Entanglement

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Entanglement is what makes quantum mechanics different from classical mechanics.

not entangled $|\uparrow\uparrow\rangle$

entangled $\frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right)$

Entanglement: correlations between outcomes of simultaneous, acausal measurements.

To quantify the amount of entanglement between two subsystems A and B entanglement entropy is a useful notion.

~number of EPR pairs between A and B

Definition:

Given a state ρ on $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ the entanglement entropy of A wrt B is

$$S(A) = -\mathrm{Tr}(\rho_A \log \rho_A) \qquad \rho_A = \mathrm{Tr}_{\mathcal{H}_B} \rho$$

ρ_A is the reduced density matrix

$$\rho = \rho_{ab}^{a'b'} |a\rangle |b\rangle \langle b'| \langle a'| \Longleftarrow \rho_A = \rho_{ab}^{a'b} |a\rangle \langle a'|$$

$$\frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right)$$

$$\rho_A = \frac{1}{2}|\uparrow\rangle\langle\uparrow| + \frac{1}{2}|\downarrow\rangle\langle\downarrow|$$

$$S(A) = -\frac{1}{2}\log\frac{1}{2} - \frac{1}{2}\log\frac{1}{2} = \log 2$$

Quantum information theory is (among others) about the interpretation and uses of this notion of entropy.

Eigenvalues of ρ_A are p_i :

$$S_A = -\sum_i p_i \log p_i$$

Classical interpretation (Shannon):

$$S_A = \lim_{N \to \infty} \frac{1}{N} \log \begin{pmatrix} & N & \\ p_1 N \, p_2 N & \ldots & p_k N \end{pmatrix}$$

This is the amount of information per bit in a string where the probability that i appears is p_i .

Entanglement entropy inequalities:

$$|S(A) - S(B)| \leq S(A \cup B) \leq S(A) + S(B)$$

$$S(A \cup B) + S(A \cap B) \leq S(A) + S(B)$$

(strong subadditivity)

Some other relevant quantum information theoretic quantities:

$$I(A,B) = S(A) + S(B) - S(A \cup B)$$

Mutual information

$$S^{(n)}(\rho) = \frac{1}{1-n} \log \mathrm{Tr}\rho^n$$

Renyi entropy

$$S(\rho|\sigma) = \mathrm{Tr}(\rho \log \rho) - \mathrm{Tr}(\rho \log \sigma)$$

Relative entropy

$$K = -\log \rho$$

Modular Hamiltonian

$$S_\alpha(\rho||\sigma) = \frac{1}{\alpha - 1} \log \left[\mathrm{Tr} \left(\sigma^{\frac{1-\alpha}{2\alpha}} \rho \sigma^{\frac{1-\alpha}{2\alpha}} \right)^\alpha \right]$$

Quantum Renyi relative entropy

Some other relevant quantum information theoretic quantities:

FINITE

$$I(A,B) = S(A) + S(B) - S(A \cup B)$$

Mutual information

REPLICA

$$S^{(n)}(\rho) = \frac{1}{1-n} \log \mathrm{Tr} \rho^n$$

Renyi entropy

COMPARE

$$S(\rho|\sigma) = \mathrm{Tr}(\rho \log \rho) - \mathrm{Tr}(\rho \log \sigma)$$

Relative entropy

THERMAL

$$K = -\log \rho$$

Modular Hamiltonian

$$S_\alpha(\rho||\sigma) = \frac{1}{\alpha - 1} \log \left[\mathrm{Tr} \left(\sigma^{\frac{1-\alpha}{2\alpha}} \rho \sigma^{\frac{1-\alpha}{2\alpha}} \right)^\alpha \right]$$

Quantum Renyi relative entropy

Monotonicity of relative entropy

$$S(\rho_A|\sigma_A) \leq S(\rho|\sigma)$$

The less you know, the more similar two states are. Cf RG-flow.

Implies strong subadditivity

Idea of proof (Petz):

Define

$$\begin{array}{rclcrcl} \Delta(X) & = & \sigma X \rho^{-1} & & B(\mathcal{H}) & \to & B(\mathcal{H}) \\ \Delta_A(X_A) & = & \sigma_A X \rho_A^{-1} & & B(\mathcal{H}_A) & \to & B(\mathcal{H}_A) \end{array}$$

$$V(X_A) = (X_A \rho_A^{-1/2} \otimes \mathbb{I})\rho^{1/2} \qquad B(\mathcal{H}_A) \to B(\mathcal{H})$$

$$V^* \Delta V = \Delta_A$$

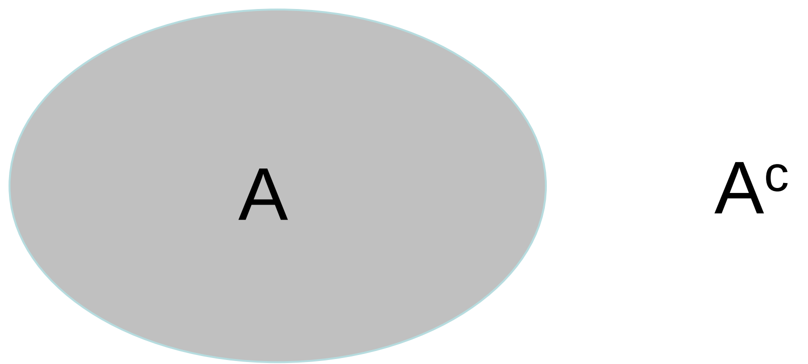
$$\frac{1}{\Delta_A + t} = \frac{1}{V^* \Delta V + t} \leq V^* \frac{1}{\Delta + t} V$$

$$\langle \sqrt{\rho_A} \left(\frac{1}{\Delta_A + t} - \frac{1}{1+t} \right) \sqrt{\rho_A} \rangle \leq \langle \sqrt{\rho} \left(\frac{1}{\Delta + t} - \frac{1}{1+t} \right) \sqrt{\rho} \rangle$$

Integrate over t and done……

Entanglement entropy in quantum field theory

Typical situation: consider degrees of freedom associated to a spatial domain and its complement



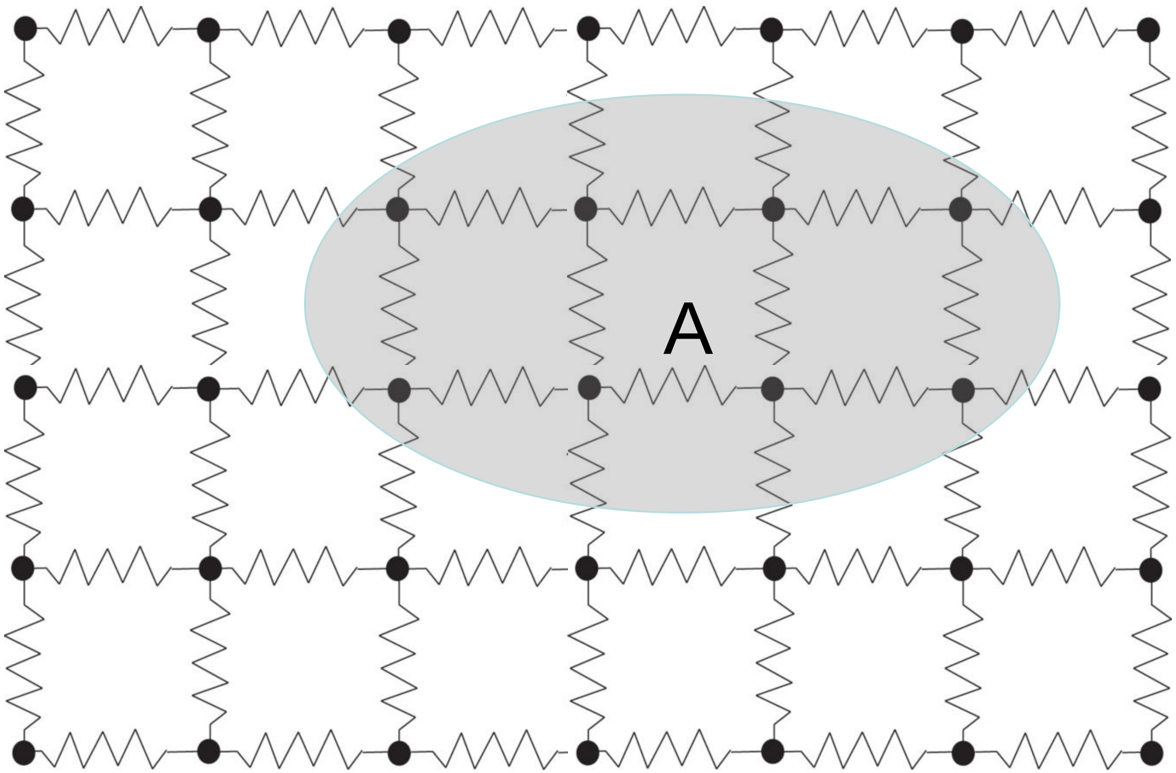
$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{A^c}$$

Bombelli, Koul, Lee, Sorkin `86
Srednicki `93

(various caveats)

Entanglement entropy = infinite in continuum field theory.

