Helicity Evolution at Small x

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work with Dan Pitonyak and Matt Sievert,
arXiv:1511.06737 [hep-ph]
+ 2 more papers in preparation

Outline

- Goal: understanding proton spin at small x
- Observables: quark helicity TMD & PDF at small-x, g_1 structure function
- Small-x evolution for the "polarized dipole":
  - New helicity evolution evolution equations at small x
  - Large-N_C limit
  - Large N_C & N_f limit
- Solution of the large-N_C evolution equations:
  - small-x asymptotics of the g_1 structure function, quark hPDFs and helicity TMDs
  - impact on proton spin

Our Goals

Proton Spin Puzzle

- Helicity sum rule (Jaffe & Manohar form): $\frac{1}{2} = S_q + L_q + S_g + L_g$

  with the net quark and gluon spin

$$S_q(Q^2) = \frac{1}{2}\int_0^1 dx\,\Delta\Sigma(x,Q^2) \qquad S_g(Q^2) = \int_0^1 dx\,\Delta G(x,Q^2)$$

- The helicity parton distributions are (f = G, u, d, s, …)

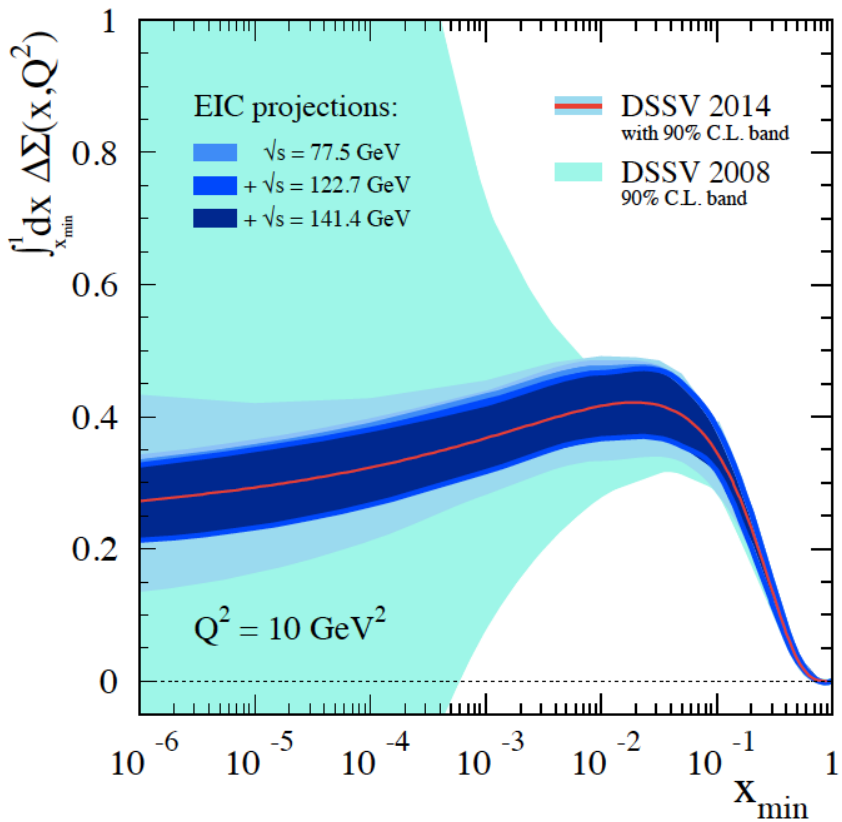
$$\Delta f(x,Q^2) \equiv f^+(x,Q^2) - f^-(x,Q^2)$$

