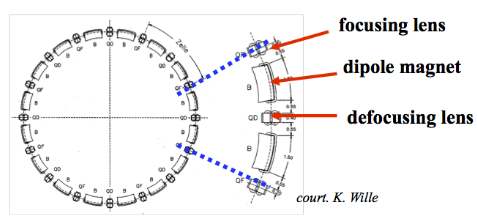
This approximation (yet quite accurate, in large machines) is useful for fast calculations... (e.g. for the guided studies!)

Transformation through a system of lattice elements

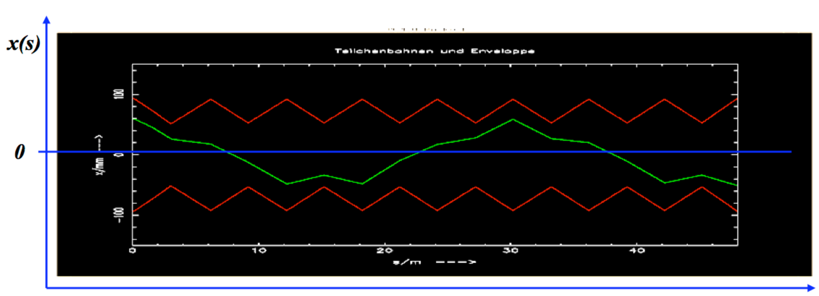
One can compute the solution of a system of elements, by multiplying the matrices of each single element:

$$M_{\text{total}} = M_{\text{QF}} \cdot M_{\text{D}} \cdot M_{\text{Bend}} \cdot M_{\text{D}} \cdot M_{\text{QD}} \cdot \quad \cdots$$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_2} = M_{s_1 \to s_2} \cdot M_{s_0 \to s_1} \cdot \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0}$$



In each accelerator element the particle trajectory corresponds to the movement of a harmonic oscillator.



...typical values are:

$$x \approx \text{mm}$$

$$x' \leq \text{mrad}$$

Properties of the transfer matrix M

The transfer matrix M has two important properties:

- (with no acceleration) its determinant is 1

  $$\det(M) = 1$$

  (Liouville's theorem)

- provides a stable motion over N turns, with $N \to \infty$, if and only if:

  $$\text{trace}(M) \leq 2$$

  (Stability condition)

Extra: Stability condition

Question: Given a periodic lattice with generic transport map M,

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

under which condition the matrix M provides stable motion after N turns (with $N \to \infty$)?

$$x_N = \underbrace{M \cdot \ldots \cdot M \cdot M \cdot M}_{N \text{ turns, with } N \to \infty} x_0 = M^N x_0$$

The answer is simple: the motion is stable when all elements of M^N are finite, with $N \to \infty$. But... how do we compute M^N with $N \to \infty$?

Remember:

- $\det(M) = ad - bc = 1$
- $\text{trace}(M) = a + d$

If we diagonalise M, we can rewrite it as:

$$M = U \cdot \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \cdot U^T$$

where U is some unitary matrix, λ_1 and λ_2 are the eigenvalues.

Extra: Stability condition (cont.)

What happens if we consider N turns?

$$M^N = U \cdot \begin{pmatrix} \lambda_1^N & 0 \\ 0 & \lambda_2^N \end{pmatrix} \cdot U^T$$

Notice that λ_1 and λ_2 can be complex numbers. Given that $\det(M) = 1$, then

$$\lambda_1 \cdot \lambda_2 = 1 \quad \rightarrow \lambda_1 = \frac{1}{\lambda_2} \quad \rightarrow \lambda_{1,2} = e^{\pm i\,x}$$

$\Rightarrow$ to have a stable motion, x must be real: $x \in \mathrm{R}$.
Now we can find the eigenvalues through the characteristic equation:

$$\det(M - \lambda I) = \det \begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix} = 0$$

$$\lambda^2 - (a + d)\,\lambda + (ad - bc) = 0$$
$$\lambda^2 - \mathrm{trace}\,(M)\,\lambda + 1 = 0$$

$$\mathrm{trace}\,(M) = \lambda + 1/\lambda =$$
$$= e^{ix} + e^{-ix} = 2\cos x$$

From which derives the stability condition:

$$\text{since } x \in \mathrm{R} \quad \rightarrow \quad |\mathrm{trace}\,(M)| \; \leq \; 2$$

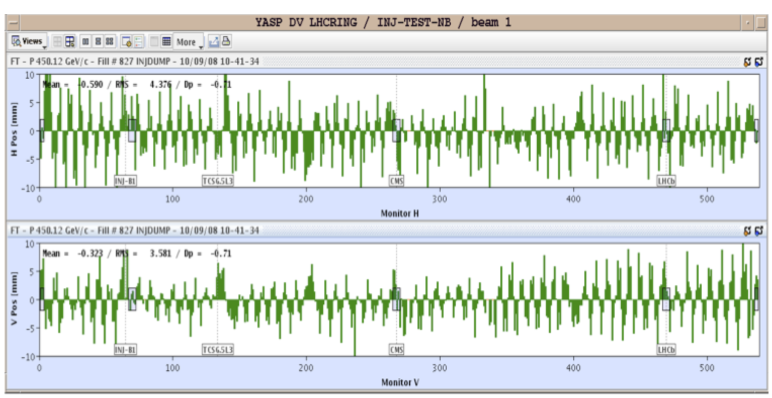
Orbit and tune

Tune: the number of oscillations per turn.

Example:

64.31

59.32



Relevant for beam stability studies is : the non-integer part

Extra: Summary of momenta and angles definitions

$P = P_0\,(1+\delta)$ total momentu w.r.t. reference momentum

$P = \sqrt{P_x^2 + P_y^2 + P_z^2}$ total momentum

- General convention: lower-case momenta: normalised to P_0

$$p = \frac{P}{P_0} = 1 + \delta$$

$$p_x = \frac{P_x}{P_0}$$

$$p_y = \frac{P_y}{P_0}$$

$$p_z = \frac{P_z}{P_0} = \frac{\sqrt{P^2 - P_x^2 - P_y^2}}{P_0} = \sqrt{(1+\delta)^2 - p_x^2 - p_y^2} \approx$$

$$\approx (1+\delta)\left(1 - \frac{1}{2}\frac{p_x^2 + p_y^2}{(1+\delta)^2}\right) =$$

$$= 1 + \delta - \frac{1}{2}\frac{p_x^2 + p_y^2}{1+\delta} \approx 1 + \delta \text{ for small } p_x \text{ and } p_y$$

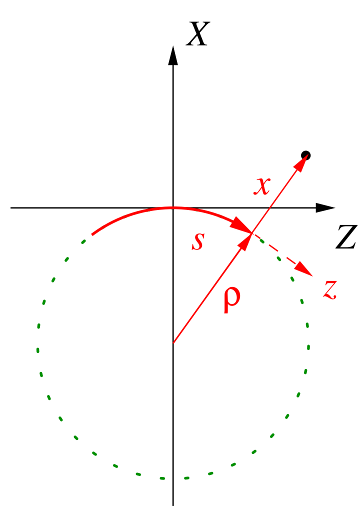
$$x' = \frac{\mathrm{d}x}{\mathrm{d}s} = \frac{V_x}{V_z} = \frac{P_x}{P_z} \approx \frac{P_x}{P_0}$$

$$y' = \frac{\mathrm{d}y}{\mathrm{d}s} = \frac{V_y}{V_z} = \frac{P_y}{P_z} \approx \frac{P_y}{P_0}$$

Extra: From a Cartesian to a curved reference system

We use a Curved Reference System: the Frenet–Serret rotating frame

Curvilinear → Cartesian $(x,\ y,\ z) \to (X,\ Y,\ Z)$	**Cartesian → Curvilinear** $(X,\ Y,\ Z) \to (x,\ y,\ z)$
$z = s - \beta ct$	$s = \rho \arctan \frac{Z}{X+\rho}$
$X = (\rho + x)\cos\frac{s}{\rho} - \rho$ $Y = y$ $Z = (\rho + x)\sin\frac{s}{\rho}$	$x = \sqrt{(X+\rho)^2 + Z^2} - \rho$ $y = Y$ $z = s - \beta ct$
$P_x = P_X \cos\frac{s}{\rho} + P_Z \sin\frac{s}{\rho}$ $P_y = P_Y$	$P_X = P_x \cos\frac{s}{\rho} - P_z \sin\frac{s}{\rho}$ $P_Y = P_y$



The y and Y axes are parallel and orthogonal to this page.