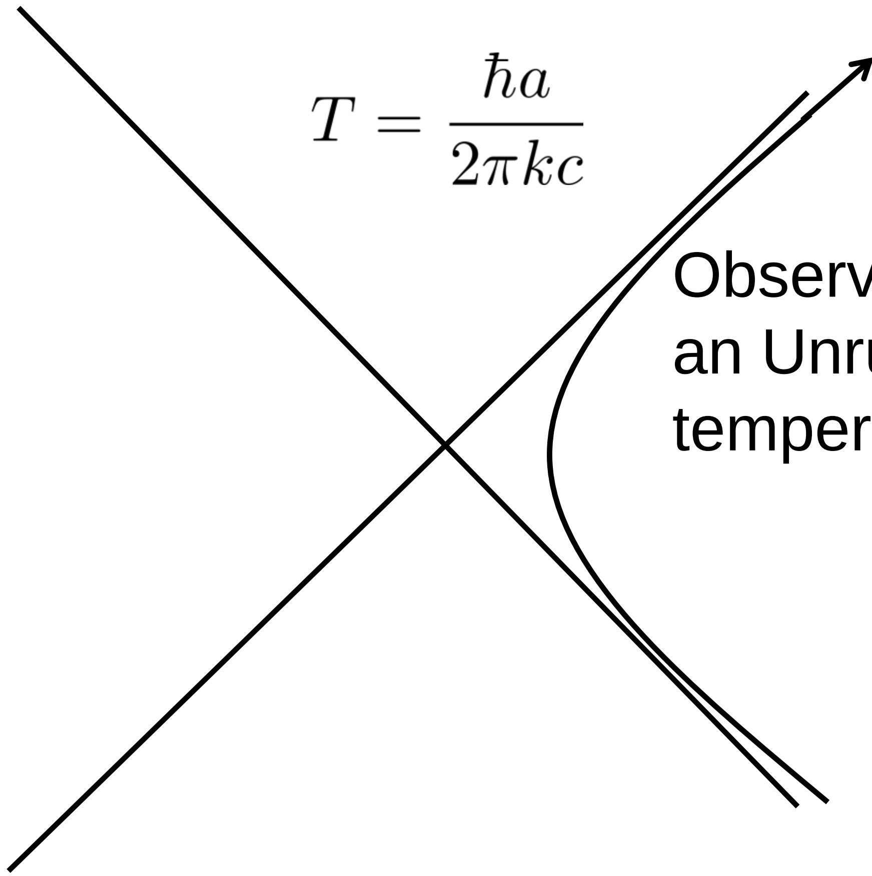
Needs to be regulated: short distance regulator a.



$$S(A) = \frac{\text{area}(A)}{a^{D-2}} + \ldots$$

Another example of entanglement: accelerated observers

Accelerated observers do not see left wedge. Effective ground state = reduced density matrix.



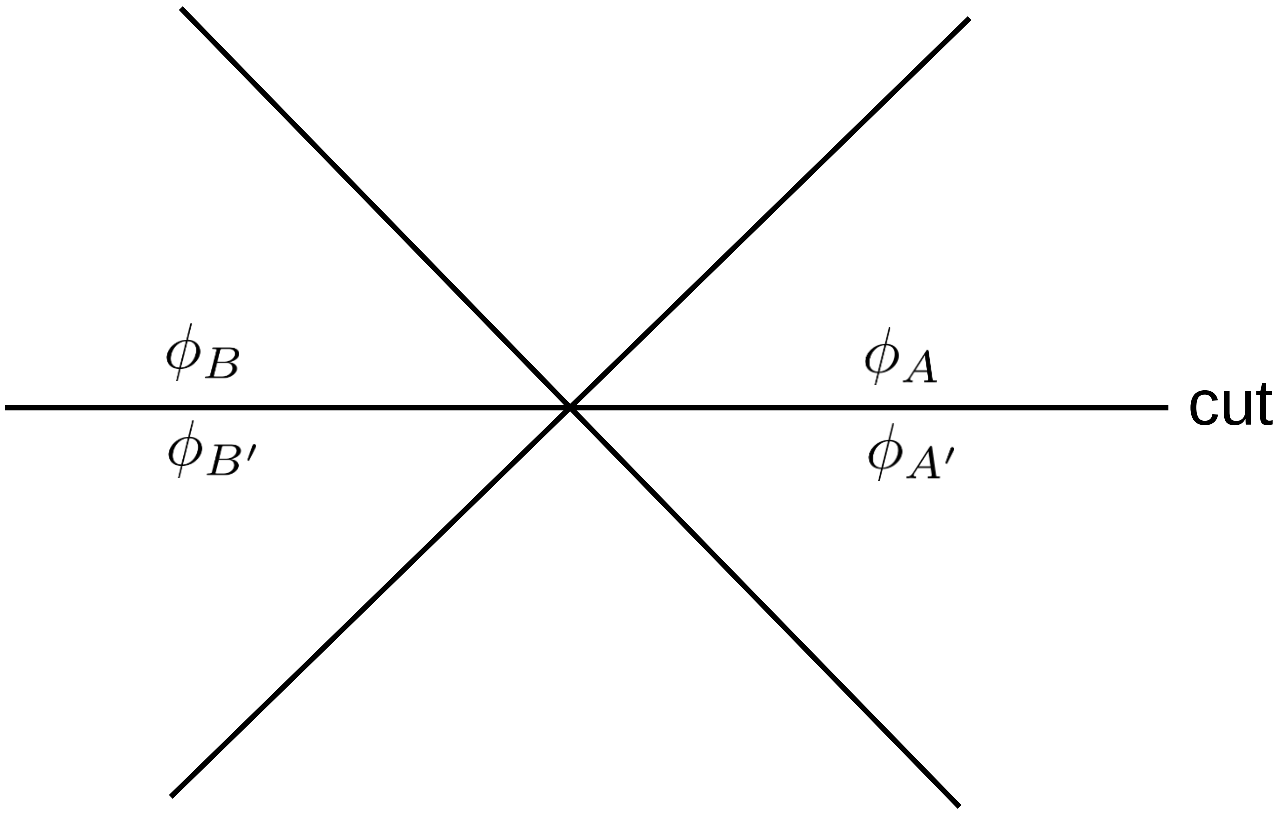
$$T = \frac{\hbar a}{2\pi k c}$$

Observer measures an Unruh temperature

$$|0\rangle_M \, {}_M\langle 0| = \sum_E e^{-\frac{E}{T}} |E\rangle_L |E\rangle_R \, {}_R\langle E|_L\langle E|$$