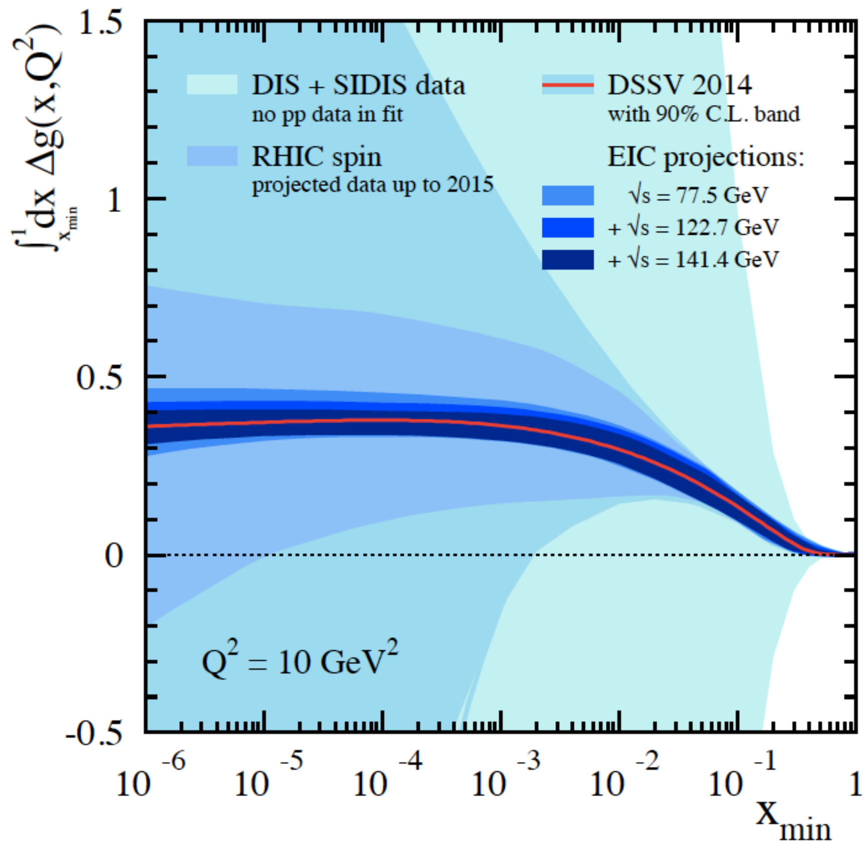
  with the net quark helicity distribution

$$\Delta\Sigma \equiv \Delta u + \Delta\bar{u} + \Delta d + \Delta\bar{d} + \Delta s + \Delta\bar{s}$$

- L_q and L_g are the quark and gluon orbital angular momenta

How much spin is at small x?





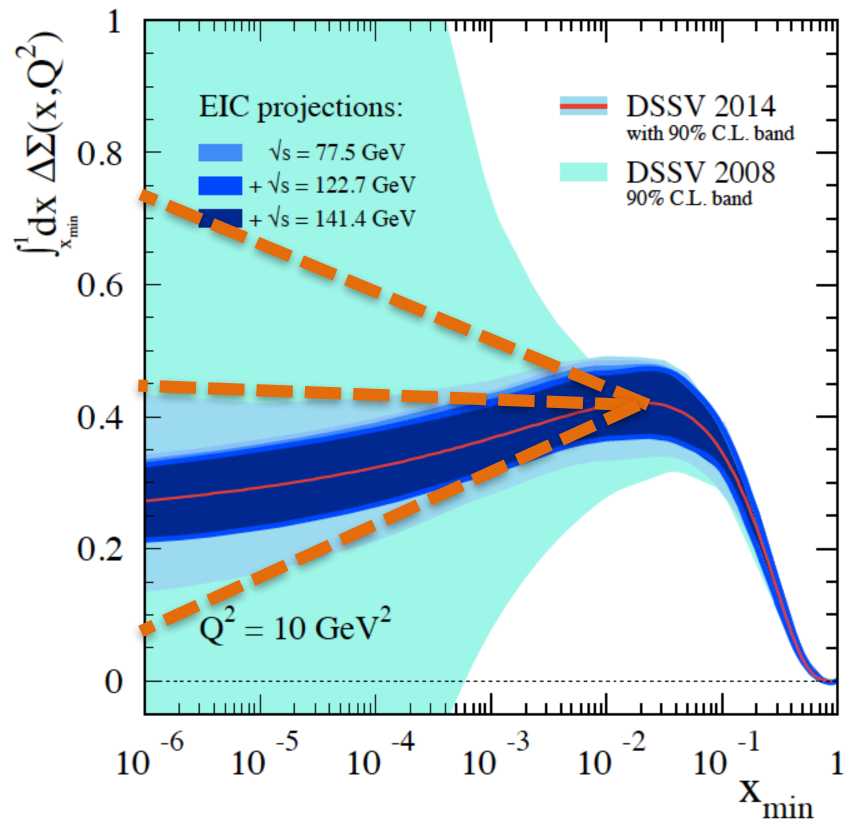
- E. Aschenaur et al, arXiv:1509.06489 **[hep-ph]**
- Uncertainties are very large!

