The conformal bootstrap

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16 March 2017

The ubiquity of CFTs

CFTs are conformal field theories. In the landscape of QFTs the are special because their physics can be described without using fundamental length scales.

There are many reasons for studying CFTs:

- Condensed matter systems
  - Vapor-liquid critical points
  - Superfluid He^4
  - Magnets at their Curie point
  - Quantum spin liquids
  - ...
- Conformal window of gauge theories
- Quantum gravity
- *Signposts* in the landscape of QFTs

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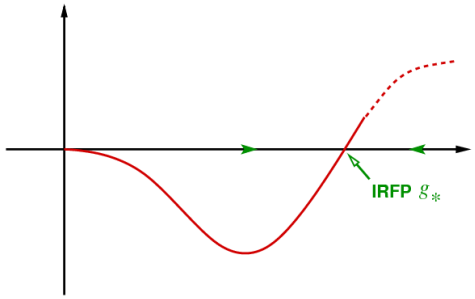

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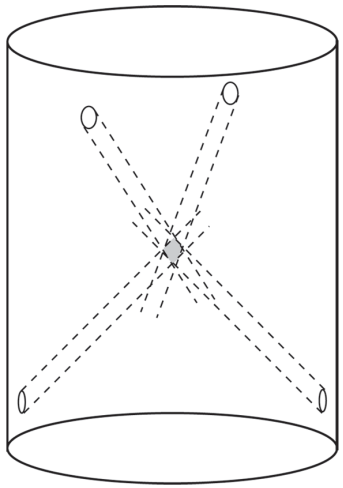

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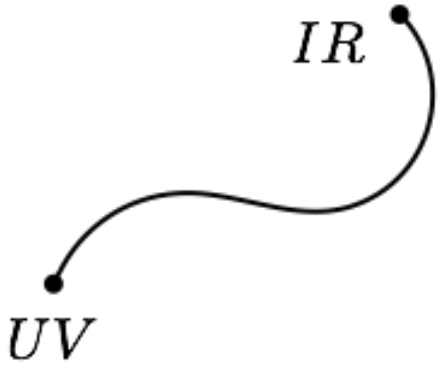


The diversity of CFTs

Let us look at three animals in the zoo of CFTs in $d > 2$.

1. The critical $O(N)$ models: defined as the IR fixed point of

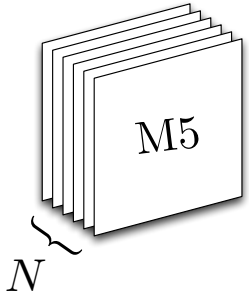
$$S = \int d^3x \left(\partial_\mu \phi_i \partial^\mu \phi^i + g(\phi^i \phi_i)^2\right) \qquad i \in \{1, \ldots, N\}$$

   or defined via the n-vector model on the lattice.

2. $\mathcal{N} = 4$ SYM

$$S = \int d^4x \, \mathrm{tr} \left(\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \ldots\right)$$

3. The six-dimensional $(2, 0)$ theories: *no* Lagrangian



Notice that these CFTs have very different microscopic descriptions.

Bootstrap

CFTs are ubiquitous and diverse.

"Grand question" : can we *classify* and *solve* all CFTs?

Clearly approaching the problem using microscopic descriptions like a lattice or a Lagrangian seems clumsy. Furthermore there is always the risk of missing exotic theories.

An alternative approach can be based on the *bootstrap* philosophy, which is the aspiration that a theory can be completely determined using only basic properties like:

- Unitarity
- Global symmetries
- General consistency conditions

We envisage some kind of procedure that takes these conditions as input and gives the set of consistent CFTs as output.

Is it possible?

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Is it possible?

Outline

CFTs

We will specialize to correlation functions of local operators in flat $\mathbb{R}^d$:

$$\langle \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \rangle$$

At this level CFTs can be <u>defined</u> as an *operator algebra* with:

- The (infinite) set of local primary operators $\mathcal{O}_i$ with $so(d+1,1)$ quantum numbers: scaling dimension Δ_i and spin $\vec{\ell}_i$.
- The *operator product expansion* with the schematic form

$$\mathcal{O}_1(x_1)\mathcal{O}_2(x_2) = \sum_k \lambda_{12}^k C^{(\Delta_k,\vec{\ell}_k)}_{(\Delta_1,\vec{\ell}_1),(\Delta_2,\vec{\ell}_2)}[x_1 - x_2, \partial_2]\mathcal{O}_k(x_2)$$

  Here the position dependence is captured by the $C[x_1 - x_2, \partial_2]$ which are completely fixed by conformal invariance. Example:

$$\mathcal{O}_i(x)\mathcal{O}_j(y) = \frac{\delta_{ij}}{|x-y|^{2\Delta_i}}\mathbf{1} + \lambda_{ij}^k \frac{1 + \#(x-y)\cdot\partial_y + \dots}{|x-y|^{\Delta_i+\Delta_j-\Delta_k}}\mathcal{O}_k(y)$$

  In a CFT the OPE has finite radius of convergence.

