TMDs: entering the precision era

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In collaboration with
I. Scimemi and A. Vladimirov

[MGE, Scimemi, Vladimirov, 1509.06392]
[MGE, Scimemi, Vladimirov, 1511.05590]
[MGE, Scimemi, Vladimirov, 1604.07869]

1. Definition of TMDs
2. Evolution of TMDs at NNLL'
3. Refactorization of TMDs
4. Soft function at NNLO
5. Unpolarized TMDs at NNLO
6. Conclusions & Outlook
1. Definition of TMDs

TMD factorization

- Take Drell-Yan as a benchmark process:

\[ q^2 = Q^2 \gg q_T^2 \]

- Same story applies to all processes with “at most two hadrons” (for the moment):

\[
\begin{align*}
H_1 + H_2 & \rightarrow h + X \\
H_1 + H_2 & \rightarrow [Q\bar{Q}] + X \\
e^- + H_1 & \rightarrow e^- + H_2 + X \\
e^- + H_1 & \rightarrow e^- + Q + \bar{Q} + X \\
e^+ + e^- & \rightarrow H_1 + H_2 + X \\
e^+ + e^- & \rightarrow [Q\bar{Q}] + H_1 + X \\
\text{etc...}
\end{align*}
\]

- It’s important to have a correct definition in order to properly connect different processes
TMD factorization: soft and collinear

- The cross-section is given in terms of collinear and soft matrix elements:

\[ d\sigma = \sigma_0(\mu) H(Q^2, \mu) \, dy \, \frac{d^2 q_\perp}{(2\pi)^2} \int d^2 y_\perp \, e^{-i q_\perp y_\perp} \, J_n(x_A, y_\perp, \mu) \, S(y_\perp, \mu) \, J_{\bar{n}}(x_B, y_\perp, \mu) \]

\[ J_n(0^+, y^-, \bar{y}_\perp) = \frac{1}{2} \sum_{\sigma_1} \langle N_1(P, \sigma_1) | \bar{\chi}_n(0^+, y^-, \bar{y}_\perp) \frac{\not{n}}{2} \chi_n(0) | N_1(P, \sigma_1) \rangle |_{zb \ subtracted} \]

\[ S(0^+, 0^-, \bar{y}_\perp) = \langle 0 | Tr \, \bar{T} [S_{\bar{n}}^{T\dagger} S_{\bar{n}}^T] (0^+, 0^-, \bar{y}_\perp) T [S_{\bar{n}}^{T\dagger} S_{\bar{n}}^T] (0) | 0 \rangle \]

\[ J_{\bar{n}}(y^+, 0^-, \bar{y}_\perp) = \frac{1}{2} \sum_{\sigma_2} \langle N_2(\bar{P}, \sigma_2) | \bar{\chi}_{\bar{n}}(0) \frac{\not{n}}{2} \chi_{\bar{n}}(y^+, 0^-, \bar{y}_\perp) | N_2(\bar{P}, \sigma_2) \rangle |_{zb \ subtracted} \]

*But these matrix elements individually are *ill-defined*!!

They contain mixed UV/Rapidity divergences...
Definition of TMDs

- Their definition is a bit tricky:

\[ k_n \sim Q(1, \lambda^2, \lambda) \]
\[ k_{\bar{n}} \sim Q(\lambda^2, 1, \lambda) \]
\[ k_s \sim Q(\lambda, \lambda, \lambda) \]

\[ k_{\bar{\eta}}^2 \sim k_{\bar{n}}^2 \sim k_{\bar{s}}^2 \sim Q^2 \lambda^2 \quad \text{Same invariant mass!} \]

\[ y = \frac{1}{2} \ln \left| \frac{k^+}{k^-} \right| \quad \text{Different rapidities} \]
\[ \text{(mixed under boosts)} \]