Spin at small x

- The goal of this project is to provide theoretical understanding of helicity PDF's at very small x.

- Our work would provide guidance for future hPDF's parametrizations.

- Strictly-speaking we only talk about quark helicity, but most likely our analysis applies to gluon hPDF's as well.

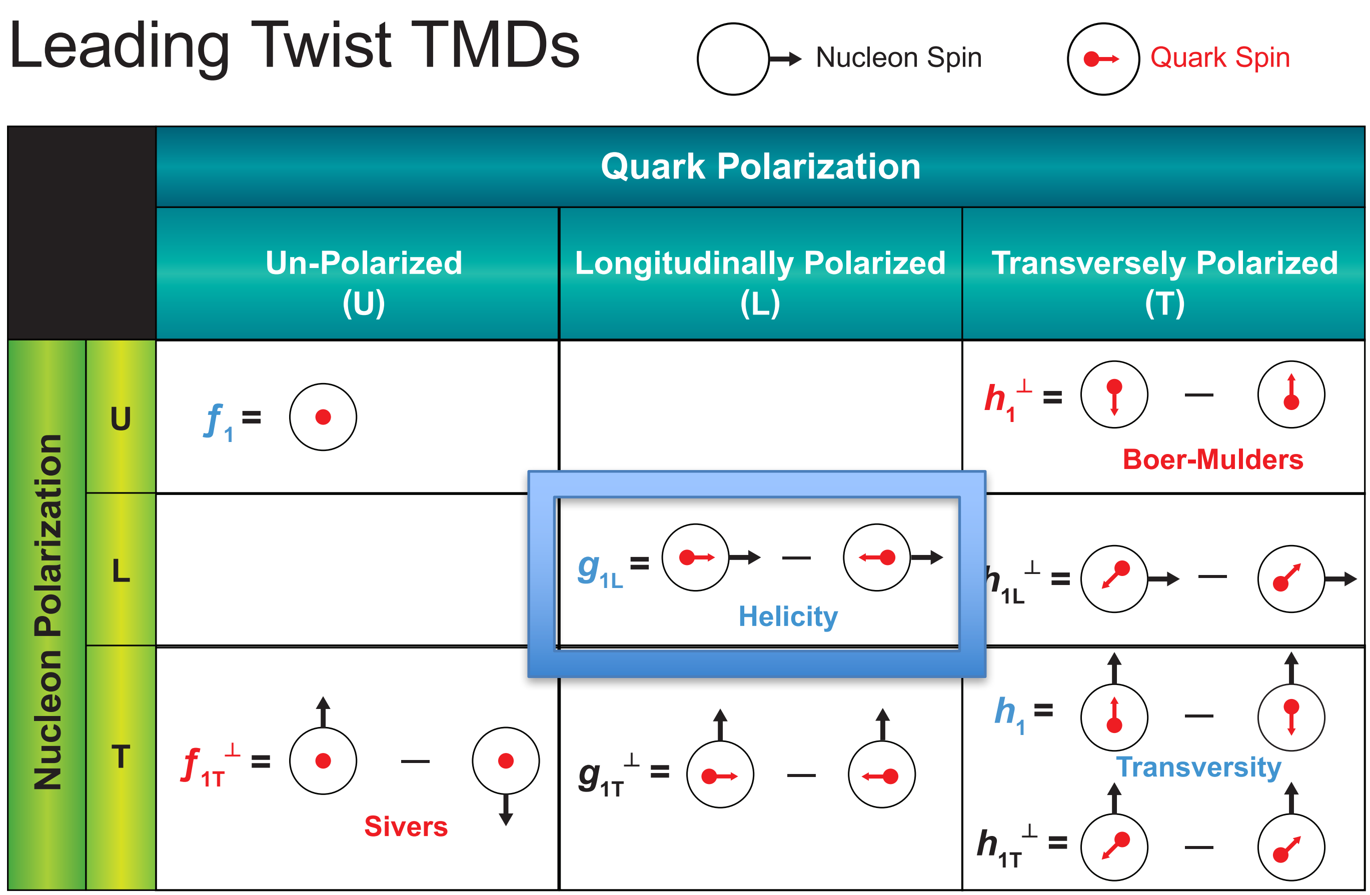


Helicity Observables

Yu.K., M. Sievert, arXiv:1505.01176 [hep-ph]
Yu.K., D. Pitonyak, M. Sievert, arXiv:1511.06737 [hep-ph]

