Lattice Nucleon 
GPDs & Form Factors

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- Axial Charge & FFs
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MOTIVATION
Lattice QCD meets Nature

Rich experimental activities in major facilities
Electron Ion Collider
The Next QCD Frontier

“Understanding the glue that binds us all”

[A. Accardi et al., EIC white paper, arXiv:1212.1701]
Electron Ion Collider  
The Next QCD Frontier

“Understanding the glue that binds us all”  
[A. Accardi et al., EIC white paper, arXiv:1212.1701]

Lattice QCD necessary for EIC measurements

EIC program

- structure & interactions of gluon-dominated matter
- Measurements will probe the region of sea quarks
- parton imaging with high statistics and with polarization in a wide range of small to moderate-x

Lattice QCD

- Study of Gluon Observables is now feasible
- Simulations of the full theory with physical values of the $m_q$
  - Unpolarized, Polarized and Transversity Distributions can be computed from first principles
What does the Lattice Community try to achieve?

★ Make contact with well-known experimental data
★ Provide input for quantities not easily accessible in experiments
★ Guide New Physics searches
INTRODUCTION
Lattice formulation of QCD

★ Space-time discretization on a finite-sized 4-D lattice
  ● Quark fields on lattice points
  ● Gluons on links
Lattice formulation of QCD

★ Space-time discretization on a finite-sized 4-D lattice
  - Quark fields on lattice points
  - Gluons on links

Why Lattice QCD?

★ Only non-perturbative approach to solve *ab initio* QCD
  (starting from original Lagrangian)
Lattice formulation of QCD

★ Space-time discretization on a finite-sized 4-D lattice
  - Quark fields on lattice points
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Why Lattice QCD?

★ Only non-perturbative approach to solve ab initio QCD (starting from original Lagrangian)

Technical Aspects

★ Parameters (define cost of simulations):
  - quark masses (aim at physical values)
  - lattice spacing (ideally fine lattices)
  - lattice size (need large volumes)

★ Discretization not unique:
  - Wilson, Clover, Twisted Mass, Staggered, Overlap, Domain Wall
Advances in Lattice QCD

Huge computational power needed & Algorithmic improvements

Cost of 1000 configurations at physical $m_q$ is currently $\mathcal{O}(10)$ TFlops $\times$ year

$32^3 \times 64$

5000 configs

$L=2.1\text{fm}$

1000 configs
Nucleon on the Lattice in a nutshell

Topologies:

1. Connected
2. Disconnected Quark loop
3. Disconnected Gluon loop
Nucleon on the Lattice in a nutshell

Topologies:

- Connected
- Disconnected Quark loop
- Disconnected Gluon loop

Computation of 2pt- and 3pt-functions:

2pt: \( G(\vec{q}, t) = \sum_{\vec{x}_f} e^{-i\vec{x}_f \cdot \vec{q}} \Gamma_0^{\beta\alpha} \langle J_\alpha(\vec{x}_f, t_f) J_\beta(0) \rangle \)

3pt: \( G_O(\Gamma^\kappa, \vec{q}, t) = \sum_{\vec{x}_f, \vec{x}} e^{i\vec{x} \cdot \vec{q}} e^{-i\vec{x}_f \cdot \vec{p}'} \Gamma^{\kappa\beta\alpha} \langle J_\alpha(\vec{x}_f, t_f) O(\vec{x}, t) J_\beta(0) \rangle \)

\[ \Gamma^0 \equiv \frac{1}{4}(1 + \gamma_0) \]
\[ \Gamma^2 \equiv \Gamma^0 \cdot \gamma_5 \cdot \gamma_i \]
and other variations
Construction of optimized ratio:

\[ R_{\mu}^{\Omega}(\Gamma, \vec{q}, t) = \frac{G_{\Omega}(\Gamma, \vec{q}, t)}{G(\vec{0}, t_f)} \times \sqrt{\frac{G(-\vec{q}, t_f-t)G(\vec{0}, t)G(\vec{0}, t_f)}{G(\vec{0}, t_f-t)G(-\vec{q}, t)G(-\vec{q}, t_f)}} \]