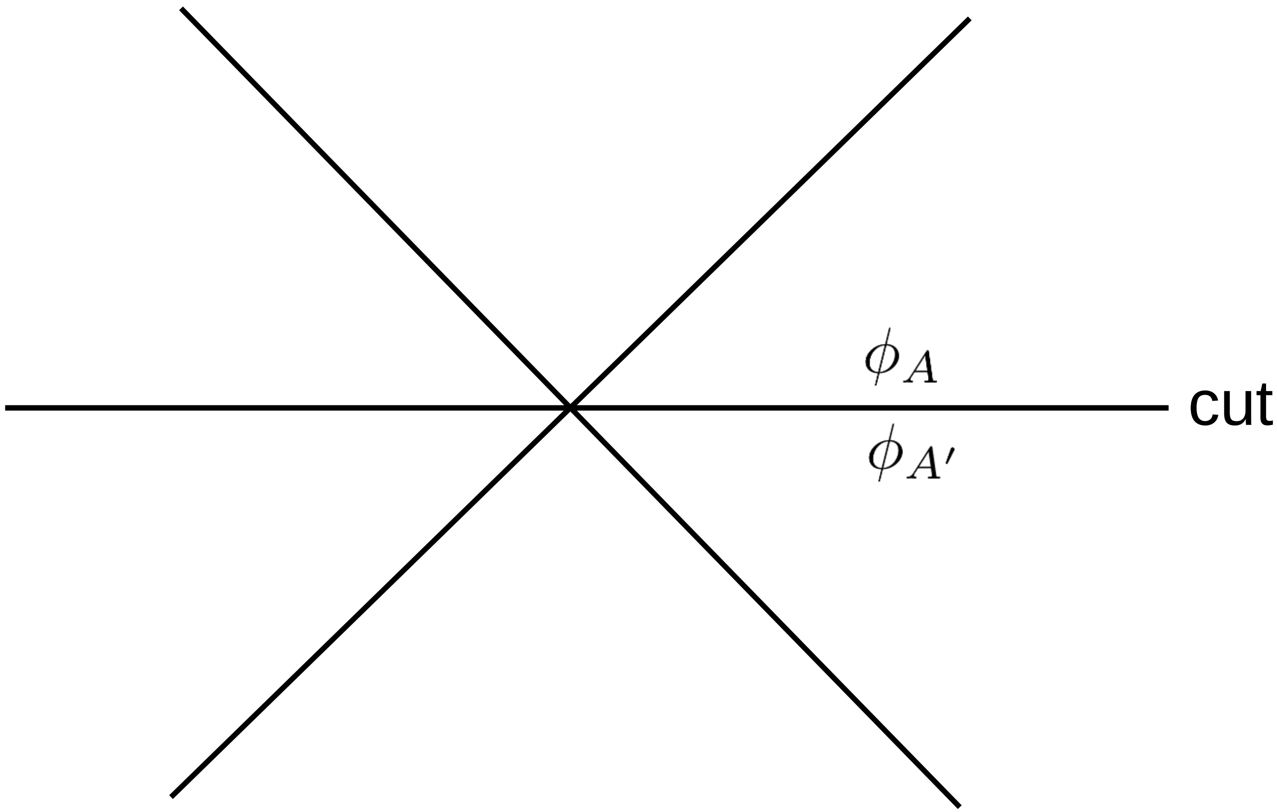
Path integral with boundary conditions computes transition elements.

$$|0\rangle\langle 0| = \sum_{\phi_A,\phi_{A'},\phi_B,\phi_{B'}} |\phi_{B'}\rangle|\phi_{A'}\rangle\langle\phi_A|\langle\phi_B|$$

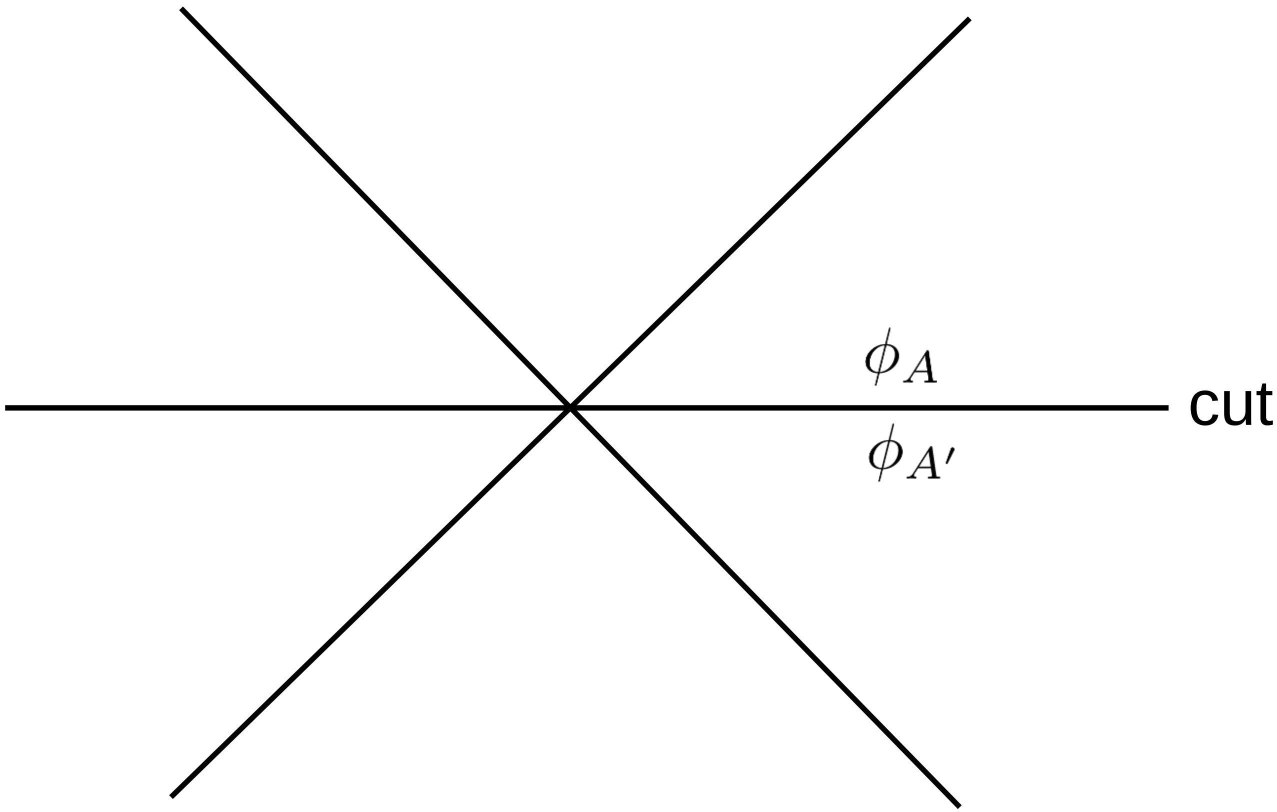


Path integral with boundary conditions computes transition elements.

$$\rho_A = \sum_{\phi_A, \phi_{A'}} |\phi_{A'}\rangle\langle\phi_A|$$



$$\rho_A = \sum_{\phi_A, \phi_{A'}} |\phi_{A'}\rangle\langle\phi_A| = e^{2\pi \frac{\partial}{\partial \varphi_E}} = e^{-2\pi H_{\rm obs}}$$



Unfortunately, it is in general very hard to compute entanglement entropy in a qft (even in a free qft).

Entanglement entropy is also not an observable.

So why care?

- Probe of ground states (e.g. diagnostic of Fermi surfaces).
- Probe of topological phases with no local order parameter.
- Helps determine properties of critical points for e.g. spin chains.
- Helps construct variational ansatze for ground states.
- Useful notion in quantum information theory.
- Sheds light on nature of thermalization/entropy production.
- Can be computed for strongly coupled field theories with a holographic dual.
- Use to prove interesting properties of QFT’s
- Seems to play a fundamental role in quantum gravity and holography.

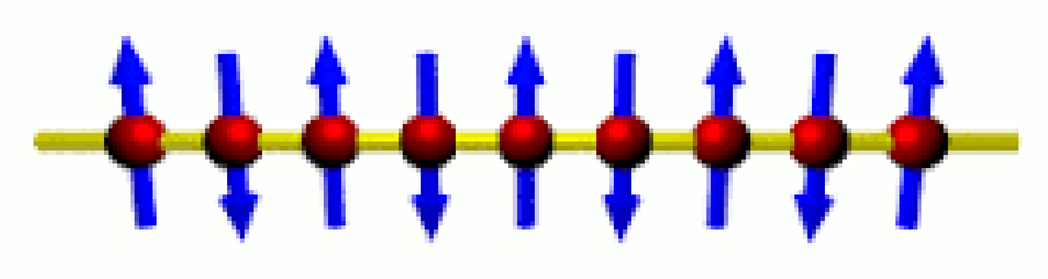
Exception: 1+1 dimensional conformal field theory.

$$S = \frac{c}{3} \log \left(\frac{\ell}{a_{UV}} \right)$$

Holzhey, Larsen, Wilczek `94
Cardy, Calabrese `07

ℓ: length of interval

c: central charge



Compute entanglement entropy as function of ℓ and extract c

Topological entanglement entropy
In 2+1 dimensions.