Observables

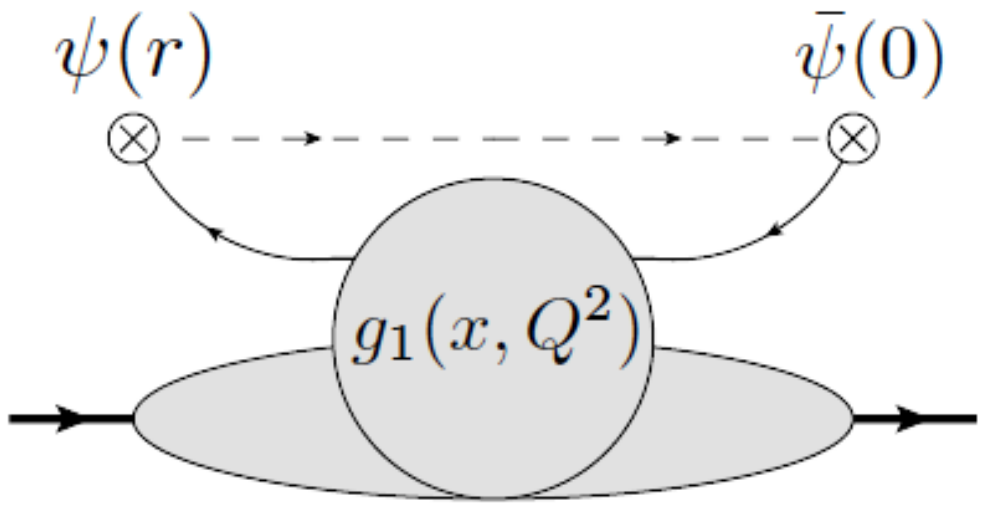
- We want to calculate quark helicity PDF and TMD and the g_1 structure function.



Quark Helicity TMD

- We could start by simply calculating quark TMD's using the operator definition:

$$g_1(x, Q^2) = \int dr^- \, e^{ixp^+r^-} \langle pS|\bar{\psi}(0)\frac{\gamma^+\gamma^5}{2}\psi(r)|pS\rangle$$

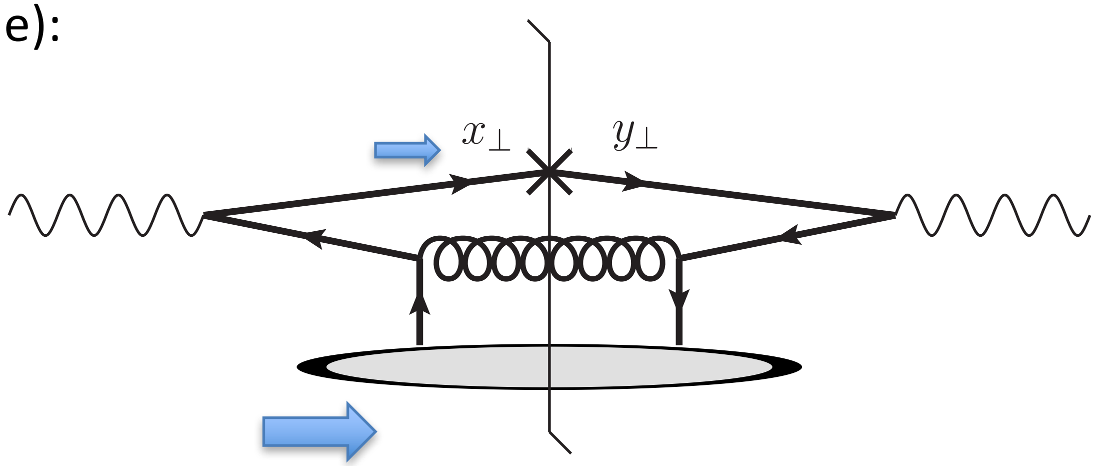


- Instead we will find the TMDs from the SIDIS cross section.

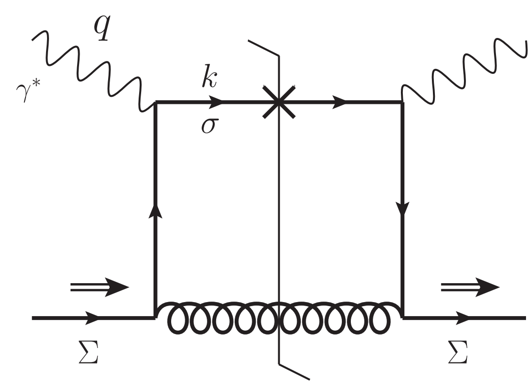
SIDIS on a Spin-Dependent Target

To transfer spin information between the polarized target and the produced quark we either need to exchange quarks in the t-channel, or non-eikonal gluons.