Summary

beam rigidity: $B\rho = \frac{P}{q}$

bending strength of a dipole: $\frac{1}{\rho}\ [m^{-1}] = \frac{0.2998 \cdot B_0\ [\mathrm{T}]}{P\ [\mathrm{GeV/c}]}$

focusing strength of a quadruple: $k\ [m^{-2}] = \frac{0.2998 \cdot g}{P\ [\mathrm{GeV/c}]}$

focal length of a quadrupole: $f = \frac{1}{k \cdot L_Q}$

equation of motion: $x'' + \left(\frac{1}{\rho^2} + k\right) x = 0$

solution of the eq. of motion: $x_{s_2} = M \cdot x_{s_1}$ . . . with $M \equiv \begin{pmatrix} C & S \\ C' & S' \end{pmatrix}$

$$\text{e.g.:}\quad M_{\mathrm{QF}} = \begin{pmatrix} \cos\left(\sqrt{K}L\right) & \frac{1}{\sqrt{K}}\sin\left(\sqrt{K}L\right) \\ -\sqrt{K}\sin\left(\sqrt{K}L\right) & \cos\left(\sqrt{K}L\right) \end{pmatrix},$$

$$M_{\mathrm{QD}} = \begin{pmatrix} \cosh\left(\sqrt{|K|}L\right) & \frac{1}{\sqrt{|K|}}\sinh\left(\sqrt{|K|}L\right) \\ \sqrt{|K|}\sinh\left(\sqrt{|K|}L\right) & \cosh\left(\sqrt{|K|}L\right) \end{pmatrix},\quad M_{\mathrm{D}} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$

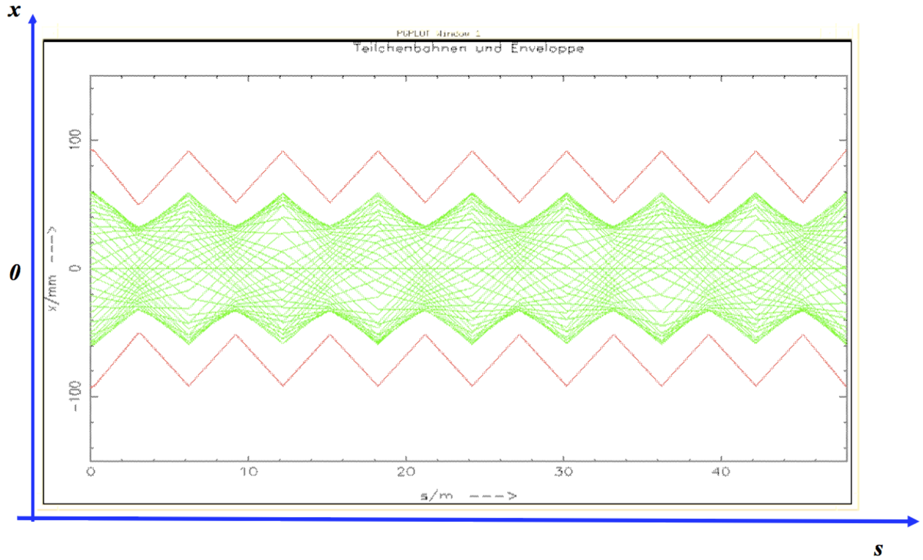
Part 2.

Optics Functions and Twiss Parameters

Envelope

So far we have studied the motion of a particle.
Question: what will happen, if the particle performs a second turn ?

- ... or a third one or ... 10^{10} turns ...



The Hill's equation

In 19th century George William Hill, one of the greatest master of celestial mechanics of his time, studied the differential equation for "motions with periodic focusing properties": the "Hill's equation"

$$x''(s) + K(s)\,x(s) = 0$$

with:

- a restoring force $\neq$ const
- $K(s)$ depends on the position s
- $K(s+L) = K(s)$ periodic function, where L is the "lattice period"

We expect a solution in the form of a quasi harmonic oscillation: amplitude and phase will depend on the position s in the ring.

The beta function

General solution of Hill's equation:

$$x(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}\cos(\mu(s) + \mu_0) \quad (1)$$

ε, μ_0 =integration constants determined by initial conditions

$\beta(s)$ is a periodic function given by the focusing properties of the lattice $\leftrightarrow$ quadrupoles

$$\beta(s + L) = \beta(s)$$

Inserting Eq. (1) in the equation of motion, we get (Floquet's theorem) the following result

$$\mu(s) = \int_0^s \frac{ds}{\beta(s)}$$

$\mu(s)$ is the "phase advance" of the oscillation between the points 0 and s along the lattice. For one complete revolution, $\mu(s)$ is the number of oscillations per turn, or "tune" when normalised to 2π

$$Q = \frac{1}{2\pi}\oint \frac{ds}{\beta(s)}$$

ε is the Courant-Snyder invariant

The beam ellipse

General solution of the Hill's equation

$$\begin{cases} x(s) &= \sqrt{\varepsilon}\sqrt{\beta(s)}\cos(\mu(s) + \mu_0) \quad (1) \\ x'(s) &= -\frac{\sqrt{\varepsilon}}{\sqrt{\beta(s)}}\{\alpha(s)\cos(\mu(s) + \mu_0) + \sin(\mu(s) + \mu_0)\} \quad (2) \end{cases}$$

From Eq. (1) we get

$$\cos(\mu(s) + \mu_0) = \frac{x(s)}{\sqrt{\varepsilon}\sqrt{\beta(s)}} \qquad \begin{aligned} \alpha(s) &= -\frac{1}{2}\beta'(s) \\ \gamma(s) &= \frac{1 + \alpha(s)^2}{\beta(s)} \end{aligned}$$

Insert into Eq. (2) and solve for ε

$$\varepsilon = \gamma(s)\, x(s)^2 + 2\alpha(s)\, x(s)\, x'(s) + \beta(s)\, x'(s)^2$$

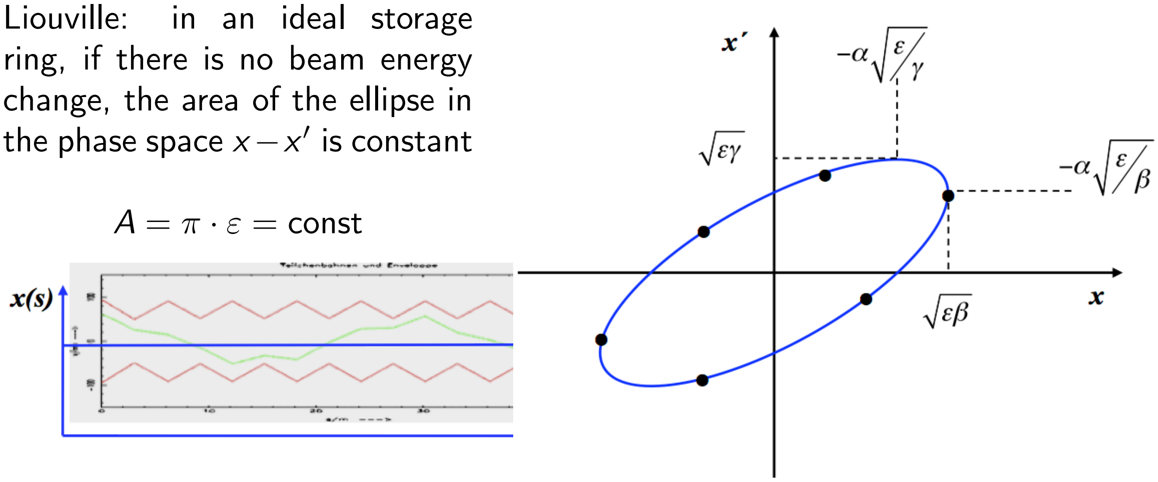
- ε is a constant of the motion, i.e. the Courant-Snyder invariant or Action
- it is a parametric representation of an ellipse in the xx' space
- the shape and the orientation of the ellipse are given by α, β, and γ $\Rightarrow$ these are the Twiss parameters

Learning from the phase-space ellipse

$$\varepsilon = \gamma(s)\,x(s)^2 + 2\alpha(s)\,x(s)\,x'(s) + \beta(s)\,x'(s)^2$$

Liouville: in an ideal storage ring, if there is no beam energy change, the area of the ellipse in the phase space $x - x'$ is constant

$$A = \pi \cdot \varepsilon = \text{const}$$



The area of ellipse, $\pi \cdot \varepsilon$, is an intrinsic beam parameter and cannot be changed by the focal properties.