Levin, Wen `05
Kitaev, Preskill ‘05

Entanglement entropy for a disc with radius ℓ

$$S = \alpha \frac{\ell}{a_{UV}} - \gamma + \ldots$$

area law

universal coefficient

General structure for spherical domain

d=even
c~ conformal anomaly
c-theorem

$$S = a_{d-2}\left(\frac{\ell}{a_{UV}}\right)^{d-2} + a_{d-4}\left(\frac{\ell}{a_{UV}}\right)^{d-4} + \ldots + \left\{ \begin{array}{c} c\log\frac{\ell}{a_{UV}} \\ c \end{array} \right. + \ldots$$

In 1+1 and 2+1 dimensions can prove c-theorem and F-theorem using entanglement (Casini, Huerta)

d=odd
c~ F
F-theorem

The divergences in entanglement entropy look very similar to those in quantum field theory.

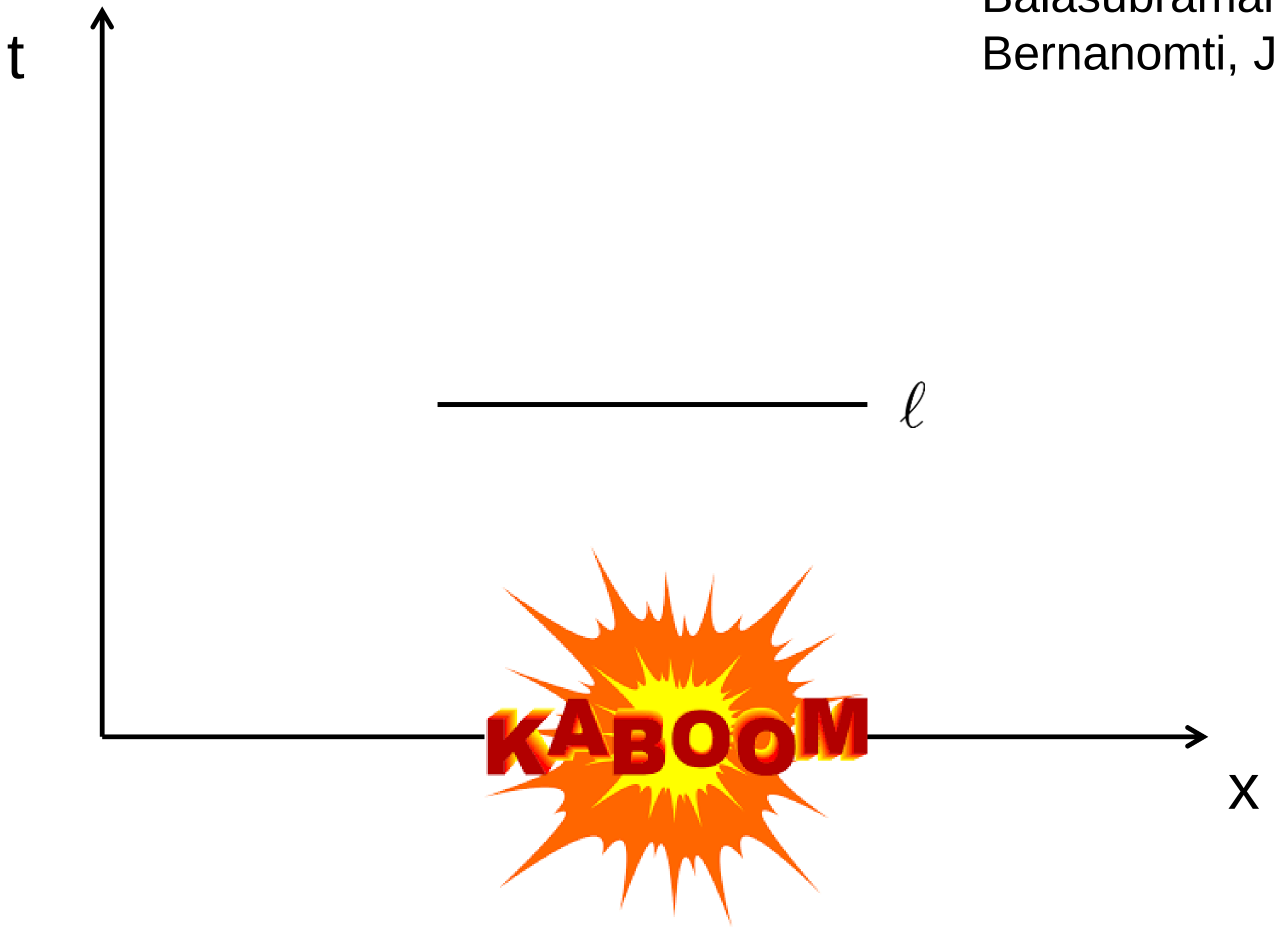
It is tempting to holographically renormalize entanglement entropy (e.g. Taylor, Woodhead).

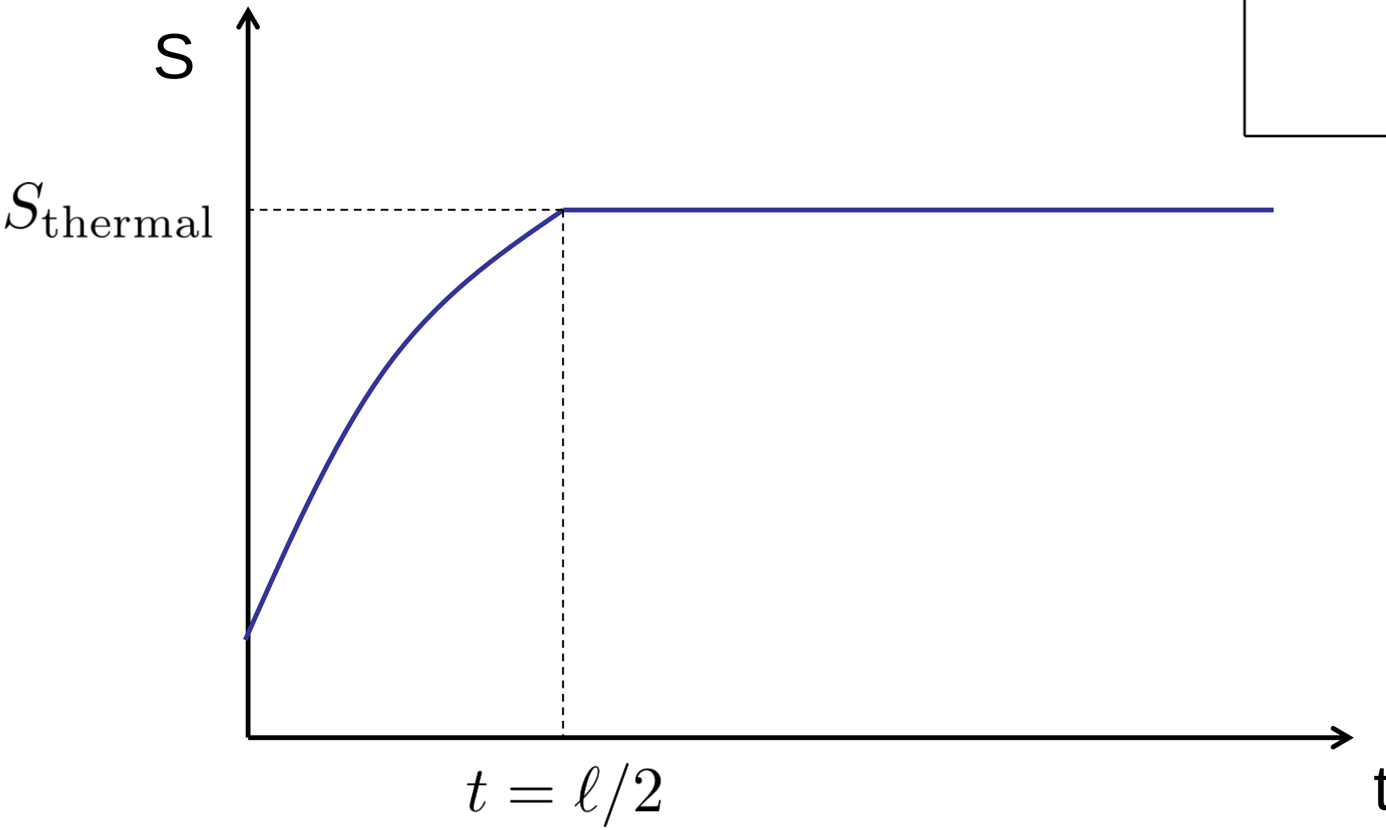
From the effective action and replica trick point of view this looks natural, but interpretation from quantum information theoretic point of view is less clear, unless we compute finite quantities.