- Cancel spurious rapidity divergences:

\[ \zeta_A = (p^+)^2 e^{-2y_c} \]

\[ \tilde{T}_n(x_A, \tilde{k}_{n \perp}, S_A; \zeta_A, \mu) = \tilde{J}_n \sqrt{\tilde{S}} \]

\[ \tilde{T}_n(x_B, \tilde{k}_{\bar{n} \perp}, S_B; \zeta_B, \mu) = \tilde{J}_n \sqrt{\tilde{S}} \]

\[ \zeta_B = (\bar{p}^-)^2 e^{+2y_c} \]

[MGE, Idilbi, Scimemi 1111.4996, 1211.1947, 1402.0869]
[MGE, Kasemets, Mulders, Pisano 1502.05354]
[Collins’ book ’11]
2. Evolution of TMDs at NNLL’

Universal TMD evolution kernel (1/3)

- TMDs depend on two scales: renormalization and rapidity scales
- We know the evolution of all (un)polarized quark/gluon TMDPDFs and TMDFFs

\[
\tilde{T}^{[p_{\text{ol}}]}_{j\leftrightarrow A}(x, b_\perp, S_A; \xi_A, f, \mu_f) = \tilde{T}^{[p_{\text{ol}}]}_{j\leftrightarrow A}(x, b_\perp, S_A; \xi_A, i, \mu_i) \tilde{R}_j^i(b_T; \xi_A, i, \mu_i, \xi_A, f, \mu_f)
\]

The dependence on the renormalization scale is:

\[
\frac{d}{d\ln \mu} \ln \tilde{T}^{[p_{\text{ol}}]}_{j\leftrightarrow A}(x, b_\perp, S_A; \xi_A, \mu) = \gamma_j \left( \alpha_s(\mu), \ln \frac{\xi_A}{\mu^2} \right)
\]

Known at 3-loops both for quark and gluon TMDs

[Moch, Vermaseren, Vogt ’04, ’05]
2. Evolution of TMDs at $\text{NNLL}'$

Universal TMD evolution kernel (1/3)

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\[
\tilde{T}_{j \leftrightarrow A}^{[\text{pol}]}(x, b_\perp, S_A; \zeta_A, f, \mu_f) = \tilde{T}_{j \leftrightarrow A}^{[\text{pol}]}(x, b_\perp, S_A; \zeta_A, i, \mu_i) \tilde{R}^j(b_T; \zeta_A, i, \mu_i, \zeta_A, f, \mu_f)
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Known at 3-loops both for quark and gluon TMDs

$$[\text{Moch, Vermaseren, Vogt '04, '05}]$$
Universal TMD evolution kernel (2/3)

- The dependence on the rapidity scale is:

\[
\frac{d}{d \ln \zeta_A} \ln \tilde{T}_{j \leftrightarrow A}[p_{\text{out}}](x, b_\perp, S_A; \zeta_A, \mu) = -D_j(b_T; \mu)
\]

Known at NLO. Recently at NNLO.

\[
\frac{dD_j}{d \ln \mu} = \Gamma_{\text{cusp}}^j(\alpha_s(\mu))
\]

Cusp anomalous dimension does not completely determine \(D_j\). One needs the soft function (or cross-section)

- Combining the evolution in both scales:

\[
\tilde{R}^j(b_T; \zeta_A, i, \mu_i, \zeta_A, f, \mu_f) = \exp \left[ \int_{\mu_i}^{\mu_f} \frac{d \hat{\mu}}{\hat{\mu}} \gamma_j \left( \alpha_s(\hat{\mu}), \ln \frac{\zeta_A, f}{\hat{\mu}^2} \right) \left( \frac{\zeta_A, f}{\zeta_A, i} \right)^{-D_j(b_T; \mu_i)} \right]
\]

The evolution itself contains some non-perturbative input
(in the \(D_j\) term at large \(b_T\))
Universal TMD evolution kernel (2/3)

- The dependence on the rapidity scale is:

\[
\frac{d}{d\ln \zeta_A} \ln \tilde{T}_{j \to A}^{[pol]}(x, b, S_A; \zeta_A, \mu) = -D_j(b_T; \mu)
\]

Known at NLO. Recently at NNLO.