CFTs

By repeated application of the OPE we reduce correlation functions to the only non-vanishing one-point function:

$$\langle \mathbf{1} \rangle = 1$$

In this way an OPE of the form

$$\mathcal{O}_i(x)\mathcal{O}_j(y) = \frac{\delta_{ij}}{|x-y|^{2\Delta_i}}\mathbf{1} + \lambda_{ij}^k \frac{1 + \#(x-y)\cdot\partial_y + \dots}{|x-y|^{\Delta_i+\Delta_j-\Delta_k}}\mathcal{O}_k(y)$$

results in

$$\begin{aligned}
\langle \mathcal{O}_i(x)\mathcal{O}_j(y)\rangle &= \frac{\delta_{ij}}{|x-y|^{2\Delta}} \\
\langle \mathcal{O}_i(x)\mathcal{O}_j(y)\mathcal{O}_k(z)\rangle &= \sum_l \lambda_{ij}^l C[x-y,\partial_y]\langle \mathcal{O}_l(y)\mathcal{O}_k(z)\rangle \\
&= \lambda_{ij}^k C[x-y,\partial_y]\frac{1}{|y-z|^{2\Delta_k}} \\
&= \frac{\lambda_{ij}^k}{|x_1-x_2|^{\Delta_{12,3}}|x_1-x_3|^{\Delta_{13,2}}|x_2-x_3|^{\Delta_{23,1}}}
\end{aligned}$$

with $\Delta_{ij,k} = \Delta_i + \Delta_j - \Delta_k$.

Bootstrap

For four-point functions we find:

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle =$$

$$\sum_k \lambda_{12}^k \lambda_{34}^k C[x_1 - x_2, \partial_2] C[x_3 - x_4, \partial_4] \langle \mathcal{O}_k(x_2)\mathcal{O}_k(x_4)\rangle$$

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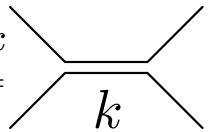

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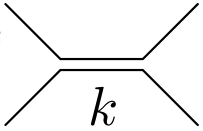
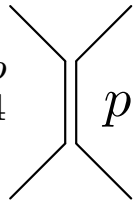


$$= \sum_p \lambda_{13}^{\ p}\lambda_{24}^{\ p}$$



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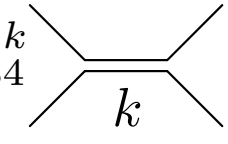
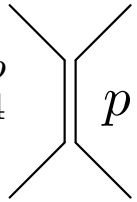
$$\sum_k \lambda_{12}^{\ k}\lambda_{34}^{\ k}$$  $$= \sum_p \lambda_{13}^{\ p}\lambda_{24}^{\ p}$$ 

The *crossing symmetry* equations represent the *associativity* of the operator algebra and give constraints for (Δ_i, ℓ_i) and $\lambda_{ij}^{\ k}$.

Refined "grand question" : can we obtain all the (unitary) solutions to the crossing symmetry equations?

[Ferrara, Gatto, Grillo (1973); Polyakov (1974)]

Note: classical and beautiful results in $d = 2$: minimal models and rational CFTs. This talk will not be about those results.

[BPZ; Moore, Seiberg; . . .]

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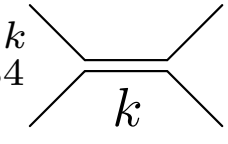
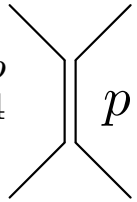
$$\sum_k \lambda_{12}^{\ k}\lambda_{34}^{\ k}$$  $$= \sum_p \lambda_{13}^{\ p}\lambda_{24}^{\ p}$$ 

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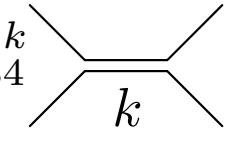
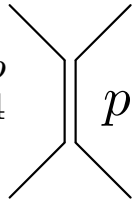
$$\sum_k \lambda_{12}^{\ k}\lambda_{34}^{\ k}$$  $$= \sum_p \lambda_{13}^{\ p}\lambda_{24}^{\ p}$$ 

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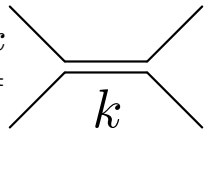
Outline

Implementation

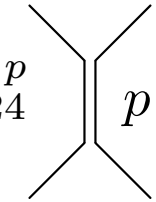
Even for a single correlation function there are infinitely many crossing constraints for infinitely many variables. To date no exact non-trivial solution in $d > 2$ is known.

Instead, we make the problem manageable by taking *derivatives* of the crossing equations:

$$0 = \partial_u^m \partial_v^n \left(\sum_k \lambda_{12}^{\ k} \lambda_{34}^{\ k} \right.$$



$$- \sum_p \lambda_{13}^{\ p} \lambda_{24}^{\ p}$$



$$\left. \right) \Bigg|_{u=v=1/4}$$

with $m, n \leq \Lambda$. We get finitely many equations, but each of them must necessarily be satisfied.

$\rightarrow$ we can (merely) exclude inconsistent points.