Learning from the phase-space ellipse

Given the particle trajectory:

$$x(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}\cos(\mu(s) + \mu_0)$$

- the max. amplitude is:

$$\hat{x}(s) = \sqrt{\varepsilon\beta}$$

- the corresponding angle, in $\hat{x}(s)$, can be found putting $\hat{x}(s) = \sqrt{\varepsilon\beta}$ in Eq.

$$\varepsilon = \gamma(s)\,x(s)^2 + 2\alpha(s)\,x(s)\,x'(s) + \beta(s)\,x'(s)^2$$

and solving for x':

$$\varepsilon = \gamma \cdot \epsilon\beta + 2\alpha\sqrt{\varepsilon\beta} \cdot x' + \beta x'^2$$

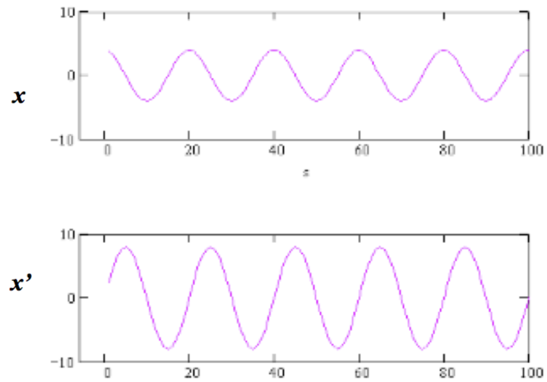
$$\rightarrow \quad \hat{x}' = -\alpha\sqrt{\frac{\varepsilon}{\beta}} \quad \leftarrow$$

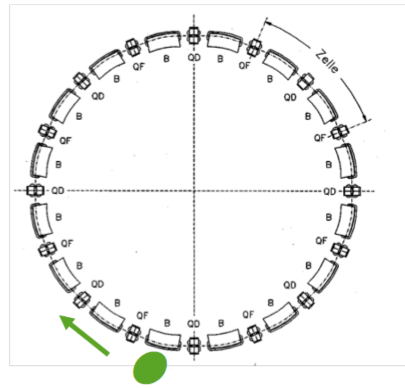
Important remarks:

- A large β-function corresponds to a large beam size and a small beam divergence
- wherever β reaches a maximum or a minimum, $\alpha = 0$ (and $x' = 0$)

Particle tracking in a storage ring

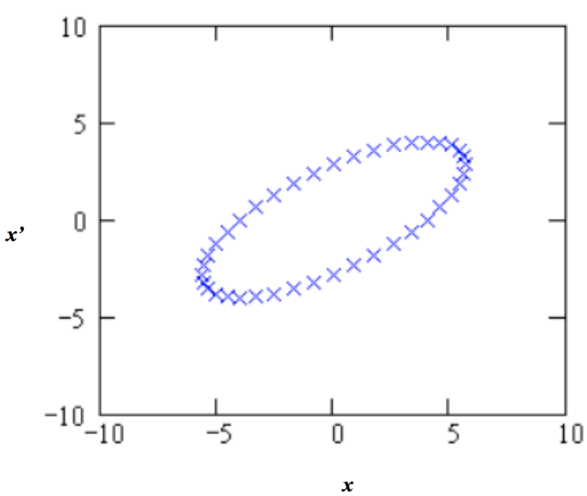
Computation of x and x' for each linear element, according to matrix formalism. We plot x and x' as a function of s

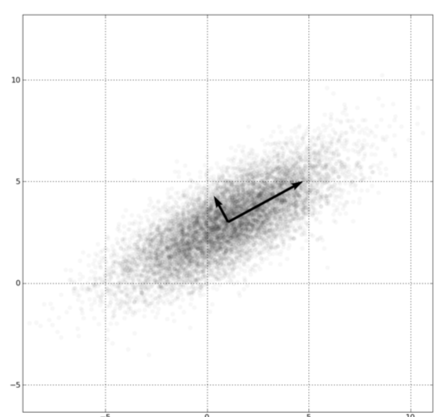




Particle tracking and beam ellipse

For each turn x, x' at a given position s_1 and plot in the phase-space diagram



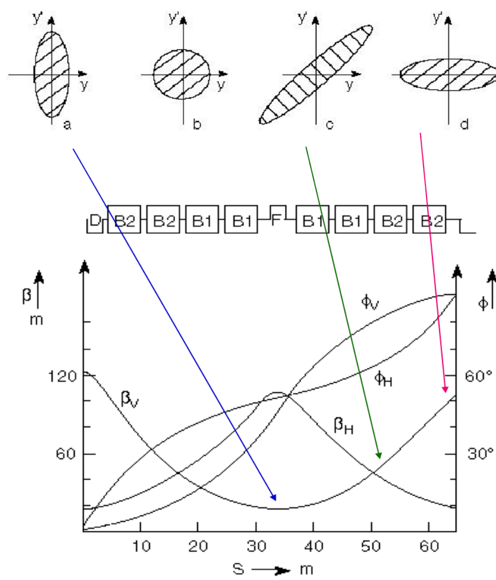


Plane: $x - x'$

Evolution of the phase-space ellipse

Let's repeat the remarks:

- A large β-function corresponds to a large beam size and a small beam divergence
- In the middle of a quadrupole, β is maximum, and $\alpha = 0 \Rightarrow x' = 0$



[VIDEOS!]

Particles distribution, beam matrix, and emittance

To track a beam of particles, let's assume with Gaussian distribution, the beam ellipse can be characterised by a "beam matrix" Σ

The equation of an ellipse can be written in matrix form:

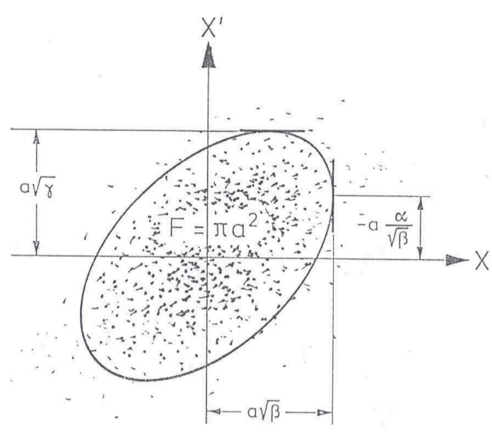
$$X^T \Omega^{-1} X = \varepsilon$$

with $X = \begin{pmatrix} x \\ x' \end{pmatrix}$ and $\Omega = \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}$.

For many particles we can define Σ as:

$$\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} = \begin{pmatrix} \langle x^2 \rangle & \langle x x' \rangle \\ \langle x' x \rangle & \langle x'^2 \rangle \end{pmatrix} = \epsilon \Omega$$

the covariance matrix of the particles distribution represents an ellipse.