Indirect: [Becher, Neubert 1007.4005 ]
Direct: [MGE, Scimemi, Vladimirov 1511.05590]

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Cusp anomalous dimension does not completely determine \( D_j \). One needs the soft function (or cross-section)

- Combining the evolution in both scales:

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\]

The evolution itself contains some non-perturbative input
(in the \( D_j \) term at large \( b_T \))
Universal TMD evolution kernel (3/3)

- Currently known perturbative ingredients allow NNLL' evolution:

\[
D = C_F \sum_{n=1}^{\infty} \left( \frac{\alpha_s}{4\pi} \right)^n \sum_{k=0}^{n} L^k_{\mu} \delta^{(n,k)}
\]

<table>
<thead>
<tr>
<th>Order</th>
<th>$\Gamma_{cusp}^j$</th>
<th>$\gamma_{nc}^j$</th>
<th>$D_j$</th>
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<tr>
<td>LL</td>
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<tr>
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<td>NNLL</td>
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<td>$\alpha_s^2$</td>
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<tr>
<td>NNNLL</td>
<td>$\alpha_s^4$</td>
<td>$\alpha_s^3$</td>
<td>$\alpha_s^3$</td>
</tr>
</tbody>
</table>

[Li, Zhu 1604.01404]

\[
d^{(3,0)} = \frac{-1}{2} C_A^2 \left( -\frac{176}{3} \zeta_3 \zeta_2 + \frac{6392 \zeta_2}{81} + \frac{12328 \zeta_3}{27} + \frac{154 \zeta_4}{3} - 192 \zeta_5 - \frac{297029}{729} \right) - C_A T_r N_f \left( -\frac{824 \zeta_2}{81} - \frac{904 \zeta_3}{27} + \frac{20 \zeta_4}{3} + \frac{62626}{729} \right) - 2 T_r^2 N_f^2 \left( -\frac{32 \zeta_3}{9} - \frac{1856}{729} \right)
\]

\[
- C_F T_r N_f \left( -\frac{304 \zeta_3}{9} - 16 \zeta_4 + \frac{1711}{27} \right)
\]
3. Refactorization of TMDs

- $k_T$ dependence is perturbatively calculable when $k_T$ is large:

$$\tilde{T}_{i\rightarrow A}(x, b_T; \zeta, \mu) = \sum_{j=q, \bar{q}, g} \tilde{C}_{i\rightarrow j}^T(x, b_T; \zeta, \mu) \otimes t_{j\rightarrow A}(x; \mu) + O(b_T \Lambda_{QCD})$$

- For each TMD we have a different OPE. For example:

$$\tilde{f}_{1}^{q/A}(x, b_T; \zeta, \mu) = \sum_{j=q, \bar{q}, g} \int_{x}^{1} \frac{d\bar{x}}{\bar{x}} \tilde{C}_{q/j}^f(\bar{x}, b_T; \zeta, \mu) f_{j/A}(x/\bar{x}; \mu)$$

$$\tilde{h}_{1}^{1/2 g/A (2)}(x, b_T; \zeta, \mu) = \sum_{j=q, \bar{q}, g} \int_{x}^{1} \frac{d\bar{x}}{\bar{x}} \tilde{C}_{g/j}^h(\bar{x}, b_T; \zeta, \mu) f_{j/A}(x/\bar{x}; \mu)$$

$$\tilde{g}_{1 L}^{g/A}(x, b_T; \zeta, \mu) = \sum_{j=q, \bar{q}, g} \int_{x}^{1} \frac{d\bar{x}}{\bar{x}} \tilde{C}_{g/j}^g(\bar{x}, b_T; \zeta, \mu) g_{j/A}(x/\bar{x}; \mu)$$

$$\tilde{f}_{1T}^{1 g/A (1)}(x, b_T; \zeta, \mu) = \sum_{j=q, \bar{q}, g} \int_{x}^{1} \frac{d\bar{x}_1}{\bar{x}_1} \frac{d\bar{x}_2}{\bar{x}_2} \tilde{C}_{g/j}^{\text{sivers}}(\bar{x}_1, \bar{x}_2, b_T; \zeta, \mu) T_{F j/A}(x_1/\bar{x}_1, x_2/\bar{x}_2; \mu)$$
3. Refactorization of TMDs