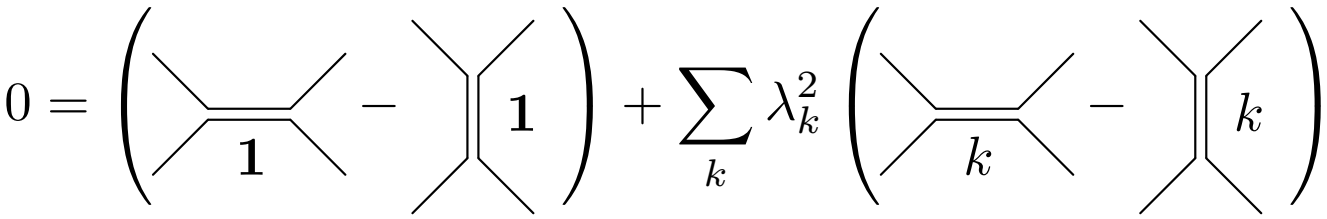
[Rattazzi,Rychkov,Tonni,Vichi (2008)]

Implementation

More precisely, consider identical scalar operators of dimension δ

$$\langle \mathcal{O}_\delta(x_1)\mathcal{O}_\delta(x_2)\mathcal{O}_\delta(x_3)\mathcal{O}_\delta(x_4)\rangle$$

and rewrite its corresponding crossing equation as:



After some reshuffling this becomes a *sum rule* of the form

$$1 = \sum_k \lambda_k^2 h^\delta_{\Delta_k,\ell_k}(u,v)$$

with known functions $h^\delta_{\Delta_k,\ell_k}(u,v)$ which depends *only* on:

- the quantum numbers (Δ_k, ℓ_k) of operator k;
- the 'external' dimension δ;
- the positions x_i via $u = x_{12}^2 x_{34}^2 x_{13}^{-2} x_{24}^{-2}$ and $v = x_{14}^2 x_{23}^2 x_{13}^{-2} x_{24}^{-2}$.

Implementation

$$1 = \sum_k \lambda_k^2 h^{\delta}_{\Delta_k,\ell_k}(u,v)$$

Toy example: suppose the set of *possible* blocks is discrete - let's label them by a set $\{I\}$. We want to know if a particular block $h_{\hat{I}}$ is present or not.

To do so we act on both sides with a *linear functional* ϕ, for example

$$\phi\left[f(u,v)\right] = \sum_{m,n=0}^{\Lambda} \phi_{m,n}\partial_u^m \partial_v^n f(u,v)|_{u=v=1/4}$$

Suppose that we can tune the $\phi_{m,n}$ such that

$$\begin{aligned} \phi[1] &= 0 \\ \phi[h_I(u,v)] &> 0 \qquad I \neq \hat{I} \end{aligned} \tag{1}$$

Then $\phi[h_{\hat{I}}(u,v)] < 0$ and $\lambda_{\hat{I}}^2 \neq 0$, so the block for $\mathcal{O}_{\hat{I}}$ *must* be present.

Implementation

$$1 = \sum_k \lambda_k^2 h^\delta_{\Delta_k,\ell_k}(u,v)$$

Real-world case: the set of possible blocks is *continuous* and labelled by (Δ, ℓ).

We will act with the same type of functional but now try to tune the ϕ_{mn} such that, for fixed δ and Δ_*,

$$\begin{aligned} \phi[1] &= 0 \\ \phi[h^\delta_{\Delta,\ell}(u,v)] &> 0 \qquad \ell > 0 \vee (\ell = 0 \wedge \Delta > \Delta_*) \end{aligned} \tag{2}$$

Then there must exist $\Delta_k < \Delta_*$ for which $\phi[h^\delta_{\Delta_k,0}(u,v)] < 0$ and for which this block *must* be present $\rightarrow$ an *upper bound* on Δ_k.

Finding the $\phi_{m,n}$ is a problem in *convex optimization* which we solve numerically.

(Note the importance of the positivity of λ^2.)