Here's an example of the quark exchange (we work in the $A^+=0$ light cone gauge of the projectile):

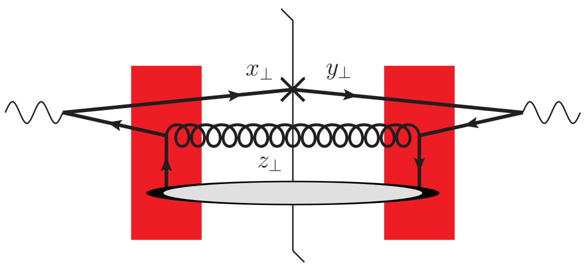


This is in addition to the standard handbag diagram which does not evolve under our small-x evolution:

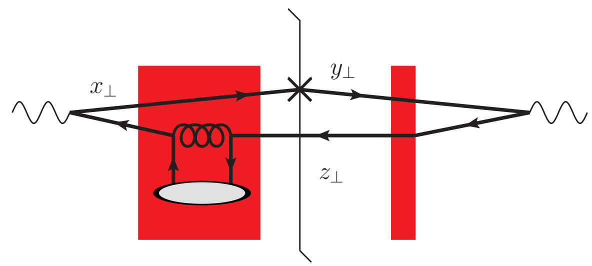


Target Spin-Dependent SIDIS

It is straightforward to include multiple shock wave interactions into the polarized SIDIS cross section:

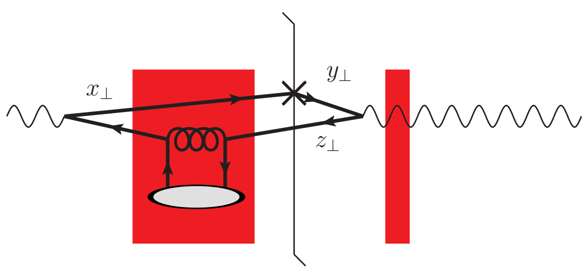


$+$



$+$ c.c. $=0$

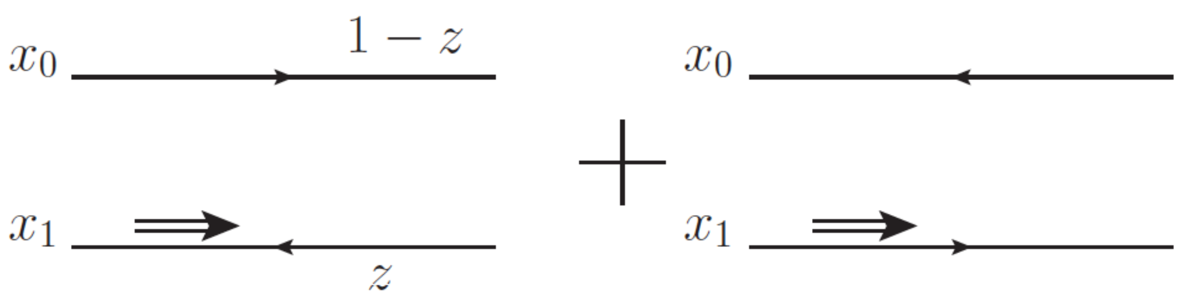
$+$



$+$ c.c. = the answer

Polarized Dipole

- All flavor singlet small-x helicity observables depend on one object, "polarized dipole amplitude":



$$G_{10}(z) \equiv \frac{1}{2N_c} \left\langle\!\left\langle \mathrm{tr}\left[V_{\underline{0}} V_{\underline{1}}^{pol\,\dagger} \right] + \mathrm{tr}\left[V_{\underline{1}}^{pol} V_{\underline{0}}^{\dagger} \right] \right\rangle\!\right\rangle (z)$$