- $k_T$ dependence is perturbatively calculable when $k_T$ is large:

$$\tilde{T}_{i \leftrightarrow A}(x, b_T; \zeta, \mu) = \sum_{j=q, \bar{q}, g} \tilde{C}^{T}_{i \leftrightarrow j}(x, b_T; \zeta, \mu) \otimes t_{j \leftrightarrow A}(x; \mu) + O(b_T \Lambda_{QCD})$$

- For each TMD we have a different OPE. For example:

$$\tilde{f}^{q/A}_1(x, b_T; \zeta, \mu) = \sum_{j=q, \bar{q}, g} \int_x^1 \frac{d\bar{x}}{\bar{x}} \tilde{C}^{f}_{q/j}(\bar{x}, b_T; \zeta, \mu) f_{j/A}(x/\bar{x}; \mu)$$

$$\tilde{h}^{1/g/A}_1(2)(x, b_T; \zeta, \mu) = \sum_{j=q, \bar{q}, g} \int_x^1 \frac{d\bar{x}}{\bar{x}} \tilde{C}^{h}_{g/j}(\bar{x}, b_T; \zeta, \mu) f_{j/A}(x/\bar{x}; \mu)$$

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Resummation of large logs

- This is how a resummed TMD looks like:

\[
\tilde{T}_{i \leftrightarrow A}(x, b_T; \zeta, \mu) = \sum_{j=q, \bar{q}, g} \tilde{C}_{i \leftrightarrow j}^{T}(x, \hat{b}_T; \mu_b^2, \mu_b) \otimes t_{j \leftrightarrow A}(x; \mu_b) \\
\times \exp \left[ \int_{\mu_b}^{\mu} \frac{d\hat{\mu}}{\hat{\mu}} \gamma_j \left( \alpha_s(\hat{\mu}), \ln \frac{\zeta}{\hat{\mu}^2} \right) \right] \left( \frac{\zeta}{\mu_b^2} \right)^{-D_j(\hat{b}_T; \mu_b)} \\
\times \tilde{T}_{i \leftrightarrow A}^{NP}(x, b_T; \zeta)
\]

- **General philosophy:** only parametrize what cannot be calculated

- The non-perturbative part of $D_j$ is universal (for all (un)polarized TMD PDFs and FFs)

- The non-perturbative part of $D_j$ seems not well-constrained by current data

- Higher-order calculations allow better determination of non-perturbative ingredients

- At low $b_T$ the TMDs are neither supposed to be correct ($q_T > Q$ region)

- The determination of non-perturbative pieces is not easy (Fourier transform mixes regions, overlap of perturbative and non-perturbative)
4. **TMD Soft Function at NNLO**

**Modified δ-Regulator**

- The old-fashion δ-regulator…

\[
\frac{1}{k^- + i0} \rightarrow \frac{1}{k^- + i\delta}
\]

At NNLO and beyond:

- Violates non-abelian exponentiation
- Zero-bin ≠ Soft Function

**Modified δ-regulator:**

- **TMDPDF:**
  \[ P \exp \left[ ig \int_0^\infty d\sigma n \cdot A(\sigma n) \right] \rightarrow P \exp \left[ ig \int_0^\infty d\sigma n \cdot A(\sigma n) e^{-\delta \sigma x} \right] \]

- **TMDFF:**
  \[ P \exp \left[ ig \int_0^\infty d\sigma n \cdot A(\sigma n) \right] \rightarrow P \exp \left[ ig \int_0^\infty d\sigma n \cdot A(\sigma n) e^{-\delta \sigma / z} \right] \]

- **Soft Function:**
  \[ P \exp \left[ ig \int_0^\infty d\sigma n \cdot A(\sigma n) \right] \rightarrow P \exp \left[ ig \int_0^\infty d\sigma n \cdot A(\sigma n) e^{-\delta \sigma} \right] \]

δ-regulator violates gauge properties of Wilson lines. Only δ=0 makes sense. Note: careful with power divergent integrals!