**Plateau Method:**

\[ R_{\Omega}(\Gamma, \vec{q}, t) \xrightarrow{t \to \infty} t_f \xrightarrow{t \to t_i} \Pi^{\mu}(\Gamma, \vec{q}) \]

**Summation Method:**

\[ \sum_{t} R_{\Omega}(\Gamma, \vec{q}, t) \xrightarrow{t_f \to \infty} C + \Pi^{\mu}(\Gamma, \vec{q}) \times t_f \]
Construction of optimized ratio:

\[ R^\mu_\mathcal{O}(\Gamma, \vec{q}, t) = \frac{G_\mathcal{O}(\Gamma, \vec{q}, t)}{G(\vec{0}, t_f)} \times \sqrt{\frac{G(-\vec{q}, t_f - t)G(\vec{0}, t)G(\vec{0}, t_f)}{G(\vec{0}, t_f - t)G(-\vec{q}, t)G(-\vec{q}, t_f)}} \]

Plateau Method:

\[ R_\mathcal{O}(\Gamma, \vec{q}, t) \xrightarrow{t \to \infty} \Pi^\mu(\Gamma, \vec{q}) \]

Summation Method:

\[ \sum_t R_\mathcal{O}(\Gamma, \vec{q}, t) \xrightarrow{t_f \to \infty} \mathcal{C} + \Pi^\mu(\Gamma, \vec{q}) \times t_f \]

Renormalization:

connection to experiments

\[ \Pi^R(\Gamma, \vec{q}) = Z_\mathcal{O} \Pi(\Gamma, \vec{q}) \]

Extraction of form factors e.g. Axial current:

\[ A_\mu^3 \equiv \bar{\psi} \gamma_\mu \gamma_5 \frac{\tau^3}{2} \psi \Rightarrow \bar{u}_N(p') \left[ G_A(q^2) \gamma_\mu \gamma_5 + G_P(q^2) \frac{q_\mu \gamma_5}{2m_N} \right] u_N(p) \]
C

Nucleon FFs & GPDs

Axial Form Factor
Why is this quantity interesting?

Axial Charge

- governs the rate of $\beta$-decay
- Well-determined experimentally!
- related to the intrinsic spin $\Delta \Sigma = g_A$

Axial Form Factors

- Relevant for experiments searching neutrino oscillation

Not well control systematics (due to model-dependence)
Axial Charge

Determined directly from lattice data (no fit necessary)

Results at the physical point (disconnected diagram):

\[ g_A^{u+d}, g_A^s \]  (ETMC, 2016)
Axial Charge

Determined directly from lattice data (no fit necessary)

Results at the physical point (disconnected diagram):
\[ g_{u+d}^A, g_{s}^A \] (ETMC, 2016)

Reliable results:

- ★ Continuum extrapolation
- ★ Infinite Volume extrapolation
- ★ Excited states

Further study for larger \( T_{\text{sink}} \)
**Systematic Uncertainties (selected)**

**Excited States Contamination**

- \[ g_A \text{ Variational Comparison} \]

- [J. Dragos et al. (QCDSF/CSSM), arXiv:1606.03195]

- [C. Alexandrou et al. (ETMC), Lattice 2016]

**Proper analysis for suppression of excited states**

**Renormalization**

- \[ Z_A(\text{unsubtracted}) \]

- \[ Z_A(\text{O}(a^2 m_f)\text{-subtracted}) \]

- [M. Constantinou et al. (ETMC), arXiv:1509.00213]

- [Bhattacharya et al. (PNDME), arXiv:1606.07049]

**Sophisticated methods to eliminate lattice artifacts**
Axial Form Factor
Axial Form Factor

Extraction of axial mass:

\[ G_A(Q^2) = \frac{g_A}{\left(1 + \frac{Q^2}{M_A^2}\right)^2} \text{ dipole fit} \]

\[ G_A(Q^2) = \sum_{n=0}^{\infty} a_n z(Q^2)^n \text{ z-expansion} \]