Outline

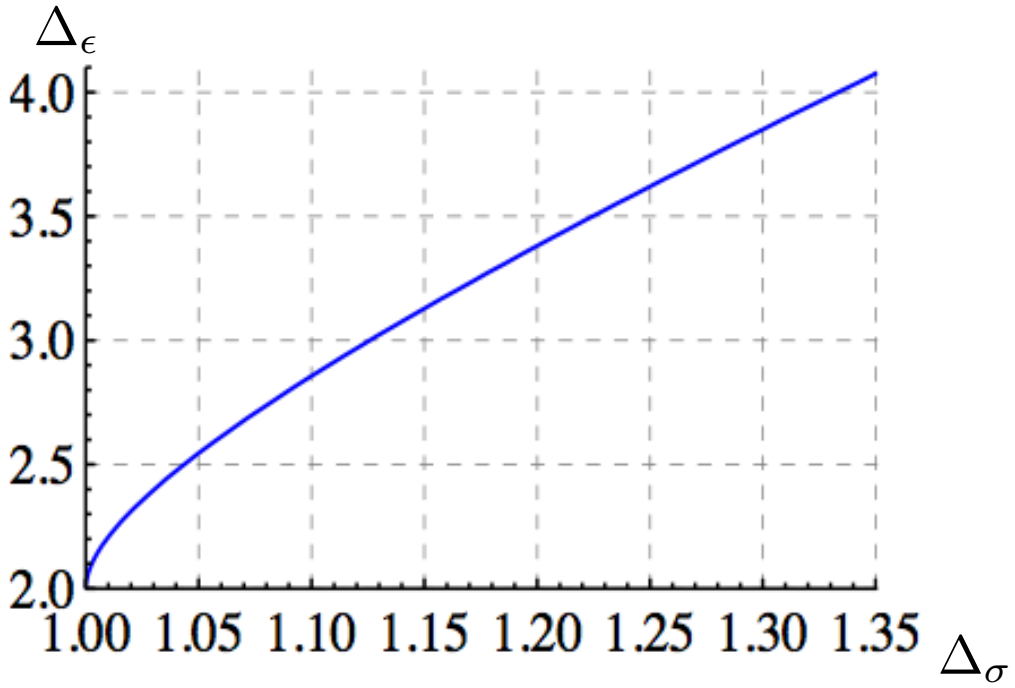
Example 0

Example: bounds for four-dimensional CFTs

Consider a four-dimensional unitary CFT with scalar operators σ and ϵ and a schematic OPE of the form

$$\sigma \times \sigma = 1 + \epsilon + \dots$$

We look at crossing constraints from $\langle \sigma\sigma\sigma\sigma \rangle$ to find the following upper bound on Δ_ϵ:



[Rattazzi, Rychkov, Tonni, Vichi (2008)]

Example 1

Example: three-dimensional Ising model

Consider a three-dimensional unitary CFT with

- 2 relevant scalar operators, σ and ϵ
- a $\mathbb{Z}_2$ symmetry such that the OPE becomes (schematically)

$$\sigma \times \sigma = 1 + \epsilon + \ldots, \quad \sigma \times \epsilon = \sigma + \ldots, \quad \epsilon \times \epsilon = 1 + \epsilon + \ldots$$

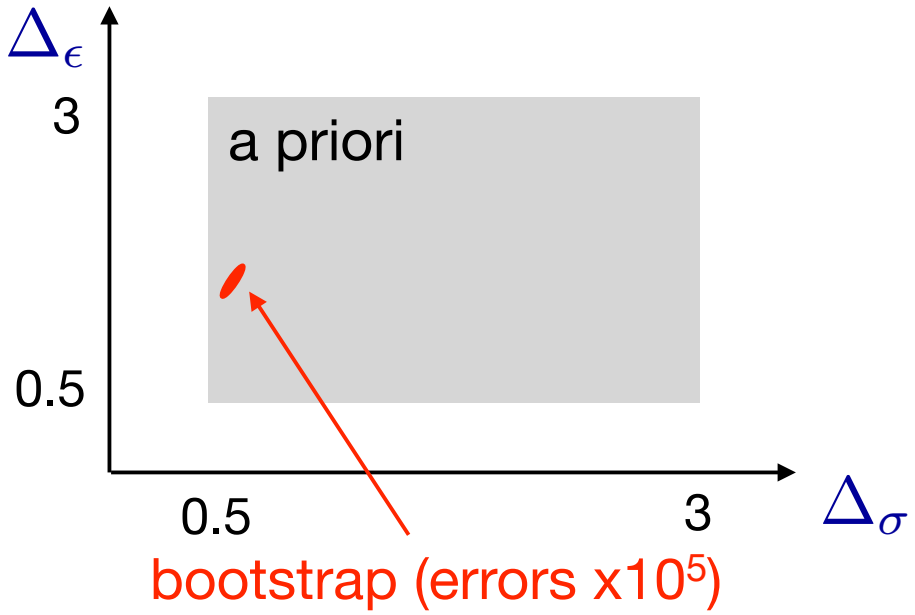
We look at crossing constraints from *three* correlators

$$\langle \sigma\sigma\sigma\sigma \rangle, \quad \langle \sigma\sigma\epsilon\epsilon \rangle, \quad \langle \epsilon\epsilon\epsilon\epsilon \rangle.$$

What do we find?

Example 1

Result from $\langle\sigma\sigma\sigma\sigma\rangle, \langle\sigma\sigma\epsilon\epsilon\rangle, \langle\epsilon\epsilon\epsilon\epsilon\rangle$ with $\Lambda = 43$:



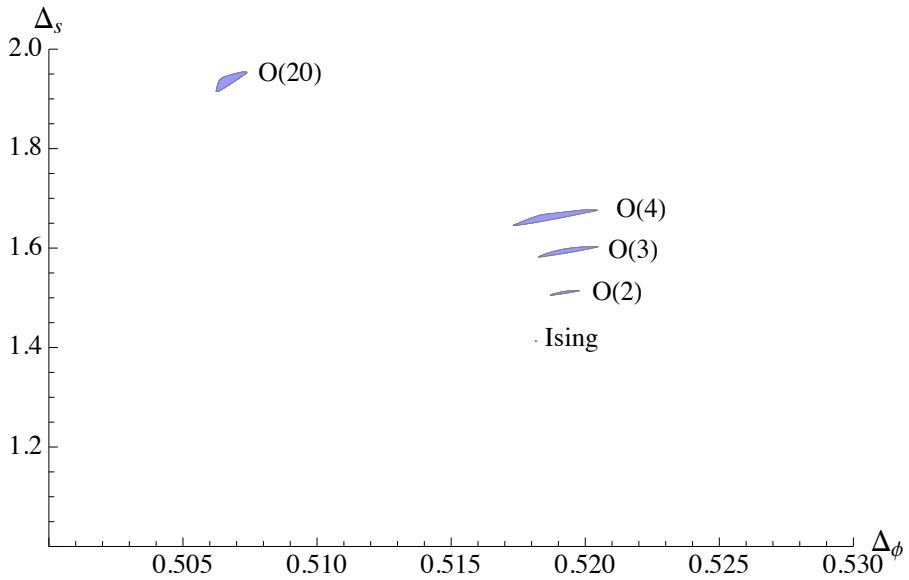
Precision data:

$$\Delta_\sigma = 0.5181489(10) \qquad \lambda_{\sigma\sigma\epsilon} = 1.0518537(41)$$
$$\Delta_\epsilon = 1.412625(10) \qquad \lambda_{\epsilon\epsilon\epsilon} = 1.532435(19)$$
$$\Delta_{\sigma'} = \ldots \qquad c = \ldots$$
$$\ldots$$

[El-Showk,Kos,Paulos,Rychkov,Poland,Simmons-Duffin,Vichi (2012-2016)]

Example 2

Similar islands exist for the $O(N)$ vector models:



- Bigger allowed regions than Ising (with $\Lambda = 35$)
- Excellent match with large N and Monte Carlo
- For $O(2)$: discrepancy with measurement of specific heat of He^4

[Kos, Poland, Simmons-Duffin, Vichi]

Example 3

Now let us consider the $(2,0)$ theories in six dimensions.

Q: Where do we start?

A: The only *universal* multiplet, the stress tensor multiplet. The superconformal primaries $\Phi^I(x)$ with $\Delta = 4$ and transform in the $\mathbf{14}$ of $so(5)_R$. So we will analyze:

$$\langle \Phi^{I_1}(x_1) \dots \Phi^{I_4}(x_4) \rangle$$

In this case there are superconformal Ward identities to solve, and superconformal blocks to compute. This is very nontrivial!

[Eden, Ferrara, Sokatchev; Arutyunov, Sokatchev; Ferrara, Sokatchev; Dolan, Gallot, Sokatchev; Heslop]

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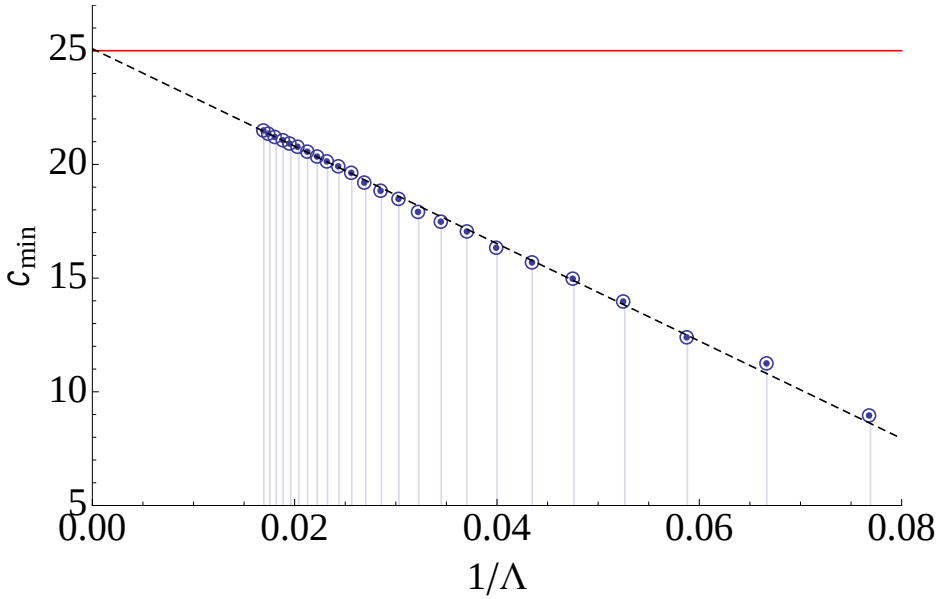
Example 3

For the $(2,0)$ theories we can find for example a lower bound on the central charge c.