- Given a particles distribution, we define the *geometric emittance* ϵ as a function of the ellipse area:

$$\epsilon = \sqrt{\det \Sigma} = \sqrt{\det\left(\text{cov}(\mathbf{x}, \mathbf{x}')\right)} = \text{Area of the ellipse}/\pi$$

  with slope $r_{21} = \sigma_{21}/\sqrt{\sigma_{11}\sigma_{22}}$

- The emittance ϵ is the area covered by the particles in the transverse x-x' phase-space, and it is preserved along the beam line (Liouville's theorem)

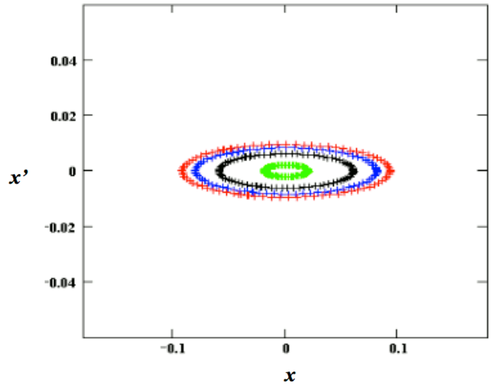
Geometric and Normalised Emittance

Example: LHC
beam parameters in the arc

$\beta(x) \approx 180\ m$

$\varepsilon \approx 5*10^{-10}\ rad \cdot m \qquad (\leftrightarrow 1\sigma)$

$\sigma = \sqrt{\varepsilon\beta} \approx 0.3\ mm$



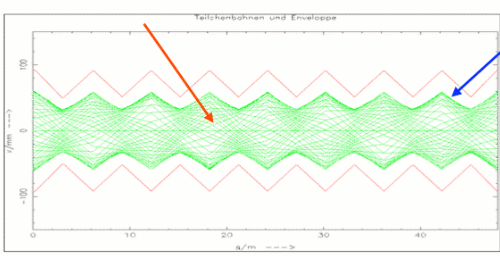
The *geometric emittance* ϵ we have seen so far, utilised e.g. to compute the beam size, is a constant of motion only when there is no acceleration ($P =$ constant).
In presence of acceleration $P_z \to P_z + \Delta P_z$, so that $x' = \frac{P_x}{P_z}$ goes to $x' = \frac{P_x}{P_z + \Delta P_z}$, and the area of the phase space shrinks. We therefore define the *normalised emittance*:

$$\epsilon_N \stackrel{\text{def}}{=} \beta_{\text{relativistic}} \cdot \gamma_{\text{relativistic}} \cdot \epsilon_{\text{geometric}}$$

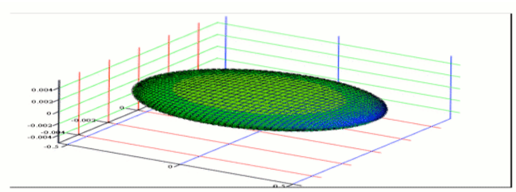
ϵ_N is a constant of motion even in case of acceleration.

Emittance of an ensemble of particles

$x(s) = \sqrt{\varepsilon}\sqrt{\beta(s)} \cdot \cos(\Psi(s) + \phi)$ $\qquad \hat{x}(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}$

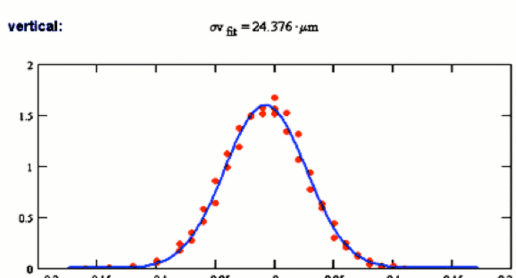


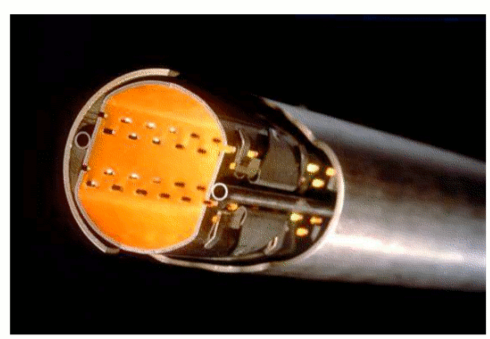
single particle trajectories, N ≈ 10 [11] ***per bunch***



Gauss Particle Distribtion: $\rho(x) = \frac{N \cdot e}{\sqrt{2\pi}\sigma_x} \cdot e^{-\frac{1}{2}\frac{x^2}{\sigma_x^2}}$

particle at distance 1 σ from centre ↔ 68.3 % of all beam particles





LHC: $\sigma = \sqrt{\varepsilon * \beta} = \sqrt{5*10^{-10} m * 180\, m} = 0.3\ mm$

aperture requirements: $r_0 \geq 10 * \sigma$

The transfer matrix M, in terms of Twiss parameters

As we have already seen, a general solution of the Hill's equation is:

$$x(s) = \sqrt{\varepsilon\beta(s)}\cos(\mu(s)+\mu_0)$$
$$x'(s) = -\sqrt{\frac{\varepsilon}{\beta(s)}}\left[\alpha(s)\cos(\mu(s)+\mu_0)+\sin(\mu(s)+\mu_0)\right]$$

Let's remember some trigonometric formulæ:

$$\sin(a\pm b) = \sin a\,\cos b \pm \cos a\,\sin b,$$
$$\cos(a\pm b) = \cos a\,\cos b \mp \sin a\,\sin b,\ \ldots$$

then,

$$x(s) = \sqrt{\varepsilon\beta(s)}\left(\cos\mu(s)\,\cos\mu_0 - \sin\mu(s)\,\sin\mu_0\right)$$
$$x'(s) = -\sqrt{\frac{\varepsilon}{\beta(s)}}\left[\alpha(s)\left(\cos\mu(s)\,\cos\mu_0 - \sin\mu(s)\,\sin\mu_0\right)+\right.$$
$$\left.+\sin\mu(s)\,\cos\mu_0 + \cos\mu(s)\,\sin\mu_0\right]$$

At the starting point, $s(0)=s_0$, we put $\mu(0)=0$. Therefore we have

$$\cos\mu_0 = \frac{x_0}{\sqrt{\varepsilon\beta_0}}$$
$$\sin\mu_0 = -\frac{1}{\sqrt{\varepsilon}}\left(x'_0\sqrt{\beta_0}+\frac{\alpha_0 x_0}{\sqrt{\beta_0}}\right)$$

If we replace this in the formulæ, we obtain:

$$\underline{x(s)} = \sqrt{\frac{\beta_s}{\beta_0}}\left\{\cos\mu_s+\alpha_0\sin\mu_s\right\}\underline{x_0}+\left\{\sqrt{\beta_s\beta_0}\sin\mu_s\right\}\underline{x'_0}$$
$$\underline{x'(s)} = \frac{1}{\sqrt{\beta_s\beta_0}}\left\{(\alpha_0-\alpha_s)\cos\mu_s-(1+\alpha_0\alpha_s)\sin\mu_s\right\}\underline{x_0}+\sqrt{\frac{\beta_0}{\beta_s}}\left\{\cos\mu_s-\alpha_s\sin\mu_s\right\}\underline{x'_0}$$

The linear map follows easily,

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = M\begin{pmatrix} x \\ x' \end{pmatrix}_0 \rightarrow M = \begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}}\left(\cos\mu_s+\alpha_0\sin\mu_s\right) & \sqrt{\beta_s\beta_0}\sin\mu_s \\ \frac{(\alpha_0-\alpha_s)\cos\mu_s-(1+\alpha_0\alpha_s)\sin\mu_s}{\sqrt{\beta_s\beta_0}} & \sqrt{\frac{\beta_0}{\beta_s}}\left(\cos\mu_s-\alpha_s\sin\mu_s\right) \end{pmatrix}$$