Exp. data differ:

- \( M_A = 1.03(2) \text{ GeV} \) (\( \nu \)-scattering, prior-1990)
- \( M_A = 1.35(17) \text{ GeV} \) (Lower energy exp., 2010)
- \( M_A = 1.01(24) \text{ GeV} \) (\( z \)-expansion, 2016)

Lattice data:

- \( M_A = 1.24(8) \text{ GeV} \) (ETMC, \( m_p = 132 \text{ MeV} \))
- \( M_A = 1.02(4) \text{ GeV} \) (PNDME, \( m_p = 130 \text{ MeV} \))
- \( M_A = 1.24(14) \text{ GeV} \) (RBC/UKQCD, \( m_p = 172 \text{ MeV} \))

Effort is needed for estimates with reliable error budgets
C
Nucleon FFs & GPDs
2
Unpolarized GPDs
Unpolarized GPDs

★ Distribution of nucleon momentum among its constituents

★ First non-trivial moment
  (moment fixed by the number of valence quarks)

★ Measured in DIS experiments
  Value uses input from phenomenological models

★ Benchmark quantity for lattice QCD calculations

Quark Momentum Fraction

\[
\langle N(p', s') | O_{DV}^{\mu \nu} | N(p, s) \rangle = \bar{u}_N(p', s') \left[ A_{20}(q^2) \gamma^{\mu P^\nu} \right. \\
+ B_{20}(q^2) \frac{i \sigma^{\mu \alpha q_\alpha P^\nu}}{2m} \\
+ C_{20}(q^2) \frac{1}{m} q^{\mu q^\nu} \left. \right] u_N(p, s)
\]

Isovector Combination

Excited States must be assessed

\[
m_\pi = 130 \text{ MeV}
\]

\[
m_\pi = 340 \text{ MeV}
\]

Excited States must be assessed

[C. Alexandrou et al. (ETMC), Lattice 2016]

[T. Rae et al. (Mainz Group), 2014]
Quark Momentum Fraction
(Disconnected: light quarks)

[C. Alexandrou et al. (ETMC), Lattice 2016]

Discrete contributions not negligible (@ physical point)!

\[ \langle x \rangle_{u+d}^{DI} = 0.21(10) \]

mixing with gluon operator

Directly at the physical point
Quark Momentum Fraction
(Disconnected: strange quark)

[C. Alexandrou et al. (ETMC), Lattice 2016]

\[ \langle x \rangle^{DI}_s = 0.08(5) \]

[M. Sun et al. (χQCD), arXiv:1502.05482]

\[ \langle x \rangle_{u/d} = 0.0285(57) \]
\[ \langle x \rangle_s = 0.0195(26) \]

chiral extrapolation, bare results

\textbf{Ratio} \( \langle x \rangle_s / \langle x \rangle_{u/d} \) consistent between lattice data and exp.

\( \langle x \rangle_s / \langle x \rangle_{u/d} = 0.76(30) \) (ETMC)
\( \langle x \rangle_s / \langle x \rangle_{u/d} = 0.78(03) \) (χQCD)

\textbf{Small } x \textbf{ region:}
• dominated by disc. sea
• Ratio \( \sim \) flat

Milestone calculations in nucleon structure!
C

Nucleon FFs & GPDs

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Gluon Momentum Fraction
Gluon Momentum Fraction

Lattice Calculations

Direct computation:

\[ O_{\mu\nu}^g = -\text{Tr} \left[ G_{\mu\rho} G_{\nu\rho} \right] \]

\[ \langle N(0) | O_{44} - \frac{1}{3} \sum_{j=1}^3 O_{jj} | N(0) \rangle = m_N \langle x \rangle_g \]