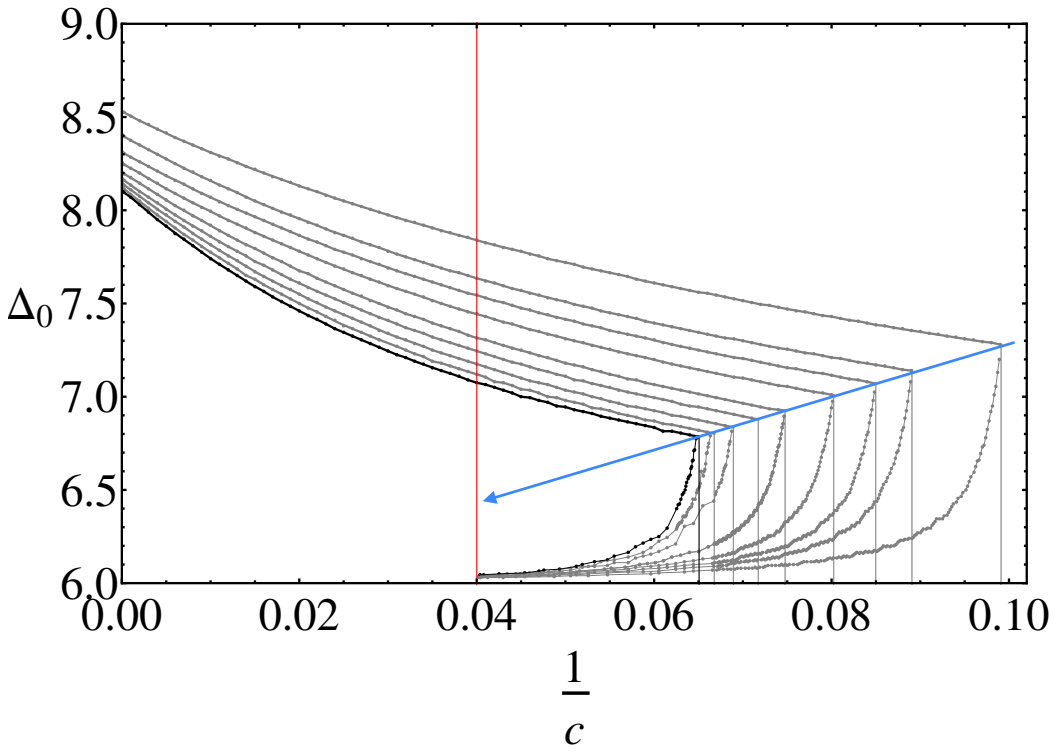


- Converges to $c = 25$, the A_1 theory of 2 parallel M5 branes.
- Conjecture: *unique* solution for $\langle \Phi^{I_1}(x_1) \ldots \Phi^{I_4}(x_4) \rangle$ at $c = 25$

[Beem, Lemos, Rastelli, BvR (2015)]

Example 3

Bounds for the dimension of an unprotected operator:

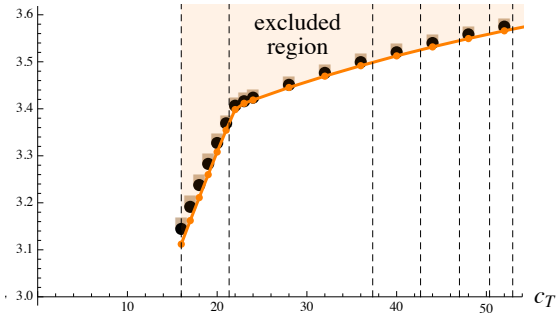


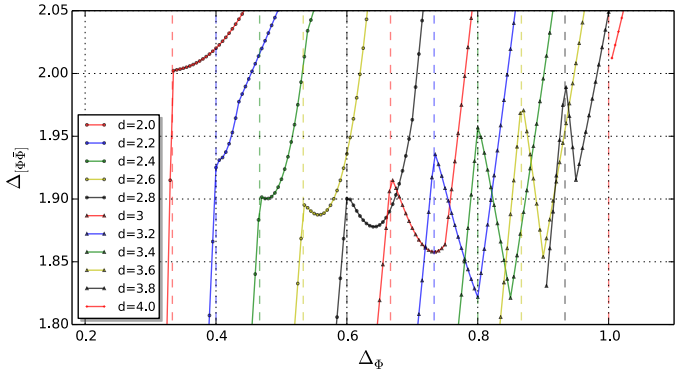
- more clearly shows uniqueness
- non-BPS observable

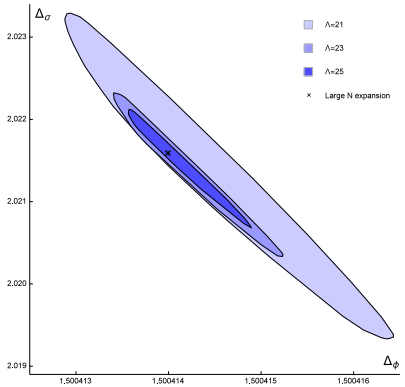
$\rightarrow$ bootstrap M-theory?

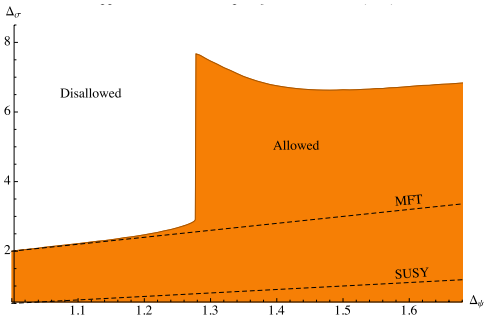
[Beem, Lemos, Rastelli, BvR (2015)]