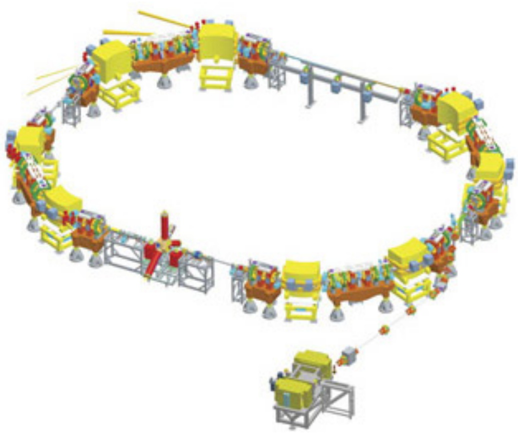
- We can compute the single particle trajectories between two locations in the ring, if we know the α, β, and γ at these positions!
- Exercise: prove that $\det(M)=1$

Periodic lattices

The transfer matrix for a particle trajectory

$$M = \begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}}\left(\cos\mu_s + \alpha_0 \sin\mu_s\right) & \sqrt{\beta_s\beta_0}\sin\mu_s \\ \frac{(\alpha_0-\alpha_s)\cos\mu_s-(1+\alpha_0\alpha_s)\sin\mu_s}{\sqrt{\beta_s\beta_0}} & \sqrt{\frac{\beta_0}{\beta_s}}\left(\cos\mu_s - \alpha_s\sin\mu_s\right) \end{pmatrix}$$

simplifies considerably if we consider one complete turn...



$$M = \begin{pmatrix} \cos\mu_L + \alpha_s \sin\mu_L & \beta_s \sin\mu_L \\ -\gamma_s \sin\mu_L & \cos\mu_L - \alpha_s \sin\mu_L \end{pmatrix}$$

where μ_L is the phase advance per period

$$\mu_L = \int_s^{s+L} \frac{ds}{\beta(s)}$$

Remember: the tune is the phase advance in units of 2π:

$$Q = \frac{1}{2\pi}\oint \frac{ds}{\beta(s)} = \frac{\mu_L}{2\pi}$$

Example: Stability of a 1-turn transfer matrix

The transfer matrix for 1 turn is:

$$M = \begin{pmatrix} \cos\mu_L + \alpha \sin\mu_L & \beta \sin\mu_L \\ -\gamma \sin\mu_L & \cos\mu_L - \alpha \sin\mu_L \end{pmatrix}$$

The stability condition is: $|\mathrm{tr}(M) = 2\cos\mu_L| \leq 2$.

Calculation for N turns:

$$M = \cos\mu_L \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\mathbf{I}} + \sin\mu_L \underbrace{\begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}}_{\mathbf{J}}$$

Given that:

$$\mathbf{I}^2 = \mathbf{I}$$
$$\mathbf{IJ} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} = \mathbf{J}$$
$$\mathbf{JI} = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} = \mathbf{J}$$
$$\mathbf{J}^2 = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} = \begin{pmatrix} \alpha^2 - \gamma\beta & \alpha\beta - \beta\alpha \\ -\gamma\alpha + \alpha\gamma & \alpha^2 - \gamma\beta \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -\mathbf{I}$$

one can compute that:

$$M^N = \mathbf{I}\cos(N\mu_L) + \mathbf{J}\sin(N\mu_L)$$

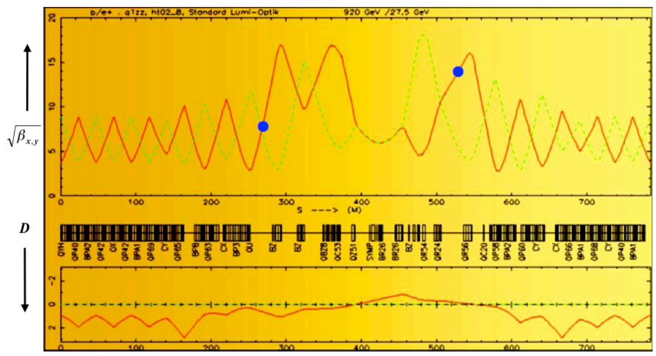
which indeeds provides stable motion:

$$\left|\mathrm{tr}\left(M^N\right) = 2\cos N\mu_L\right| \leq 2$$

The transformation for α, β, and γ

Consider two positions in the storage ring: s_0, s

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = M \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0} \quad \text{with} \quad \begin{array}{l} M = M_{\mathbf{QF}} \cdot M_{\mathbf{D}} \cdot M_{\mathbf{Bend}} \cdot M_{\mathbf{D}} \cdot M_{\mathbf{QD}} \cdot \quad \cdots \\ M = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} \qquad M^{-1} = \begin{pmatrix} S' & -S \\ -C' & C \end{pmatrix} \end{array}$$



Since the Liouville theorem holds, $\varepsilon = \text{const}$:

$$\varepsilon = \beta x'^2 + 2\alpha x x' + \gamma x^2$$
$$\varepsilon = \beta_0 x_0'^2 + 2\alpha_0 x_0 x_0' + \gamma_0 x_0^2$$

We express x_0 and x_0' as a function of x and x':

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_0} = M^{-1} \begin{pmatrix} x \\ x' \end{pmatrix}_s \quad \Rightarrow \quad \begin{array}{l} x_0 = S'x - Sx' \\ x_0' = -C'x + Cx' \end{array}$$

Inserting into ϵ we obtain:

$$\varepsilon = \beta x'^2 + 2\alpha x x' + \gamma x^2$$
$$\varepsilon = \beta_0 \left(-C'x + Cx'\right)^2 + 2\alpha_0 \left(S'x - Sx'\right)\left(-C'x + Cx'\right) + \gamma_0 \left(S'x - Sx'\right)^2$$

We need to sort by x and x':

$$\beta(s) = C^2 \beta_0 - 2SC\alpha_0 + S^2 \gamma_0$$
$$\alpha(s) = -CC'\beta_0 + \left(SC' + S'C\right)\alpha_0 - SS'\gamma_0$$
$$\gamma(s) = C'^2 \beta_0 - 2S'C'\alpha_0 + S'^2 \gamma_0$$

The transformation for α, β, and γ

The beam ellipse transformation in matrix notation:

$$T_{0\to s} = \begin{pmatrix} C^2 & -2SC & S^2 \\ -CC' & SC'+S'C & -SS' \\ C'^2 & -2S'C' & S'^2 \end{pmatrix}$$

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_s = T_{0\to s} \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_0$$

This expression is important, and useful:

1. given the twiss parameters α, β, γ at any point in the lattice we can transform them and compute their values at any other point in the ring
2. the transfer matrix is given by the focusing properties of the lattice elements, the elements of M are just those that we used to compute single particle trajectories

Beam ellipse transformation (another approach)

Let's start from the equation of Σ seen before, now for x_0:

$$X_0^T \Omega_0^{-1} X_0 = \varepsilon \qquad \text{with:} \quad \Omega_0 = \begin{pmatrix} \beta_0 & -\alpha_0 \\ -\alpha_0 & \gamma_0 \end{pmatrix}$$

At a later point if the lattice the coordinates of an individual particle are given using the transfer matrix M from s_0 to s_1:

$$X_1 = M \cdot X_0$$

Solving for X_0 , i.e. $X_0 = M^{-1} \cdot X_1$, and inserting in the first equation above, one obtains:

$$\left(M^{-1} \cdot X_1\right)^T \Omega_0^{-1} \left(M^{-1} \cdot X_1\right) = \varepsilon$$

$$\left(X_1^T \cdot \left(M^T\right)^{-1}\right) \Omega_0^{-1} \left(M^{-1} \cdot X_1\right) = \varepsilon$$

$$X_1^T \cdot \underbrace{\left(M^T\right)^{-1} \Omega_0^{-1} M^{-1}}_{\Omega_1^{-1}} \cdot X_1 = \varepsilon$$