Decomposition of Energy-momentum Tensor

\[ J^i_{q,g} = \frac{1}{2} \epsilon^{ijk} \int d^3 x \left( \mathcal{T}^{0k}_{q,g} x^j - \mathcal{T}^{0j}_{q,g} x^k \right) \]

\[ \mathcal{T}^{(E)}_{\{4i\}q} = -\frac{i}{4} \sum_f \overline{\psi}_f \left[ \gamma_4 \overrightarrow{D}_i + \gamma_i \overrightarrow{D}_4 - \gamma_4 \overleftarrow{D}_i - \gamma_i \overleftarrow{D}_4 \right] \psi_f \]

\[ \mathcal{T}^{(E)}_{\{4i\}g} = -\frac{i}{2} \sum_{k=1}^3 2\text{Tr} \left[ G_{4k}G_k + G_{ik}G_{k4} \right] \]
Lattice Results

Quenched

Feynman-Hellmann

\[ \langle x \rangle_g = 0.43(7)(5) \]

Energy-Momentum tensor

\[ \langle x \rangle_g = 0.313(56) \]

Dynamical

Smearing: improves signal

\[ N_f = 2 \text{ TM fermions, } m_\pi = 130 \text{ MeV} \]

Renormalized results require work!
Challenges

★ Disconnected diagram
  • Small signal-to-noise ratio  • Requires special techniques

★ Renormalization
  • Mixing with operator for $\langle x \rangle_{u+d}$
    Unavoidable
  • Mixing with other Operators
    Gauge Invariant, BRS transformation, vanish by e.o.m.
    Vanish in physical matrix elements
Challenges

*Disconnected diagram*
- Small signal-to-noise ratio
- Requires special techniques

*Renormalization*
- Mixing with operator for $\langle x \rangle_{u+d}$
- Unavoidable
- Mixing with other Operators
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$2 \times 2$ mixing matrix

\[
\begin{pmatrix}
\langle x \rangle_g^{\overline{MS}}(\mu) \\
\sum_q \langle x \rangle_q^{\overline{MS}}(\mu)
\end{pmatrix} =
\begin{pmatrix}
Z_{gg}^{MS}(\mu) & Z_{gq}^{MS}(\mu) \\
Z_{qg}^{MS}(\mu) & Z_{qq}^{MS}(\mu)
\end{pmatrix}
\begin{pmatrix}
\langle x \rangle_g \\
\sum_q \langle x \rangle_q
\end{pmatrix}
\]

\[\langle x \rangle_g^R = Z_{gg} \langle x \rangle_g^B + Z_{gq} \sum_q \langle x \rangle_q^B\]
\[\sum_q \langle x \rangle_q^R = Z_{qq} \sum_q \langle x \rangle_q^B + Z_{qg} \langle x \rangle_g^B\]

*Quenched case:* $Z_{qq} = 1 - Z_{qq}$, $Z_{gq} = 1 - Z_{qq}$
Challenges

★ Disconnected diagram
- Small signal-to-noise ratio
- Requires special techniques

★ Renormalization
- Mixing with operator for $\langle x \rangle_{u+d}$  
  Unavoidable
- Mixing with other Operators  Gauge Invariant, BRS transformation, vanish by e.o.m.
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2 x 2 mixing matrix

\[
\begin{pmatrix}
\langle x \rangle_{g}^{MS}(\mu) \\
\sum_q \langle x \rangle_{q}^{MS}(\mu)
\end{pmatrix}
= 
\begin{pmatrix}
Z_{gg}^{MS}(\mu) & Z_{gq}^{MS}(\mu) \\
Z_{qg}^{MS}(\mu) & Z_{qq}^{MS}(\mu)
\end{pmatrix}
\begin{pmatrix}
\langle x \rangle_{g} \\
\sum_q \langle x \rangle_{q}
\end{pmatrix}
\]