Of course physics is independent of the regulator!!
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\text{Soft Function:} & \quad P \exp \left[ ig \int_0^\infty d\sigma n \cdot A(\sigma n) \right] \rightarrow P \exp \left[ ig \int_0^\infty d\sigma n \cdot A(\sigma n) e^{-\delta \sigma} \right]
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$\delta$-regulator violates gauge properties of Wilson lines. Only $\delta=0$ makes sense. Note: careful with power divergent integrals!

Of course physics is independent of the regulator!!
**TMD soft function: splitting**

\[ \tilde{S}(b_T; \mu) = \langle 0 | T(S_n^+ S_n^-) (b_\perp) [\tilde{T}(S_n^+ S_n^-)](0) | 0 \rangle \]

\[ \tilde{S}(b_T) = e^{\alpha_s C_a (S^{[1]} + \alpha_s S^{[2]} + \ldots)} \]

- The most important property:

\[ \tilde{S}(b_T) = \exp \left( A(b_T, \epsilon) \ln |\delta^+ \delta^-| + B(b_T, \epsilon) \right) \]

- In general one expects for a diagram that (at finite epsilon):

\[ \text{Diagram} = \mu^{4\epsilon} \left( A_0 \delta^{-2\epsilon} + A_1 \delta^{-\epsilon} B^\epsilon + A_2 B^{2\epsilon} \right) + O(\delta) \]

\[ \delta = \pm \delta^+ \delta^- \]

\[ B = \frac{b_T^2}{4} \]

Virtual diagrams are irrelevant…

\[ \lim_{b_T \to 0} \tilde{S} = 0 \]

\[ A_0 = 0 \]

\[ \left( \lim_{b_T \to 0} \tilde{S} \right)_{\delta/b_T \ fixed} = 0 \]

\[ A_1 = 0 \]
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\[ \text{Diagram} = \mu^{4\epsilon} \left( A_0 \delta^{-2\epsilon} + A_1 \delta^{-\epsilon} B^\epsilon + A_2 B^{2\epsilon} \right) + \mathcal{O}(\delta) \]

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Splitting at NLO

- The SF relevant for gluon TMDs:

\[
\tilde{S}(b_T; \mu) = \frac{1}{N_c^2 - 1} \sum_{X_s} \langle 0 | (S_n^+ S_{\bar{n}}^+) \langle b_\perp | X_s \rangle \langle X_s | (S_n^+ S_{\bar{n}}^-)^{b_a} (0) | 0 \rangle
\]

\[L_T = \ln \frac{\mu^2 b_T^2}{4e^{-2\gamma_E}}\]

\[
\tilde{S} (b_T; \mu; \delta^+, \delta^-) = 1 + \frac{\alpha_s C_A}{2\pi} \left[ -\frac{2}{\varepsilon_{UV}} + \frac{2}{\varepsilon_{UV}} \ln \frac{\delta^+ \delta^-}{\mu^2} + L_T^2 + 2L_T \ln \frac{\delta^+ \delta^-}{\mu^2} + \frac{\pi^2}{6} \right]
\]

\[
\tilde{S}_-(b_T; \mu; \delta^+, \delta^-) = \tilde{S}_-(b_T; \nu, \mu; \delta^-) \tilde{S}_+(b_T; 1/\nu, \mu; \delta^+)
\]

\[
\tilde{S}_- = 1 + \frac{\alpha_s C_A}{2\pi} \left[ -\frac{1}{\varepsilon_{UV}} + \frac{1}{\varepsilon_{UV}} \ln \frac{\nu(\delta^-)^2}{\mu^2} + \frac{1}{2} L_T^2 + \frac{1}{\varepsilon_{UV}} L_T \ln \frac{\nu(\delta^-)^2}{\mu^2} + \frac{\pi^2}{12} \right]
\]
Splitting at NLO