Which gives:

$$\Omega_1 = M \cdot \Omega_0 \cdot M^T$$

Beam matrix and Twiss parameters

The beam matrix is the covariance matrix of the particle distribution

$$\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} = \begin{pmatrix} \langle x^2 \rangle & \langle x x' \rangle \\ \langle x' x \rangle & \langle x'^2 \rangle \end{pmatrix}$$

this matrix can be also expressed in terms of Twiss parameters α, β, γ and of the emittance ϵ:

$$\Sigma = \begin{pmatrix} \langle x^2 \rangle & \langle x x' \rangle \\ \langle x' x \rangle & \langle x'^2 \rangle \end{pmatrix} = \epsilon \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}$$

Given $M = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix}_{0 \to s}$, we can transport the beam matrix, or the twiss parameters, from 0 to s in two equivalent ways:

- Twiss 3×3 transport matrix:

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_s = \begin{pmatrix} C^2 & -2SC & S^2 \\ -CC' & SC' + S'C & -SS' \\ C'^2 & -2S'C' & S'^2 \end{pmatrix} \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_0$$

- Recalling that $\Sigma_s = M \, \Sigma_0 \, M^T$:

$$\begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}_s = M \cdot \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}_0 \cdot M^T$$

Exercise: Twiss transport matrix, T

Compute the Twiss transport matrix, T,

$$T = \begin{pmatrix} C^2 & -2SC & S^2 \\ -CC' & SC' + S'C & -SS' \\ C'^2 & -2S'C' & S'^2 \end{pmatrix}$$

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_s = T \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_0$$

for:

1. the identity matrix: $M = \pm \mathbf{I}$
2. a thin quadrupole with focal length $\pm f$
3. a drift of length L

Summary

Hill's equation: $x''(s) + K(s)\,x(s) = 0, \quad K(s) = K(s+L)$

general solution of the Hill's equation: $x(s) = \sqrt{\varepsilon\beta(s)}\cos(\mu(s) + \mu_0)$

phase advance & tune: $\mu_{12} = \int_{s_1}^{s_2} \frac{ds}{\beta(s)}, \quad Q = \frac{1}{2\pi}\oint \frac{ds}{\beta(s)}$

beam ellipse: $\varepsilon = \gamma(s)\,x(s)^2 + 2\alpha(s)\,x(s)\,x'(s) + \beta(s)\,x'(s)^2$

beam emittance: $\epsilon = \text{Area of the ellipse}/\pi = \sqrt{\det(\text{cov}(\mathbf{x}, \mathbf{x}'))}$

transfer matrix $s_1 \rightarrow s_2$: $M = \begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}}(\cos\mu_s + \alpha_0\sin\mu_s) & \sqrt{\beta_s\beta_0}\sin\mu_s \\ \frac{(\alpha_0-\alpha_s)\cos\mu_s - (1+\alpha_0\alpha_s)\sin\mu_s}{\sqrt{\beta_s\beta_0}} & \sqrt{\frac{\beta_0}{\beta_s}}(\cos\mu_s - \alpha_s\sin\mu_s) \end{pmatrix}$

stability criterion: $|\text{trace}(M)| \leq 2$

Summary: The transfer matrix M

- Transformation of particle coordinates:

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = M_{2\times 2}\begin{pmatrix} x \\ x' \end{pmatrix}_0$$

- using matrix notation in terms of the focusing strength K:

$$M = \begin{pmatrix} \cos\left(\sqrt{K}L\right) & \frac{1}{\sqrt{K}}\sin\left(\sqrt{K}L\right) \\ -\sqrt{K}\sin\left(\sqrt{K}L\right) & \cos\left(\sqrt{K}L\right) \end{pmatrix} = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix}$$

- in Twiss form, and for a periodic lattice (over a period):

$$M(s) = \begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}}(\cos\mu + \alpha_0\sin\mu) & \sqrt{\beta_s\beta_0}\sin\mu \\ \frac{(\alpha_0-\alpha_s)\cos\mu - (1+\alpha_0\alpha_s)\sin\mu}{\sqrt{\beta_s\beta_0}} & \sqrt{\frac{\beta_0}{\beta_s}}(\cos\mu - \alpha_s\sin\mu) \end{pmatrix}$$

  for a period: (1) phase advance: $\cos\mu = \frac{1}{2}\text{trace}(M)$; (2) stability condition: $|\text{trace}(M)| \leq 2$

- Transport of Twiss parameters:

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_s = \begin{pmatrix} C^2 & -2SC & S^2 \\ -CC' & SC' + S'C & -SS' \\ C'^2 & -2S'C' & S'^2 \end{pmatrix}\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_0$$

Part 3.

Lattice design

Lattice design in particle accelerators

Or..."how to build a storage ring"

High energy accelerators are mostly circular machines
we need to juxtapose a number of **dipole** magnets, to bend the design orbit to a closed ring, then add **quadrupole** magnets (FODO cells) to focus the beam transversely

The geometry of the system is determined by the following equality

centrifugal force = Lorentz force

$$
\begin{aligned}
\text{Lorentz force} \quad F_L &= evB \\
\text{Centrifugal force} \quad F_{\text{centr}} &= \frac{\gamma m v^2}{\rho} \\
\frac{\gamma m v^{\cancel{2}}}{\rho} &= e\cancel{v}B
\end{aligned}
$$

$$\frac{P}{q} = B\rho$$

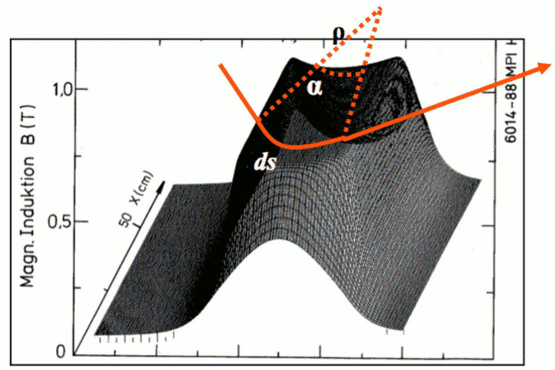
$B\rho$ is the well known beam ridigity

Lattice design: the magnetic guide

$$B\rho = P/q$$

Circular orbit: the dipole magnets define the geometry

$$\theta = \frac{ds}{\rho} \approx \frac{B\,L}{B\rho}$$



field map of a storage ring dipole magnet

The angle spanned in one revolution must be 2π, so, for a full circle:

$$\theta = \frac{\int B dl}{B\rho} = 2\pi \quad \rightarrow \quad \int B dl \approx N L_{\text{Bend}} B = 2\pi \frac{P}{q}$$

this defines the integrated dipole field around the machine.

Note that usually $\frac{\Delta B}{B} \approx 10^{-4}$ is required!

7000 GeV proton storage ring

$N = 1232$ ***dipole magnets***

$L_{Bend} = 15$ ***m***

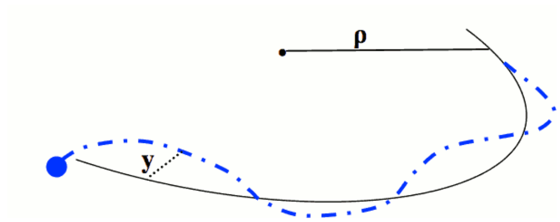
$q = +e$

$$\int B dl \approx N L_{Bend} B = 2\pi p/e$$

$$B \approx \frac{2\pi \cdot 7000 \cdot 10^{9}\ \mathbf{eV}}{1232 \cdot 15\ \mathbf{m} \cdot 3 \cdot 10^{8}\ \frac{m}{s}\ e} = 8.3\ \mathbf{T}$$

Focusing forces for single particles

$$x'' + Kx = 0$$



$$K = 1/\rho^2 + k \quad \text{hor. plane}$$
$$K = -k \quad \text{vert. plane}$$

$$\left.\begin{array}{lll} \text{dipole magnet} & \frac{1}{\rho} & = \frac{B}{P/q} \\ \text{quadrupole magnet} & k & = \frac{g}{P/q} \end{array}\right\}$$