★ Quenched case:  $Z_{gq} = 1 - Z_{qq}, Z_{gq} = 1 - Z_{qq}$

MUST compute mixing coefficients and subtract contributions

Perturbation Theory
Perturbative computation

\[ \times Z_{qq} : \quad \Lambda_{qq} = \langle q | O_q | q \rangle \]

\[ \times Z_{qg} : \quad \Lambda_{qg} = \langle q | O_g | g \rangle \]

\[ \bullet Z_{gg} : \quad \Lambda_{gg} = \langle g | O_g | g \rangle \]
Elimination of mixing

Application for TM fermions

\[ \langle x \rangle_{u+d+s}^R = Z_{qq} \langle x \rangle_{u+d+s} + Z_{qg} \langle x \rangle_g = 0.748(105) \]

\[ \langle x \rangle_g^R = Z_{gg} \langle x \rangle_g + Z_{gq} \langle x \rangle_{u+d+s} = 0.320(24) \]

Momentum Conservation

\[ \sum_{q=u,d,s} \langle x \rangle_q^R + \langle x \rangle_G^R = \langle x \rangle_{u+d}^{CI,R} + \langle x \rangle_{u+d+s}^{DI,R} + \langle x \rangle_G^R = 1.068(108) \]

Energy-momentum Tensor

Y.-B. Yang et al., (χQCD), 2016

\( m_{\pi,v} = 400 \text{ MeV}, \ m_{\pi,s} = 170 \text{ MeV} \)

Preliminary

Large gluon contribution

[27]
C
Nucleon FFs & GPDs
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Proton Spin
Proton Spin: Can we put the puzzle together?

Spin Structure from First Principles

Spin Sum Rule:

\[
\frac{1}{2} = \sum_q J^q + J^G = \sum_q \left( L^q + \frac{1}{2} \Delta \Sigma^q \right) + J^G
\]

- \( L_q \): Quark orbital angular momentum
- \( \Delta \Sigma_q \): Intrinsic spin
- \( J^G \): Gluon part
Proton Spin: Can we put the puzzle together?

Spin Structure from First Principles

Spin Sum Rule:

\[ \frac{1}{2} = \sum_q J^q + J^G = \sum_q \left( L^q + \frac{1}{2} \Delta \Sigma^q \right) + J^G \]

- \( L^q \): Quark orbital angular momentum
- \( \Delta \Sigma^q \): intrinsic spin
- \( J^G \): Gluon part

Extraction from LQCD:

\[ J^q = \frac{1}{2} (A_{20}^q + B_{20}^q) \, , \, \, L^q = J^q - \Sigma^q \, , \, \Sigma^q = g_A^q \]

★ Individual quark contributions: disconnected insertion contributes
Quark Contributions to Spin

Valence Quarks Contributions

Total Spin

Intrinsic Spin

Valence Quark carry $\sim$ half of the proton spin

Where does the rest of the spin come from?

- Sea Quark Contributions
- Gluon Contributions
Quark Contributions to Spin

Valence + Sea Quarks Contributions

Total Spin

Intrinsic Spin

★ Sea Quark contribution bring data in agreement with experiment!
Energy-Momentum Tensor

Glue Spin

- glue, 0.37(7)
- s, 0.02(2)
- d, -0.03(3)
- u, 0.64(6)

Angular Momentum

Preliminary

HYP smearing
LML: ($\mu^2 = 10$ GeV$^2$)
$S_G = 0.287(55)(16)$

1-loop pert. renormalization & normalization of gluon self-energy

Talk by Yi-Bo Yang, Mon @ 12:20pm

[Y.-B. Yang et al. (χ QCD), arXiv:1609.05937]
SUMMARY

Lattice QCD milestones:

★ Simulations of the physical world
★ Large effort on addressing the systematics
★ Calculation of more involved quantities
★ New approaches to address parton distributions e.g. quasi-PDFs (Ji’s definition)
★ Predictions related to Physics BSM
Join us!

Joint POETIC7 & CTEQ Meeting

7th International Conference on Physics Opportunities @ Electron-Ion-Collider

Temple University, November 14-18, 2016

THANK YOU