- The SF relevant for gluon TMDs:

\[
\tilde{S}(b_T; \mu) = \frac{1}{N_c^2 - 1} \sum_{X_s} \langle 0 | (S_n^\dagger S_n)^{ab}(b_\perp) | X_s \rangle \langle X_s | (S_n^\dagger S_n)^{ba}(0) | 0 \rangle
\]

\[
\tilde{S}(b_T; \mu; \delta^+, \delta^-) = 1 + \frac{\alpha_s C_A}{2\pi} \left[ -\frac{2}{\epsilon_{UV}} + \frac{2}{\epsilon_{UV}} \ln \frac{\delta^+ \delta^-}{\mu^2} + L_T^2 + 2L_T \ln \frac{\delta^+ \delta^-}{\mu^2} + \frac{\pi^2}{6} \right]
\]

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\tilde{S} (b_T; \mu; \delta^+, \delta^-) = \tilde{S}_-(b_T; \nu, \mu; \delta^-) \tilde{S}_+(b_T; 1/\nu, \mu; \delta^+)
\]

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\]

\[
L_T = \ln \frac{\mu^2 b_T^2}{4e^{-2\gamma_E}}
\]
Splitting at NNLO

\[ S^{[2]} = \frac{d_{(2,2)}^{(2,2)}}{C_F} \left( \frac{3}{\epsilon^3} + \frac{2L_\delta}{\epsilon^2} + \frac{\pi^2}{6\epsilon} + \frac{4}{3} L_\mu^3 - 2L_\mu^2 L_\delta + \frac{2\pi^2}{3} L_\mu + \frac{14}{3} \zeta_3 \right) - \frac{d_{(2,1)}^{(2,0)}}{C_F} \left( \frac{1}{\epsilon} + \frac{1}{\epsilon^2} + L_\mu^2 + 2L_\mu L_\delta - \frac{\pi^2}{4} \right) \]

\[ + \frac{656}{81} T_R N_f + C_A \left( -\frac{2428}{81} + 16 \ln 2 - \frac{7\pi^4}{18} - 28 \ln 2 \zeta_3 + \frac{4}{3} \pi^2 \ln 2 - \frac{4}{3} \ln 4 - 32 \zeta_4 \left( \frac{1}{2} \right) \right) + O(\epsilon) \]

- SF calculated at NNLO
- All cancellations shown explicitly
- Depends on \(|\delta|\): process independent

\[ l_\delta = \ln \left( \frac{\mu^2}{|\delta|} \right) \]
5. Unpolarized TMDs at **NNLO**

- We calculated at NNLO the quark/gluon TMD PDFs and FFs:

\[
F_{f/A}(x, b_T; \zeta, \mu) = \sum_{j=q, \bar{q}, g} \int_x^1 \frac{d\bar{x}}{\bar{x}} \tilde{C}_{f/j}(x/\bar{x}, b_T; \zeta, \mu) f_{j/A}(\bar{x}; \mu)
\]

\[
D_{A/f}(z, b_T; \zeta, \mu) = \sum_{j=q, \bar{q}, g} \int_z^1 \frac{d\bar{z}}{\bar{z}^{3-2\epsilon}} C_{j/f}(z/\bar{z}, b_T; \zeta, \mu) d_{A/j}(\bar{z}; \mu)
\]

- With the modified $\delta$-regulator the virtual diagrams of TMDs are zero.