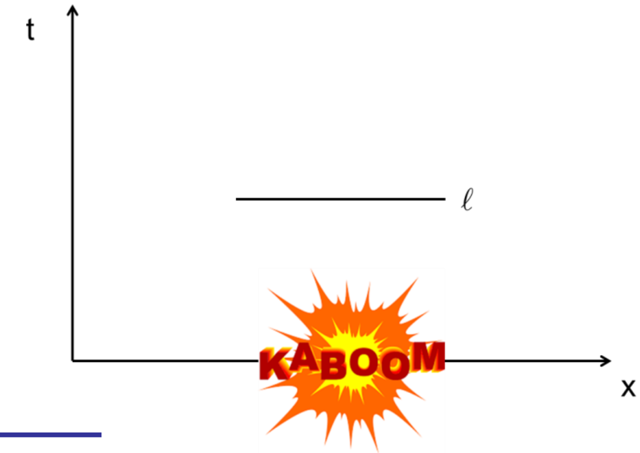
There are also theories where one can apply the replica trick, like Chern-Simons theories, but a quantum information theoretic understanding of the result does not exist.

Thermalization after a quench

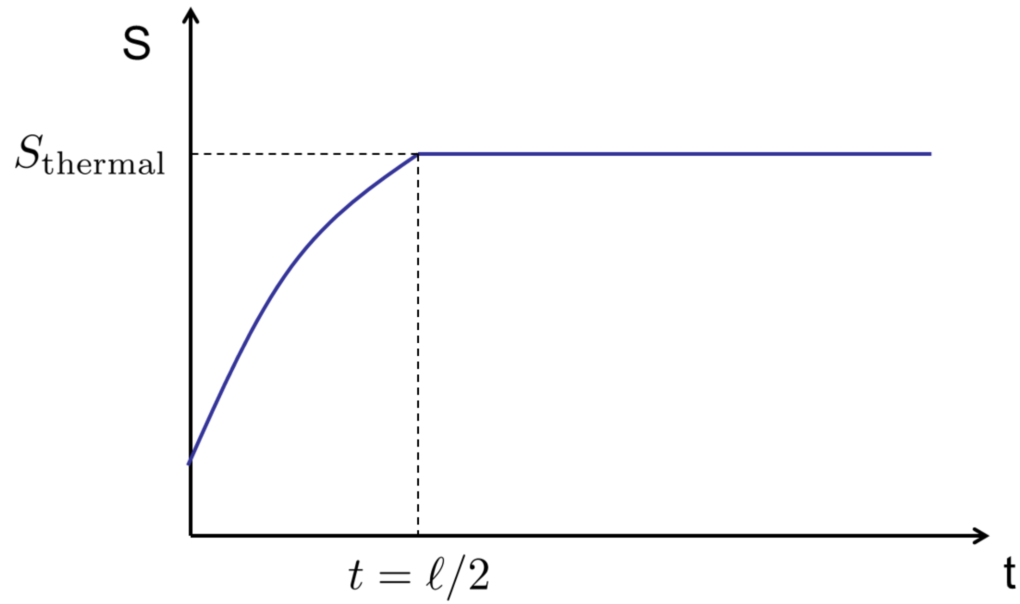
Cardy, Calabrese `07
Abajo-Arrastia, Aparicio, Lopez ‘10
Balasubramanian, Bernanomti, JdB et al `11

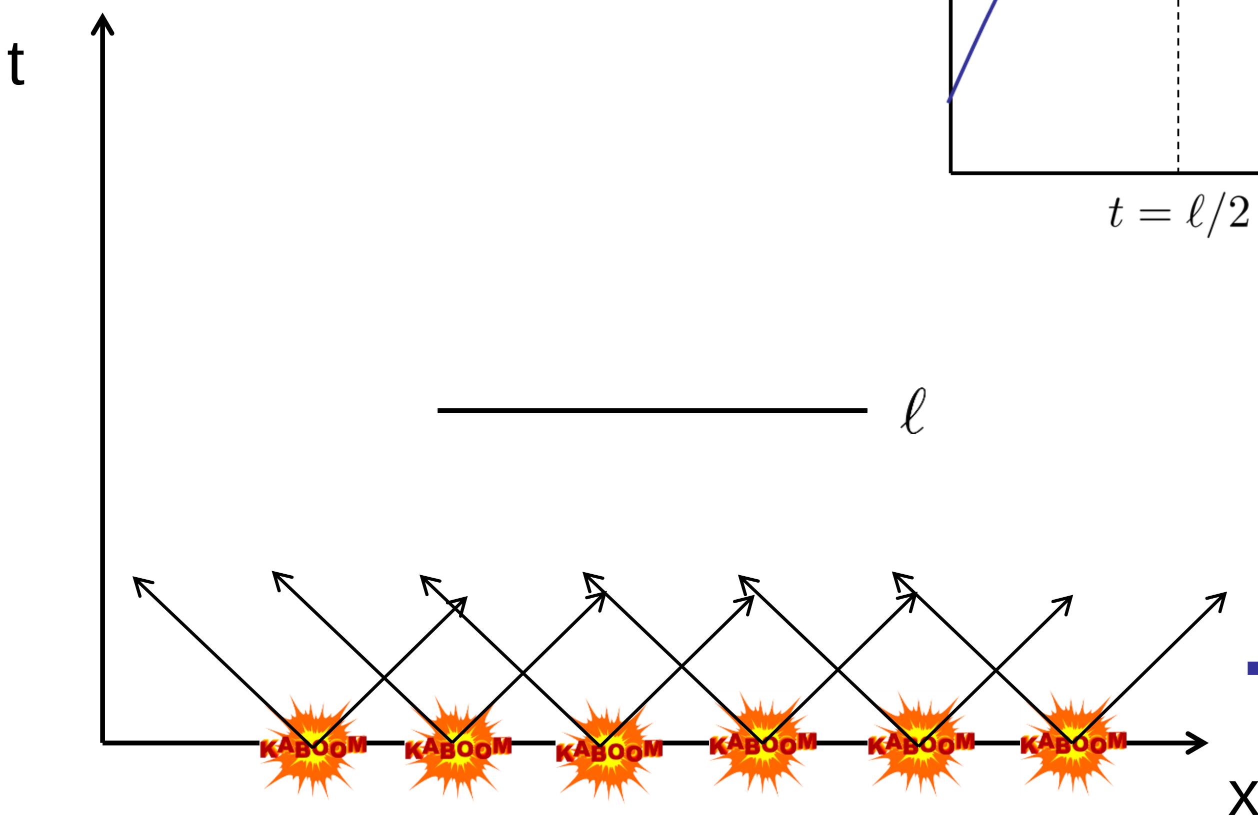



S
S_{thermal}
$t = \ell/2$
t


t
ℓ
KABOOM
x

Explanation





Entangled pairs of quasi-particles

A similar picture exists in higher dimensions for those field theories with a holographic dual.

If these capture the qualitative dynamics of thermalization after heavy ion collisions get an interesting picture:

Thermalization proceeds from the UV to the IR and not the other way around.