unpolarized quark

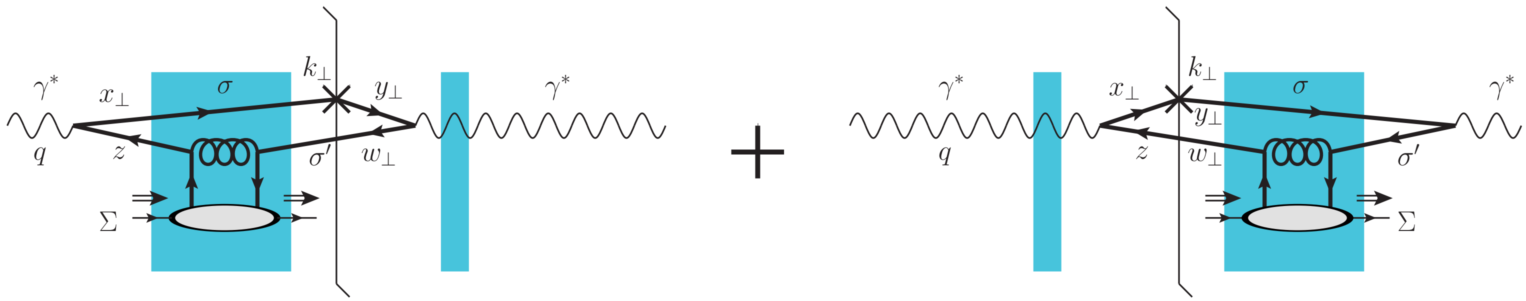
polarized quark: eikonal propagation, non-eikonal spin-dependent interaction

$$V_{\underline{x}} \equiv \mathcal{P} \exp \left[ig \int\limits_{-\infty}^{\infty} dx^+ \, A^-(x^+, 0^-, \underline{x}) \right]$$

- Double brackets denote an object with energy suppression scaled out:

$$\left\langle\!\left\langle \mathcal{O} \right\rangle\!\right\rangle (z) \equiv zs \left\langle \mathcal{O} \right\rangle (z)$$

Quark Helicity Observables at Small x



- One can show that the g_1 structure function and quark helicity PDF and TMD at small-x can be expressed in terms of the polarized dipole amplitude (flavor singlet case):

$$g_1^S(x,Q^2) = \frac{N_c\,N_f}{2\,\pi^2 \alpha_{EM}} \int\limits_{z_i}^{1} \frac{dz}{z^2(1-z)} \int dx_{01}^2 \left[\frac{1}{2} \sum_{\lambda\sigma\sigma'} |\psi^T_{\lambda\sigma\sigma'}|^2_{(x_{01}^2,z)} + \sum_{\sigma\sigma'} |\psi^L_{\sigma\sigma'}|^2_{(x_{01}^2,z)} \right] G(x_{01}^2, z),$$

$$\Delta q^S(x,Q^2) = \frac{N_c\,N_f}{2\pi^3} \int\limits_{z_i}^{1} \frac{dz}{z} \int\limits_{\frac{1}{zs}}^{\frac{1}{zQ^2}} \frac{dx_{01}^2}{x_{01}^2}\, G(x_{01}^2, z),$$

$$g_{1L}^S(x,k_T^2) = \frac{8\,N_c\,N_f}{(2\pi)^6} \int\limits_{z_i}^{1} \frac{dz}{z} \int d^2x_{01}\, d^2x_{0'1}\, e^{-i\underline{k}\cdot(\underline{x}_{01}-\underline{x}_{0'1})}\, \frac{\underline{x}_{01}\cdot\underline{x}_{0'1}}{x_{01}^2 x_{0'1}^2}\, G(x_{01}^2, z)$$

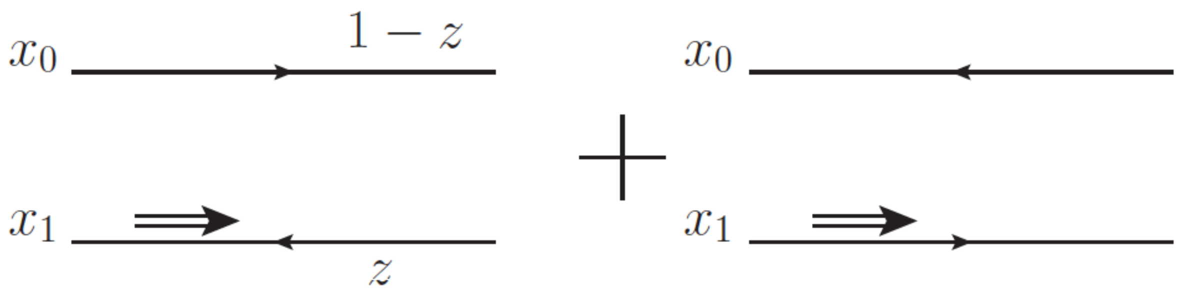
- Here s is cms enery squared, $z_i=\Lambda^2/s$, $G(x_{01}^2, z) \equiv \int d^2b\, G_{10}(z)$