- Integrated PDFs/FFs also vanish, so we just have their renormalization:

\[
f_{f/f'}^{[1]} = \frac{-1}{\epsilon} P^{(1)}_{f/f'}
\]

\[
d_{f'/f}^{[1]} = \frac{-1}{\epsilon} P^{(1)}_{f'/f}
\]

\[
f_{f\rightarrow f'}^{[2]} = \frac{1}{2\epsilon^2} \left( \sum_r P^{(1)}_{f\rightarrow r} \otimes P^{(1)}_{r\rightarrow f'} + \beta^{(1)} P^{(1)}_{f\rightarrow f'} \right) - \frac{P^{(2)}_{f\rightarrow f'}}{2\epsilon}
\]

\[
d_{f\rightarrow f'}^{[2]} = \frac{1}{2\epsilon^2} \left( \sum_r P^{(1)}_{f\rightarrow r} \otimes P^{(1)}_{r\rightarrow f'} + \beta^{(1)} P^{(1)}_{f\rightarrow f'} \right) - \frac{P^{(2)}_{f\rightarrow f'}}{2\epsilon}
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\]
Matching coefficients for all channels

- \ (~100) non-zero diagrams
- \ (~20) master integrals (calculated at finite $\epsilon$)
- Checks: RGEs, crossing TMDPDF-TMDFF, cancellation of IR divergencies by topology,…
- Algebra done with Mathematica

\begin{align*}
    & \text{At LO:} \\
    C_{f \leftarrow f'}^{[0]} &= \delta_{f f'} \delta(\bar{x}) \\
    C_{f \rightarrow f'}^{[0]} &= \delta_{f f'} \delta(\bar{z})
\end{align*}

\begin{align*}
    & \text{At NLO:} \\
    C_{f \leftarrow f'}^{[1]} &= F_{f \leftarrow f'}^{[1]} - f_{f \leftarrow f'}^{[1]} \\
    C_{f \rightarrow f'}^{[1]} &= D_{f \rightarrow f'}^{[1]} - \frac{d_{f \rightarrow f'}^{[1]}}{z^{2-2\epsilon}} \\
    C_{f \leftarrow f'}^{[2]} &= F_{f \leftarrow f'}^{[2]} - \sum_r C_{f \rightarrow r}^{[1]} \otimes f_{r \leftarrow f'}^{[1]} - f_{f \leftarrow f'}^{[2]} \\
    C_{f \rightarrow f'}^{[2]} &= D_{f \rightarrow f'}^{[2]} - \sum_r C_{f \rightarrow r}^{[1]} \otimes \frac{d_{r \rightarrow f'}^{[1]}}{z^{2-2\epsilon}} - \frac{d_{f \rightarrow f'}^{[2]}}{z^{2-2\epsilon}}
\end{align*}

\begin{align*}
    & \text{At NNLO:} \\
    C_{f \rightarrow f'}^{[2]} &= D_{f \rightarrow f'}^{[2]} - \sum_r C_{f \rightarrow r}^{[1]} \otimes \frac{d_{r \rightarrow f'}^{[1]}}{z^{2-2\epsilon}} - \frac{d_{f \rightarrow f'}^{[2]}}{z^{2-2\epsilon}}
\end{align*}
Results

- We calculated both quark/gluon TMDPDF and TMDFF, all flavor channels.
- And their matching coefficients onto their integrated counterparts.
- Coefficients for TMDPDFs coincide with [Catani et al., Gehrmann et al.]
- For TMDFFs, the results are new.

Gluon TMDFF is considered for the first time, even at one loop.
Conclusions & Outlook

• **TMD evolution is universal, and currently known at NNLL’**

• **Matching coefficients for all unpolarized TMDs calculated at NNLO** (for TMDPDFs coincide with [Catani et.al., Gehrmann et.al.]). Ready to be used!

• **Gluon TMDFF is considered for the first time (even at one-loop)**

• **We only want to parametrize what cannot be calculated!**

• **TMD pheno is a mess: non-perturbative ingredients, different regions mixed under Fourier transform, need to match TMD and collinear regions,…**

• **The more accurately we know the perturbative pieces, the better we will constrain the non-perturbative ones**

★ **We need more experimental data:** unpolarized e+e-, more unpolarized Drell-Yan, etc

★ **Push the pheno:** perform global fits exploiting all available perturbative information

★ **Calculate polarized TMDs at NNLO**