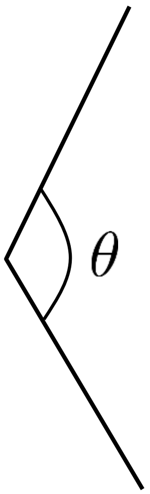
But many issues and subtleties…

Other results:

Integrated null energy condition (Faulkner, Leigh, Parrikar, Wang); Also follows from causality (Hartman, Kundu, Tajdini).

$$\langle \int dx^- T_{--} \rangle \geq 0$$

Universal features of corner entropy (many authors)

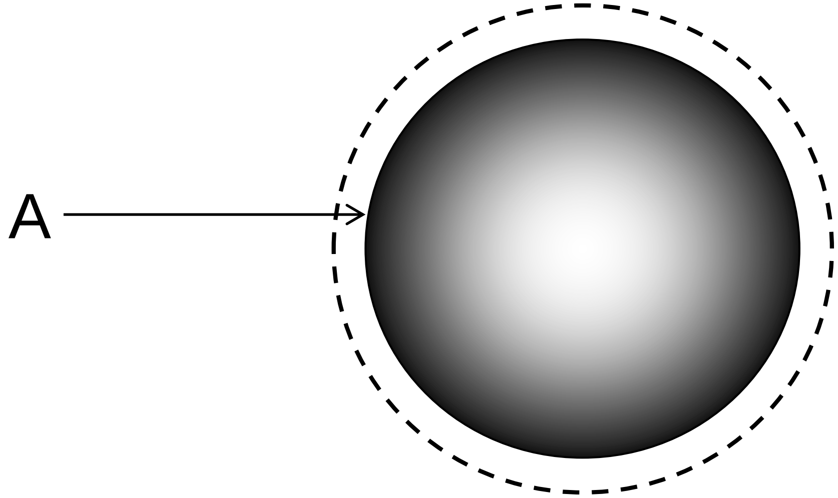


$$S = \alpha \frac{\ell}{a_{UV}} + a(\theta) \log \left(\frac{\ell}{a_{UV}} \right) + \gamma + \dots$$

$$a(\theta \to \pi) \sim \frac{C_T \pi^2}{24} (\theta - \pi)^2$$

Holography and quantum gravity.

Black holes: entropy = entanglement entropy?



$$S_{\mathrm{BH}} = \frac{A}{4G}$$

$$S_{\mathrm{EE}} \sim \frac{A}{4G} \quad (a \sim \ell_P)$$

Cooperman, Luty `03

It appears natural to write down the UV finite combination

$$S_{\text{gen}} = \frac{A}{4G_N} + S_{EE}$$

This combination appears to have a preferred meaning and features in various theorems (second law, quantum focusing conjecture).

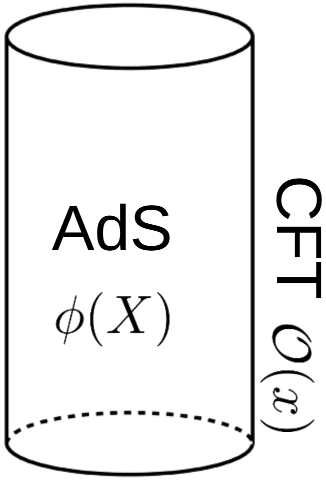
Quantum gravity and holography made precise: AdS/CFT.

Precise equivalence between a conformal field theory in d dimensions and quantum gravity in d+1 dimensions.

Matches nicely with entropy of black holes.

Bulk fields vs boundary operators

$$\phi(X)_{\text{bulk}} \simeq \int d^d x K(X,x)\mathcal{O}(x)_{\text{boundary}}$$



Important question: How does locality come about?

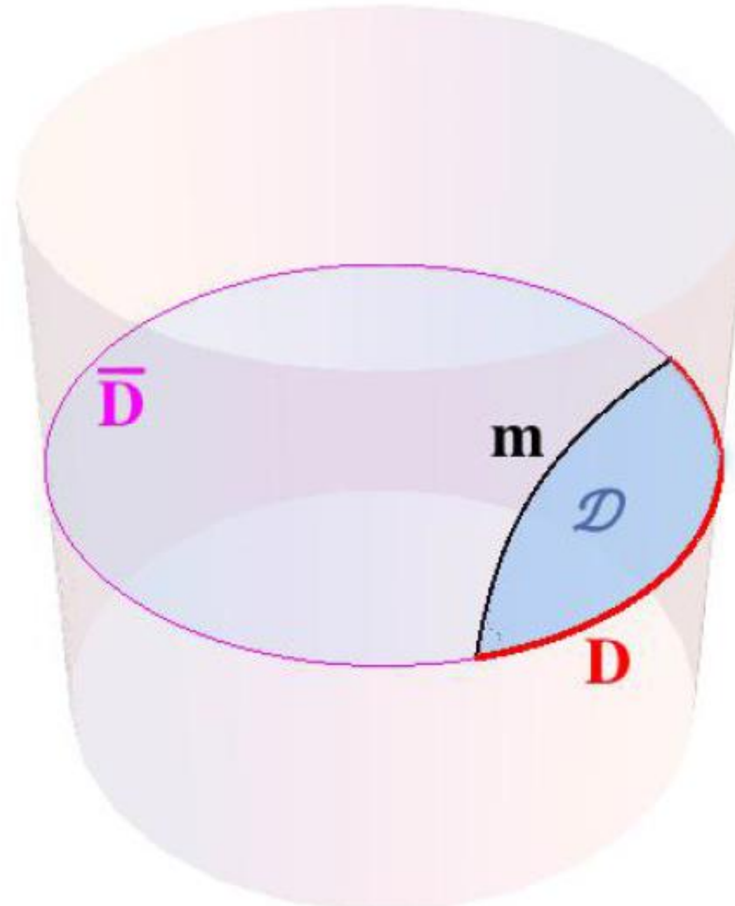
Entanglement entropy?

Ryu-Takayanagi proposed that

$$S(D) = \frac{\text{Area}(m)}{4G}$$

m: extremal surface

A fairly complete proof of this Statement has been given (Lewkowycz, Maldacena `13)

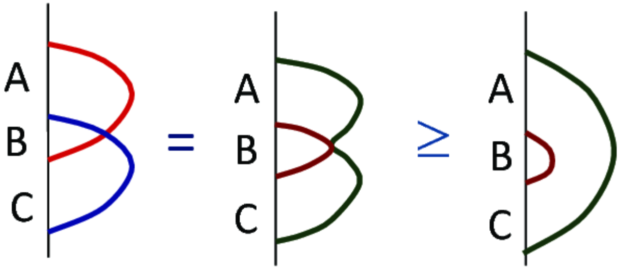


Therefore entanglement entropy is a direct probe of space-time geometry. Can perhaps probe the (non)locality of quantum gravity.

Entanglement entropy is very easy to compute, purely geometric computation!

Entanglement inequalities become geometric inequalities.

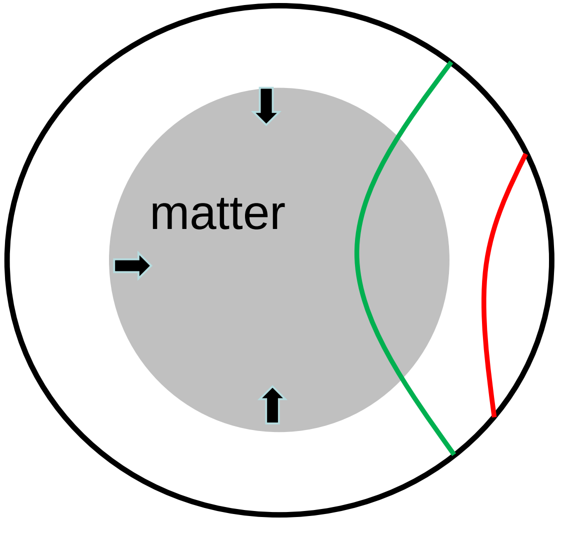
UV cutoff = radial cutoff.