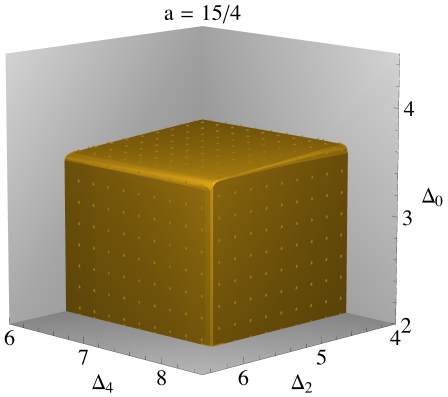
Many more examples

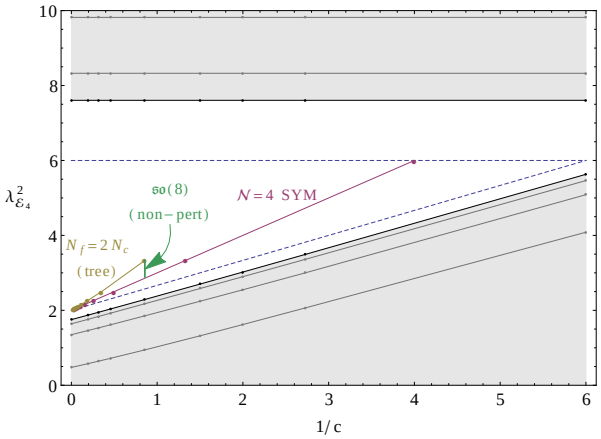












[Beem, Rastelli, BvR; Alday, Bissi; Beem, Lemos, Liendo, Rastelli, BvR; Iha, Makino, Suzuki; Lemos, Liendo; Iliesiu, Kos, Poland, Pufu, Simmons-Duffin, Yacoby; Bobev, El-Showk, Mazac, Paulos; Nakayama, Ohtsuki; Poland, Stergiou; Li, Su; Chester, Giombi, Iliescu, Klebanov, Pufu, Yacoby; Lin, Shao, Simmons-Duffin, Wang, Yin . . .]

Lots of physics!

Outlook

The numerical bootstrap efforts can go *much* further.

Current efforts include:

- Classification of CFTs with few relevant scalar operators (relevant also for IR dualities)
- Correlation functions of operators with spin, e.g.

$$\langle j_\mu(x_1)\dots j_\rho(x_4)\rangle \qquad \text{or} \qquad \langle T_{\mu\nu}(x_1)\dots T_{\rho\sigma}(x_4)\rangle$$

  Application: conformal window of non-abelian gauge theories
- More correlation functions in supersymmetric theories
- Improving the numerical methods
- Revisting the S-matrix bootstrap
- ...

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$\rightarrow$ Revisting the S-matrix bootstrap

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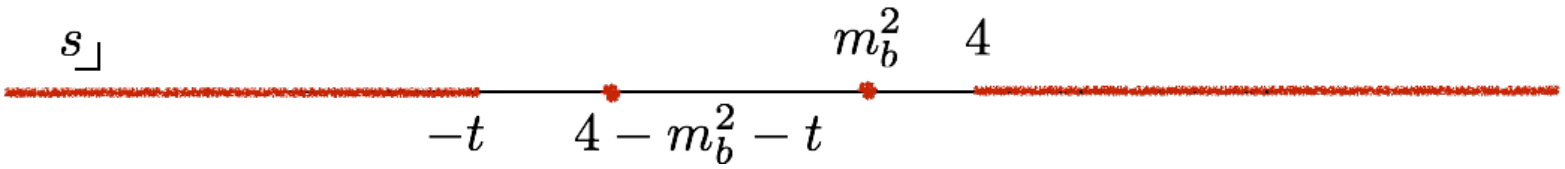
Outlook: the S-matrix bootstrap

For 2-to-2 scattering we assume a crossing symmetric function:

$$T(s,t,u) = T(t,s,u) = T(u,t,s)$$

with the usual analyticity properties:



and obeying unitarity:

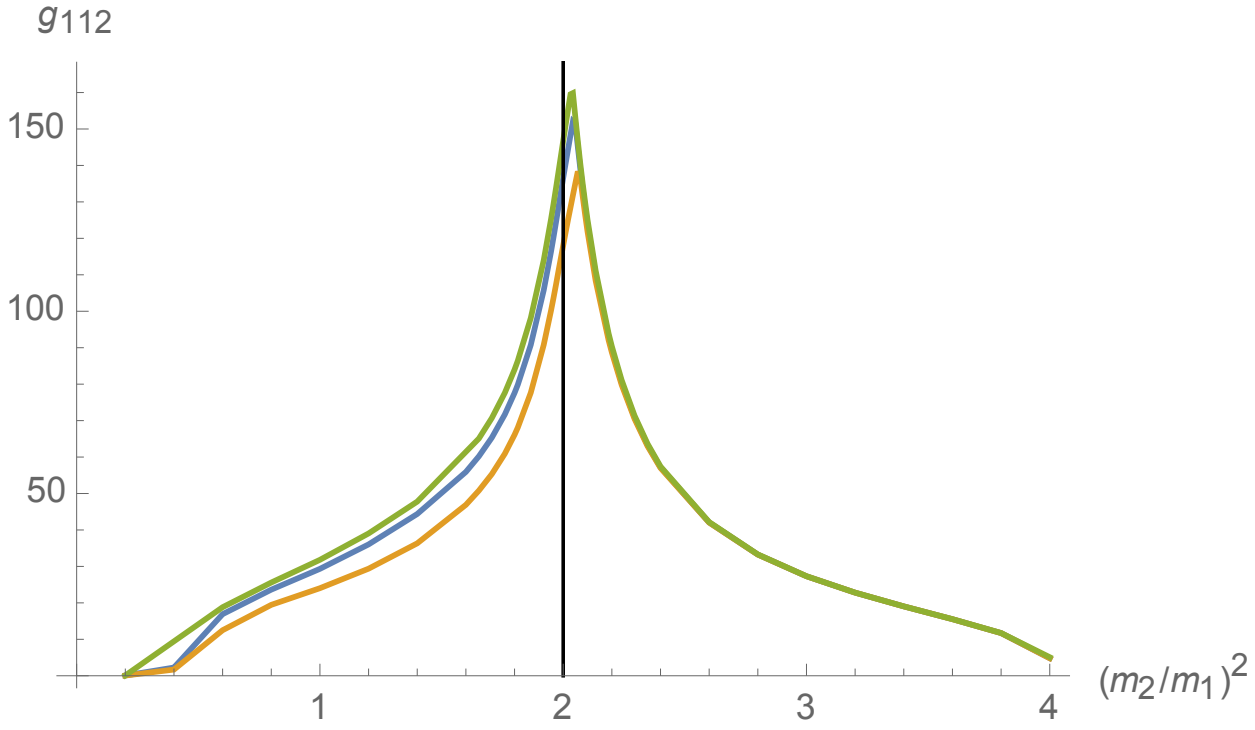
$$\text{Im}(T(s,t)) = \int [d\tau]\; T^*(s,\tilde{t}(s,\tau))T(s,\tau)$$

The same convex optimization techniques can be used for QFTs!

[Paulos, Penedones, BvR, Vieira (2016)]

Outlook: the S-matrix bootstrap

For the upper bound on the *residue* we for example find, in $d = 4$,



[Paulos, Penedones, BvR, Vieira (to appear)]

Outline

Analytic results

In parallel developments many *analytic* results have been derived. The main idea is to look for *universality*, using only general consistency conditions and little physical input.

In a nutshell:

1. At large spin every CFT becomes “holographic”.
2. Certain supersymmetric theories have *solvable* subsectors, with Virasoro and/or Kac-Moody algebras in $d = 4$ and $d = 6$.
3. Proof of the conformal collider bounds of [Hofman,Maldacena].

1. Large spin analysis

Consider a four-point function

$$\langle \mathcal{O}_\delta(0)\mathcal{O}_\delta(z,\bar{z})\mathcal{O}_\delta(1)\mathcal{O}_\delta(\infty)\rangle$$

In Lorentzian signature $0 < z, \bar{z} < 1$. The *lightcone* limit is $z \to 1$, $\bar{z}$ fixed.

In this limit one finds:

- The identity operator dominates in the first channel, which can only be recovered from the other channel by an infinite set of "double-trace" operators with fixed OPE coefficients and

$$\Delta_\ell - \ell \sim 2\delta \quad \text{as} \quad \ell \to \infty$$

- An operator of twist $\tau = \Delta - \ell$ in the first channel gives $1/\ell^\tau$ subleading corrections to both spectrum and OPE coefficients.

[Alday,Maldacena; Komargodski,Zhiboedov; Kaplan,Fitzpatrick,Simmons-Duffin,Poland; . . .]

Outlook: Could this be a new way to attack the crossing equations?

[Simmons-Duffin; Hogervorst, BvR; Caron-Huot]

2. A chiral algebra

Consider a CFT with $\mathcal{N} = 2$ supersymmetry in $d = 4$ or $(2, 0)$ supersymmetry in $d = 6$. By working in the cohomology of

$$\mathbb{Q} = \mathcal{Q} - \mathcal{S}$$

one finds that correlation functions

- of a subset of (protected) operators
- suitably 'twisted'
- restricted to a two-plane $(z, \bar{z})$

become exactly those of a chiral algebra.

In other words,

$$T(z)T(0) \sim \frac{c/2}{z^4} + \frac{2T(0)}{z^2} + \frac{\partial T(0)}{z}$$

now has a four- or six-dimensional interpretation!

We find many analytic results, for example for $\mathcal{N} = 2$ in $d = 4$,

$$c \geq \frac{11}{30}$$

[Beem,Liendo,Lemos,Peelaers,Rastelli,BvR;Liendo,Ramirez,Seo]

3. Regge limits

There are lightcone singularities at $z = 0, 1$ and at $\bar{z} = 0, 1$. By crossing these cuts and entering the strictly Lorentzian regime, one obtains, analytically:

- A proof of the bounds of [Hofman-Maldacena]

$$\frac{1}{3} \leq \frac{a}{c} \leq \frac{31}{18}$$

- The ANEC

$$\int T_{\mu\nu} u^\mu u^\nu \geq 0$$

  for interacting CFTs in $d > 2$.

- ...

[Hartman,Jain,Kundu; Hofman,Li,Meltzer,Poland,Rejon-Barrera]

The future

Both numerically and analytically we continue to make progress in understanding the utility of the crossing symmetry equations.

We seem to be edging closer to answering the “grand question”. . . let’s see how far we can get!

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