Example: the LHC ring

Bending radius: $\rho = 2.53$ km

Quad gradient: $g = 220$ T/m

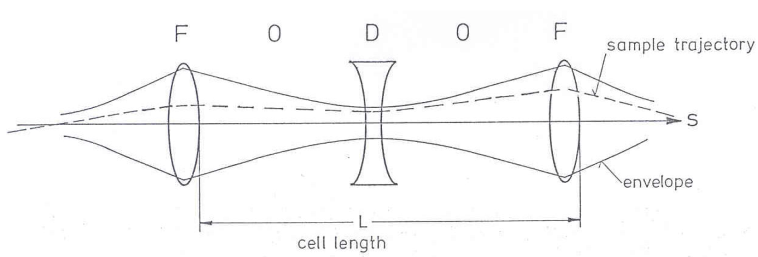
$k = 9.4 \cdot 10^{-3}$ m^{-2}

$1/\rho^2 = 1.3 \cdot 10^{-7}$ m^{-2}

For estimates, in large accelerators, the weak focusing term $1/\rho^2$ can in general be neglected

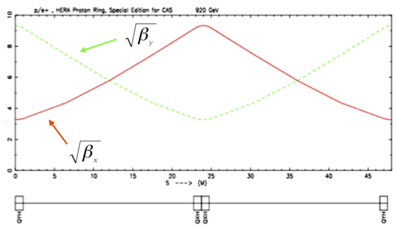
The FODO lattice

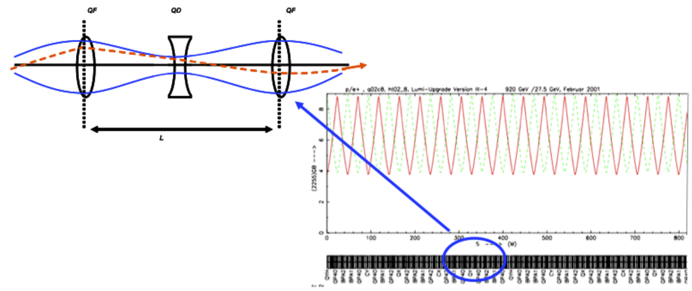
- Most high energy accelerators or storage rings have a periodic sequence of quadrupole magnets of alternating polarity in the arcs



- A magnet structure consisting of focusing and defocusing quadrupole lenses in alternating order with "nothing" in between
- Nota bene: "nothing" here means the elements that can be neglected on first sight: drift, bending magnet, RF structures ... and experiments...

Periodic solution in a FODO Cell





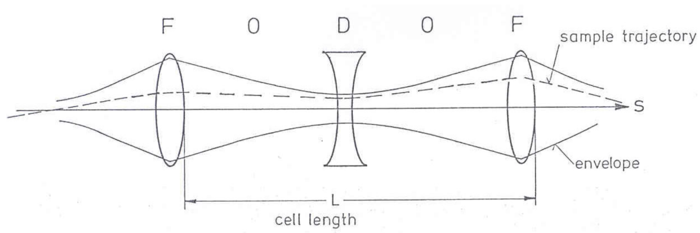
Output of MAD-X

Nr	Type	Length	Strength	β_x	α_x	φ_x	β_z	α_z	φ_z
		m	1/m2	m		1/2π	m		1/2π
0	IP	0,000	0,000	11,611	0,000	0,000	5,295	0,000	0,000
1	QFH	0,250	-0,541	11,228	1,514	0,004	5,488	-0,781	0,007
2	QD	3,251	0,541	5,488	-0,781	0,070	11,228	1,514	0,066
3	QFH	6,002	-0,541	11,611	0,000	0,125	5,295	0,000	0,125
4	IP	6,002	0,000	11,611	0,000	0,125	5,295	0,000	0,125

QX= 0,125 QZ= 0,125 → $0.125 * 2\pi = 45^0$

The FODO cell

The transfer matrix gives all the information we need.



In thin-lens approximation, we have:

$$M_{\mathrm{F}} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}; \qquad M_{\mathrm{O}} = \begin{pmatrix} 1 & L/2 \\ 0 & 1 \end{pmatrix}; \qquad M_{\mathrm{D}} = \begin{pmatrix} 1 & 0 \\ +\frac{1}{f} & 1 \end{pmatrix}$$

the transformation matrix of the cell is:

$$M_{\mathrm{FODO}} = M_{\mathrm{F}} \cdot M_{\mathrm{O}} \cdot M_{\mathrm{D}} \cdot M_{O}$$

(notice that you can also write $M = M_{\mathrm{F}/2} \cdot M_{\mathrm{O}} \cdot M_{\mathrm{D}} \cdot M_{O} \cdot M_{\mathrm{F}/2}$, or other cyclic permutations), which corresponds to

$$M_{\mathrm{FODO}} = \begin{pmatrix} 1+\frac{L}{2f} & L+\frac{L^2}{4f} \\ -\frac{2L}{f^2} & 1-\frac{L}{2f}-\frac{L^2}{4f^2} \end{pmatrix}$$

The FODO cell (cont.)

If we compare the previous matrix with the Twiss representation over one period,

$$M_{\text{FODO}} = \begin{pmatrix} 1+\frac{L}{2f} & L+\frac{L^2}{4f} \\ -\frac{2L}{f^2} & 1-\frac{L}{2f}-\frac{L^2}{4f^2} \end{pmatrix}$$

$$M_{\text{Twiss}} = \begin{pmatrix} \cos\mu+\alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu-\alpha\sin\mu \end{pmatrix} = \cos\mu \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{I} + \sin\mu \underbrace{\begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}}_{J}$$

we can derive interesting properties.

- Phase advance

$$\cos\mu = \frac{1}{2}\text{trace}\,(M) = 1-\frac{L^2}{8f^2}$$

remembering that $\cos\mu = 1-2\sin^2\frac{\mu}{2}$

$$\left|\sin\frac{\mu}{2}\right| = \frac{L}{4f}$$

This equation allows to compute the phase advance per cell from the cell length and the focal length of the quadrupoles.

The FODO cell (cont.)

- Example: compute the focal length in order to have a phase advance of 90° per cell

$$f = \frac{1}{\sqrt{2}}\frac{L}{2}$$

e.g. an emittance measurement station

- Stability requires that $|\cos\mu| < 1$, that is

$$\frac{L}{4f} < 1 \qquad \rightarrow \qquad \text{stability is for:} \quad f > L/4 \quad (\text{or } L < 4f)$$

- Compute the phase advance per cell from the transfer matrix: From $\cos\mu = \frac{1}{2}\text{trace}\,(M)$

$$\mu = \arccos\left(\frac{1}{2}\text{trace}\,(M)\right)$$

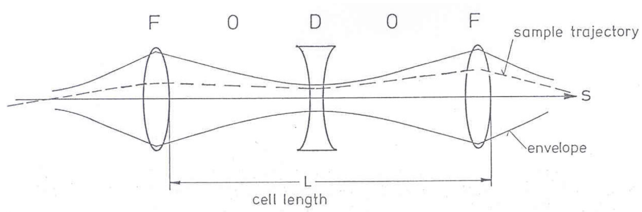
- Compute β-function and α parameter

$$\beta = \frac{M_{12}}{\sin\mu}$$

$$\alpha = \frac{M_{11}-\cos\mu}{\sin\mu}$$

The FODO cell: useful formulæ

For a FODO cell like in figure, with two thin quads separated by length $L/2$



one has:

$$f = \frac{L}{4 \sin \frac{\mu}{2}}$$

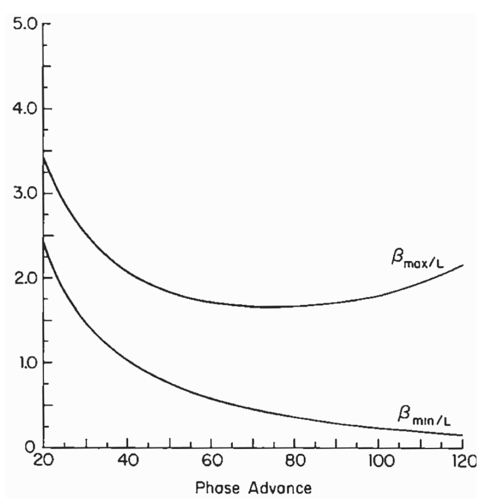
$$\beta^{\pm} = \frac{L\left(1 \pm \sin \frac{\mu}{2}\right)}{\sin \mu}$$

$$\alpha^{\pm} = \frac{\mp 1 - \sin \frac{\mu}{2}}{\cos \frac{\mu}{2}}$$

$$D^{\pm} = \frac{L\theta\left(1 \pm \frac{1}{2} \sin \frac{\mu}{2}\right)}{4 \sin^2 \frac{\mu}{2}}$$

θ is the total bending angle of the whole cell.