Helicity Evolution at Small x

Yu.K., D. Pitonyak, M. Sievert, arXiv:1511.06737 [hep-ph]

Polarized Dipole

- Our goal now is to construct a small-x evolution equation for the "polarized dipole amplitude".



$$G_{10}(z) \equiv \frac{1}{2N_c} \left\langle\!\!\left\langle \mathrm{tr}\left[V_{\underline{0}} V_{\underline{1}}^{pol\,\dagger}\right] + \mathrm{tr}\left[V_{\underline{1}}^{pol} V_{\underline{0}}^{\dagger}\right] \right\rangle\!\!\right\rangle(z)$$

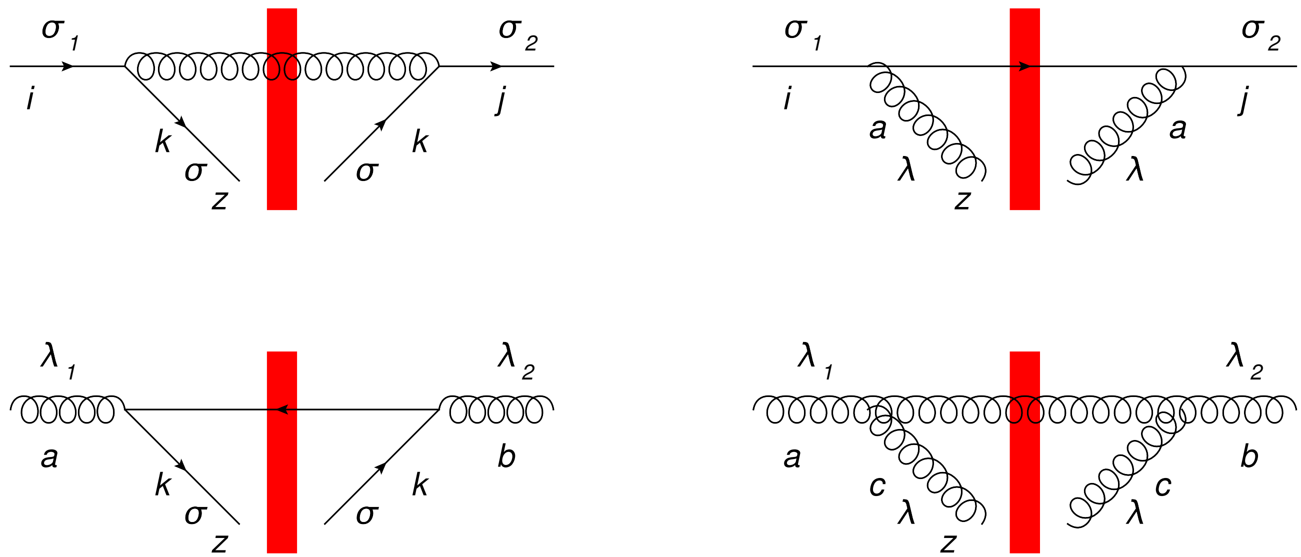
unpolarized quark polarized quark

- Double brackets denote an object with energy suppression scaled out:

$$\left\langle\!\!\left\langle \mathcal{O} \right\rangle\!\!\right\rangle(z) \equiv zs \left\langle \mathcal{O} \right\rangle(z)$$

Helicity Evolution Ingredients

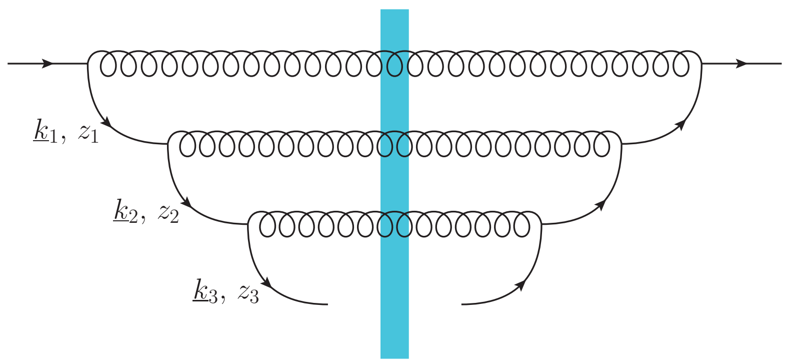
- Unlike the unpolarized evolution (glue only), in one step of helicity evolution we may emit a soft gluon or a soft quark (all in $A^+=0$ LC gauge of the projectile):