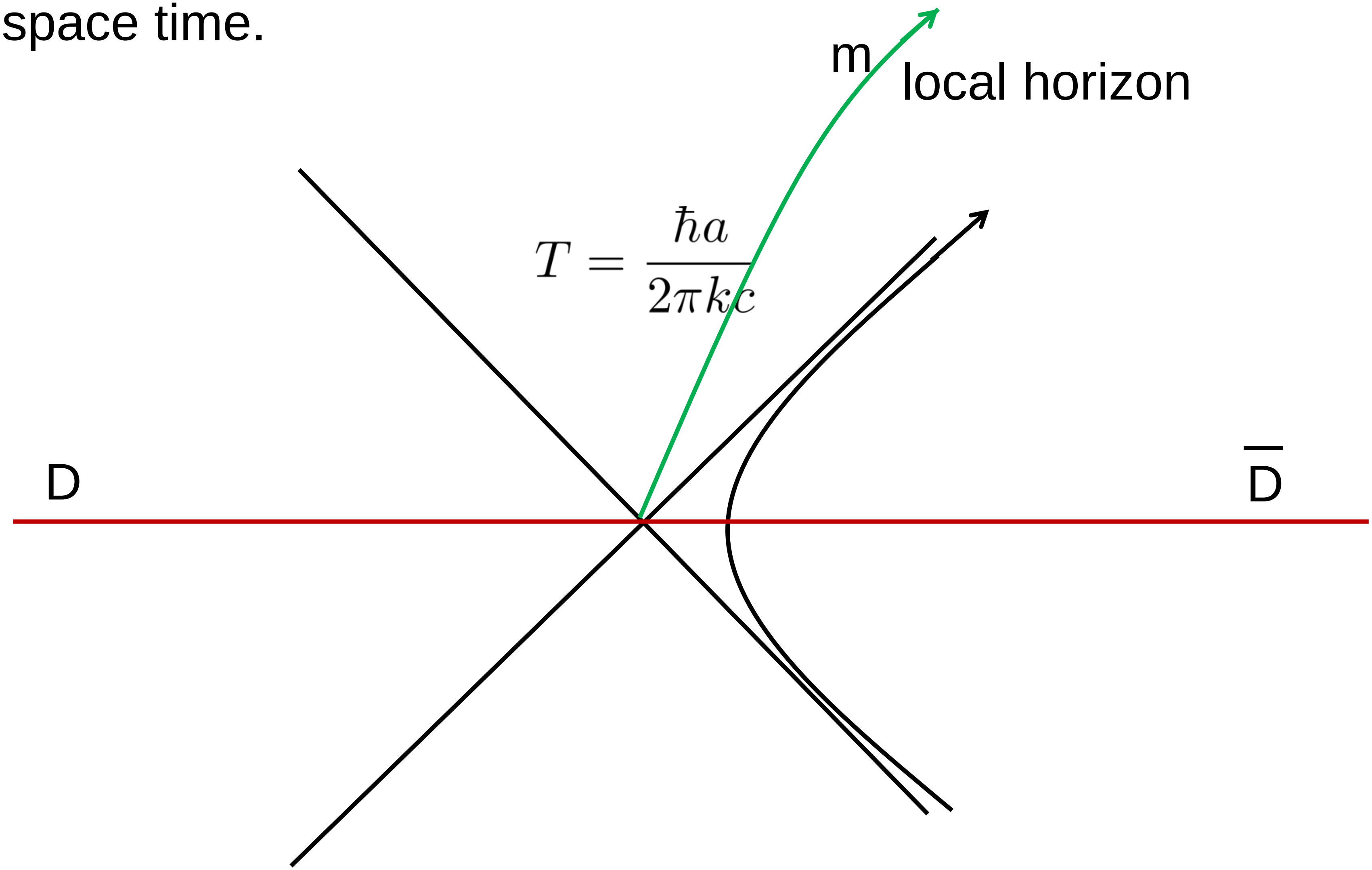
$$\Rightarrow \quad S_{A\cup B} + S_{B\cup C} \geq S_{A\cup B\cup C} + S_B$$

Universal result for 2d CFT's is reproduced.

Thermalization picture is confirmed.



Important message: entanglement is crucial to get a smooth space time.



$$|0\rangle_M \, {}_M\langle 0| = \sum_E e^{-\frac{E}{T}} |E\rangle_L |E\rangle_R \, {}_R\langle E|_L \langle E|$$

Van Raamsdonk `09

Changing the nature of the entanglement easily leads to a divergent expectation value of the energy momentum tensor on the “horizons”.

In some sense, this is a manifestation of the “firewall”.

Almheiri, Marold, Polchinski, Sully

Postulating that entanglement entropy is computed by minimal area surfaces implies the linearized Einstein equations.

Faulkner, Guica, Hartman, Myers, van Raamsdonk `13

Important ingredient: first law of entanglement entropy

$$\delta S = -\delta \mathrm{Tr}(\rho \log \rho) = -\mathrm{Tr}(\delta\rho \log \rho) = -\delta\langle \log \rho \rangle$$

For conformal field theories:

$$\delta S(B) = 2\pi \int_B d^{d-1}x' \ \frac{R^2 - |\vec{x} - \vec{x}'|^2}{2R} \langle T_{tt}(\vec{x}') \rangle \ .$$

Is quantum gravity a local theory?

If the degrees of freedom of quantum gravity were approximately local, one should be able to compute their entanglement between some spatial domain and its complement.

This requires a factorization

$$\mathcal{H} = \mathcal{H}_{\rm outside} \otimes \mathcal{H}_{\rm inside}$$

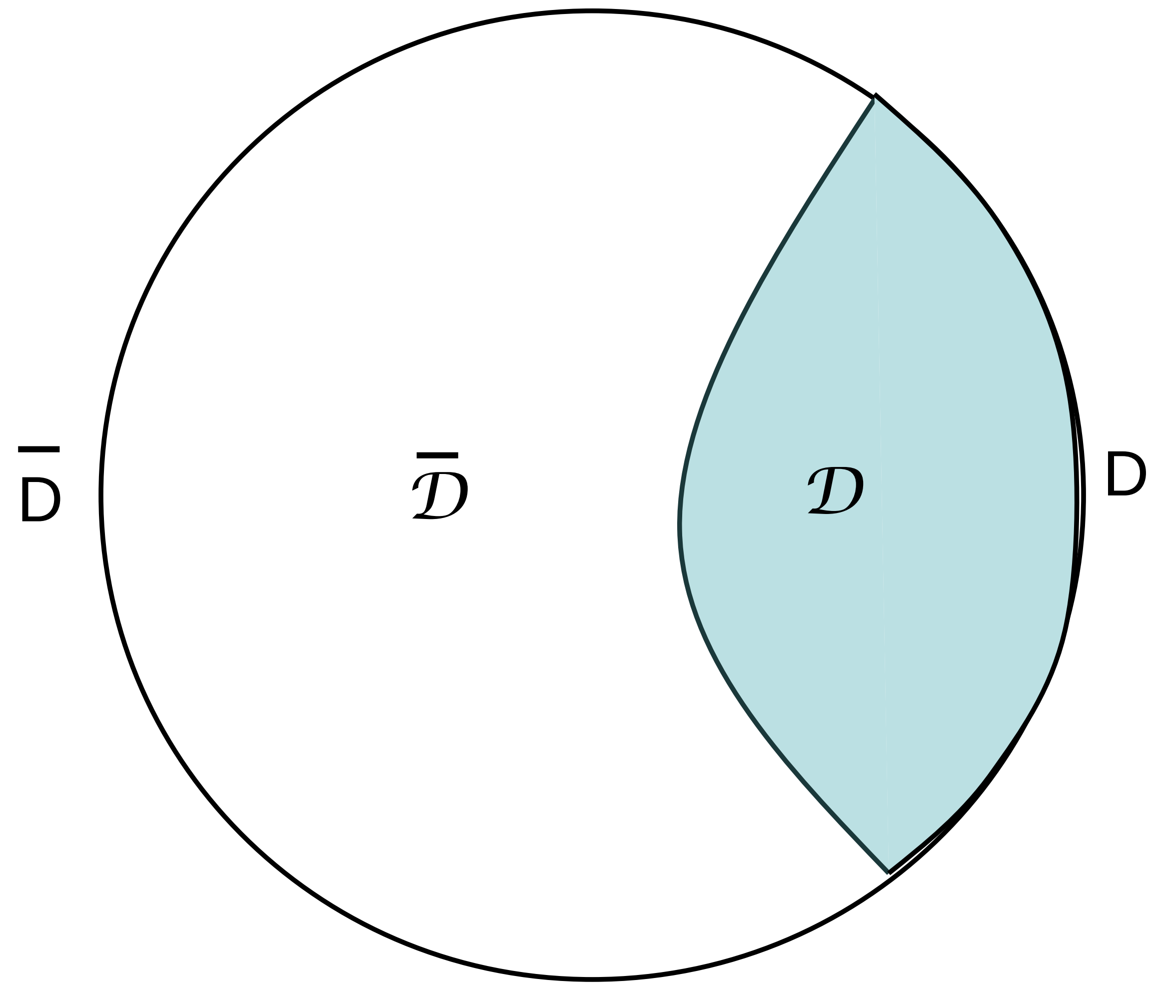
Such a factorization is often used when computing Hawking radiation, when discussing the information loss paradox, and in many arguments pertaining to the (non)existence of firewalls.