β_{max} and β_{min} as a function of μ

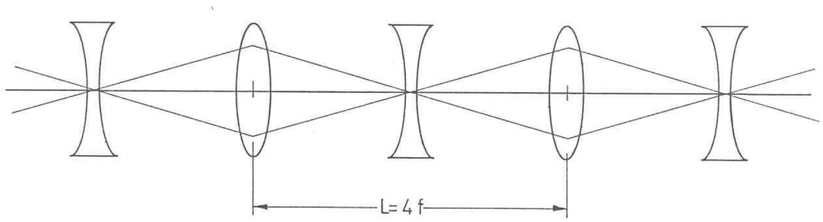


- The minimum of β_{max} can be found at $\mu_{\text{min}} = 76.345°$. (Remember: μ_{min} is such that $\frac{d\beta(\mu_{\text{min}})}{d\mu} = 0$) $\Leftarrow$ this applies only for the cases where $\epsilon_y \gg \epsilon_x$, or $\epsilon_x \gg \epsilon_y$.
- In cases where $\epsilon_x \approx \epsilon_y$ one needs to minimise $\beta_x + \beta_y$ (i.e. find the zero of $\frac{d(\beta_x+\beta_y)}{d\mu}$), which has solution $\mu_{\text{min}} = 90°$.

The FODO cell (example 1)

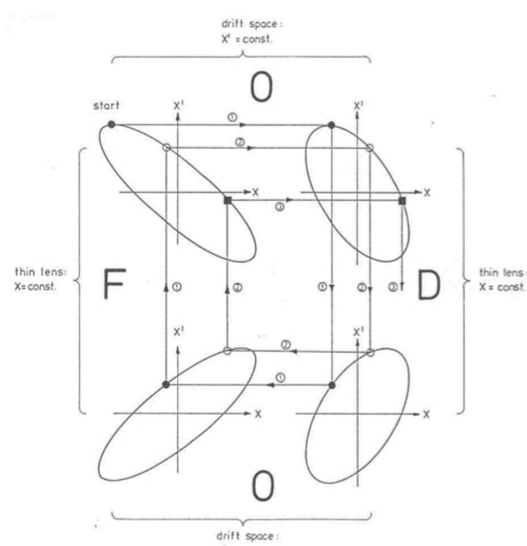
Stability condition $4f \geq L$, has a simple interpretation:

- It is well known from optics that an object at a distance $a = 2f$ from a focusing lens has its image at $b = 2f$



- The defocusing lenses have no effect if a point-like object is located exactly on the axis at distance $2f$ from a focusing lens, because they are traversed on the axis
- If however the lens system is moved further apart ($L > 4f$), this is no more true and the divergence of the light or particle beam is increased by every defocusing lens

The FODO cell (example 2)



- Phase space dynamics in a simple circular accelerator consisting of one FODO cell with two 180° bending magnets located in the drift spaces (the O's)
- The periodicity of α, β, and γ is reflected by the fact that the phase-space ellipse is transformed into itself after each turn
- An individual particle trajectory, however, which starts, for instance, somewhere on the ellipse at the exit of the focusing quadrupole (small circle), is seen to move on the ellipse from turn to turn as determined by the phase angle μ
- Thus, an individual particle trajectory is not periodic, while the envelope of a whole beam is

Exercise: phase-advance of a transfer line

We have seen that the phase advance of a periodic system is given by:

$$\mu = \arccos\left(\frac{1}{2}\text{trace}\,(M)\right)$$

<u>Question</u>: given the transfer matrix M of an arbitrary lattice, and knowing the initial Twiss parameters α_0 and β_0; compute the phase advance μ:

$$\mu = ?$$

Hint: M can be written as:

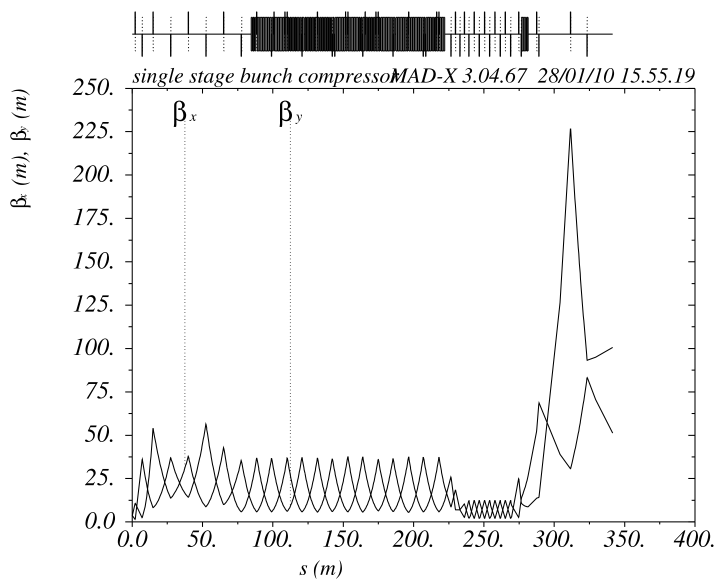
$$M(s) = \begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}}\left(\cos\mu + \alpha_0 \sin\mu\right) & \sqrt{\beta_s\beta_0}\sin\mu \\ \frac{(\alpha_0-\alpha_s)\cos\mu-(1+\alpha_0\alpha_s)\sin\mu}{\sqrt{\beta_s\beta_0}} & \sqrt{\frac{\beta_0}{\beta_s}}\left(\cos\mu - \alpha_s\sin\mu\right) \end{pmatrix}$$

Non-periodic beam optics

- In the previous sections the Twiss parameters α, β, γ, and μ have been derived for a periodic, circular accelerator. The condition of periodicity was essential for the definition of the beta function (Hill's equation)
- Often, however, a particle beam moves only **once** along a **beam transfer line**, but one is nonetheless interested in quantities like beam envelopes and beam divergence
- In a circular accelerator α, β, and γ are completely determined by the magnet optics and the condition of periodicity (beam properties are not involved - only the beam emittance is chosen to match the beam size)
- In a transfer line, α, β, and γ are no longer uniquely determined by the transfer matrix, but they also depend on initial conditions which have to be specified in an adequate way

Non-periodic optics: ILC bunch compressor (EX1)

Optics of a non-periodic system including non-periodic optics. “Matching” sections connect parts with different periodic conditions.



The matrix

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_s = M_{3\times 3} \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_0$$

with

$$M_{3\times 3} = \begin{pmatrix} C^2 & -2SC & S^2 \\ -CC' & SC' + S'C & -SS' \\ C'^2 & -2S'C' & S'^2 \end{pmatrix}$$

allows to compute the magnets parameters for the matching sections

Note: even if the β functions are very large, the beam size keeps small: $\sigma = \sqrt{\beta\epsilon}$, with

$$\epsilon_y = \frac{\epsilon_{y,N}}{\gamma_{\text{rel}}} = \frac{5 \times 10^{-9}\ \text{m}}{5\ \text{GeV}/\ 0.5\ \text{MeV}} = 10^{-13}\ \text{m}$$

Non-periodic optics: final focus of a HEP experiment (EX2)