Factorization fails in gauge theories as well: no gauge invariant separation between degrees of freedom in A and its complement. Answer becomes a sum over superselection sectors (see e.g. Soni, Trivedi)

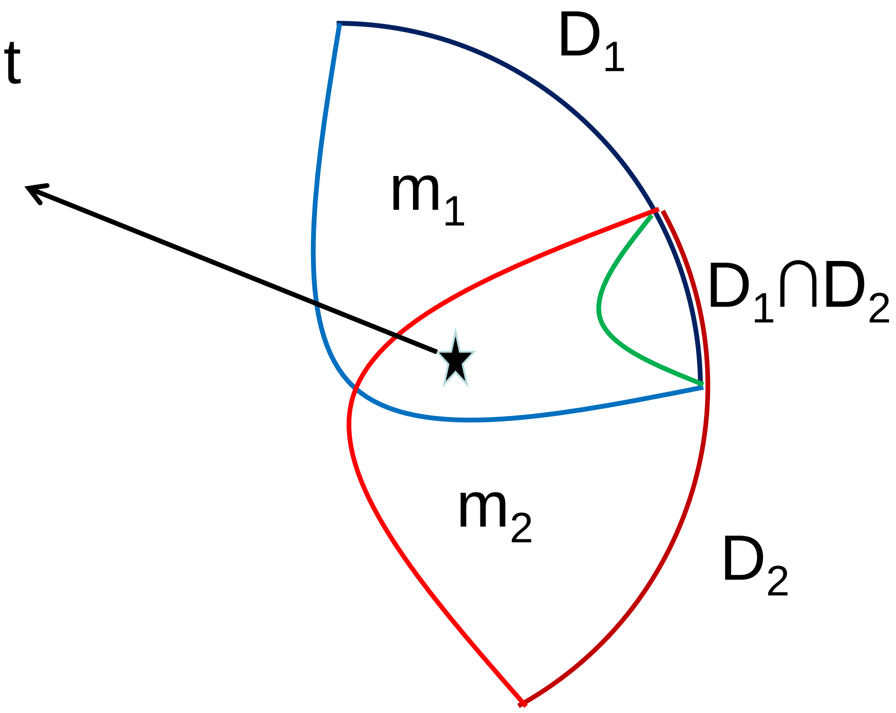
$$S_{\mathrm{EE}} = -\sum_k p_k \log p_k + \sum p_k \log D_k - \sum_k p_k \mathrm{Tr}(\rho_k \log \rho_k)$$

Consistent with dualities.

$$\mathcal{H} = \mathcal{H}_D \otimes \mathcal{H}_{\bar{D}} \simeq \mathcal{H}_{\mathcal{D}} \otimes \mathcal{H}_{\bar{\mathcal{D}}}$$



A local operator here would act entirely in D_1 but also entirely in D_2; but it does not act in $D_1 \cap D_2$. This is a contradiction. Local operators do not exist?



Connection to quantum error correction?

Local operators only act properly on a subset of the degrees of freedom: the so-called “code subspace”.

Almheiri, Dong, Harlow `14
Mintun, Polchinski, Rosenhaus `15

Summary

- Entanglement entropy is a natural “observable” for quantum gravity.
- Most fundamental degrees of freedom are non-local, only a small subset (“ripples on the sea”) are local and manifest themselves as local degrees of freedom (code subspace).
- Smooth spacetime requires that the microscopic degrees of freedom be very entangled.
- Requiring smooth physics near horizons may imply that the description of local smooth physics is state dependent. (Papadodimas-Raju).
- Standard results in quantum information theory imply e.g. the linearized Einstein equations.

Serious problem:

Minimal surfaces do not get everywhere.

They get exponentially close to the horizon for large black holes (in AdS units) but stay order one distance away for medium size black hole in AdS units.

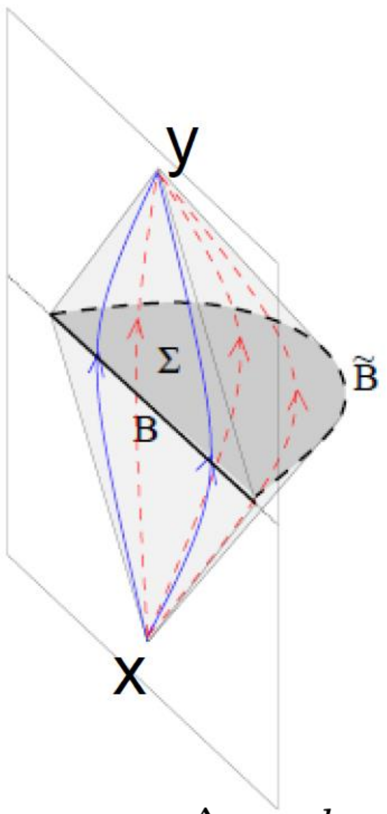
Happens even for stars.

Sometimes referred to as the “entanglement shadow”.

To get closer can use non-minimal extremal surfaces but boundary interpretation is less clear (“entwinement”).
Balasubramanian, Chowdhury, Czech, JdB

Other interesting observables in quantum gravity

Czech, Lamprou, McCandlish, Mosk, Sully
JdB, Haehl, Heller, Myers



Usually, entanglement wedge reconstruction is messy. In this case it is not.

Related to OPE blocks, explains geodesic Witten diagrams, …

$$\int_{\tilde{B}} d^{d-1}u\, \phi(u) = C \int_{D(X,Y)} d^d\xi \left(\frac{|X-\xi||\xi-Y|}{|X-Y|} \right)^{\Delta_{\mathcal{O}}-d} \langle \mathcal{O}(\xi) \rangle$$

Did not discuss many things:

- wormholes and ER=EPR
- replica trick
- complexity
- non-area behavior: Fermi liquids, holographic strange metals,…
- tensor networks
- shape dependence
- higher derivative corrections
- deformed CFT's, quenches, operator insertions, …
- entanglement with boundaries
- bulk dual of relative entropy
- non-geometric entanglement (e.g. Balasubramanian-van Raamsdonk, Taylor..)
- many subtleties and technicalities
- Quantum error correction
- …..

Replica trick relies on Carlson's Theorem: if

$$|f(z)| \leq Ce^{\tau|z|}, \qquad z \in \mathbb{C}$$

$$|f(iy)| \leq Ce^{c|y|}, \qquad y \in \mathbb{R}, \quad c < \pi$$

$$f(n) = 0, \qquad n = 0, 1, 2, \ldots$$

then f=0.

Outlook:

- Entanglement entropy is an interesting quantity with many applications. Apply to black holes? (Scrambling, chaos..)
- Generalizations: Rényi entropies, mutual information, relative entropy.. Also powerful when combined with semiclassical gravity
- Generalize entanglement entropy to deal with non-minimal bulk surfaces? Differential entropy

  Balasubramanian, Chowdhury, Czech, JdB, Heller `14
- Develop better techniques to compute entanglement entropy.
- There seems to be a deep relation between quantum information theory and locality, causality and unitarity, what is the precise relation? (eg Hartman-Kundu-Tandjini ‘16 vs Faulkner-Lee-Parrikar-Wang ‘16)
- Reformulate bulk purely in terms of entanglement entropy and related quantities? Implications?

Jacobson `95 `15;
JdB, Haehl, Heller, Myers ’16
Czech, Lamprou, McCandlish, Mosk, Sully ‘16