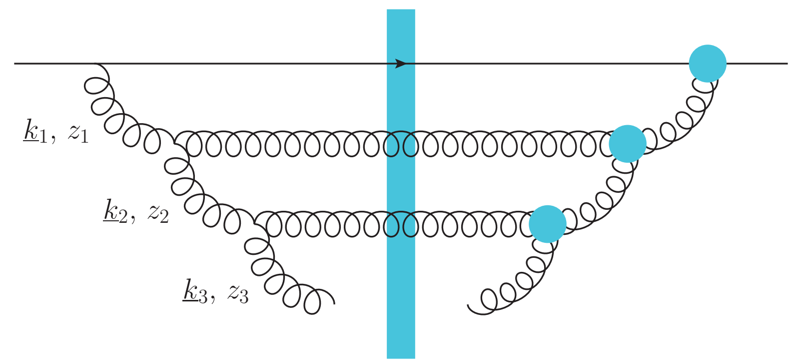


- When emitting gluons, one emitted gluon is eikonal, while another one is soft, but non-eikonal, as is needed to transfer polarization down the cascade/ladder.

Helicity Evolution: Ladders

- To get an idea of how the helicity evolution works let us try iterating the splitting kernels by considering ladder diagrams (circles denote non-eikonal gluon vertices):

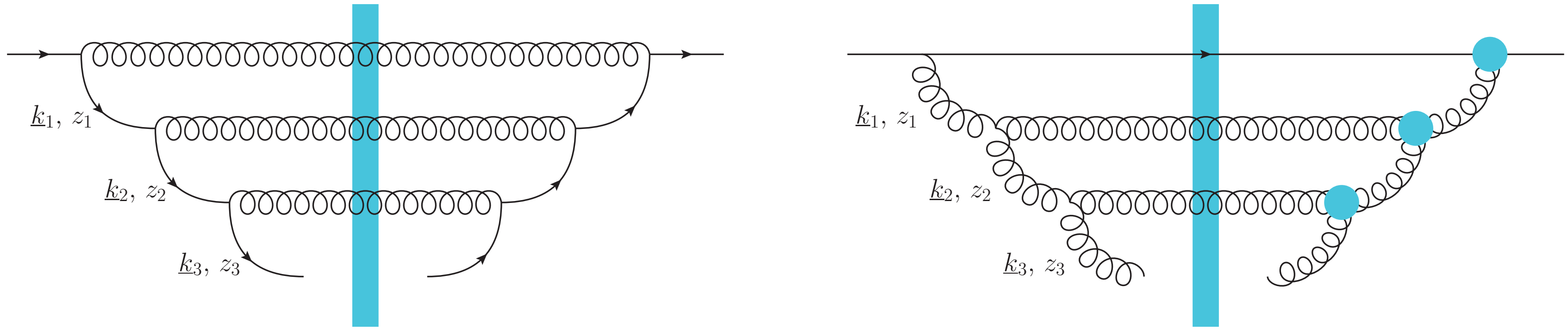




- To get the leading-energy asymptotics we need to order the longitudinal momentum fractions of the quarks and gluons (just like in the unpolarized evolution case) $1 \gg z_1 \gg z_2 \gg z_3 \gg \ldots$

obtaining a nested integral $$\alpha_s^3 \int\limits_{z_i}^{1} \frac{dz_1}{z_1} \int\limits_{z_i}^{z_1} \frac{dz_2}{z_2} \int\limits_{z_i}^{z_2} \frac{dz_3}{z_3}\, z_3 \otimes \frac{1}{z_3\, s} \sim \frac{1}{s}\, \alpha_s^3\, \ln^3 s$$

Helicity Evolution: Ladders



- However, these are not all the logs of energy one can get here. Transverse momentum (or distance) integrals have UV and IR divergences, which lead to logs of energy as well.
- If we order transverse momenta / distances as (Sudakov-β ordering)

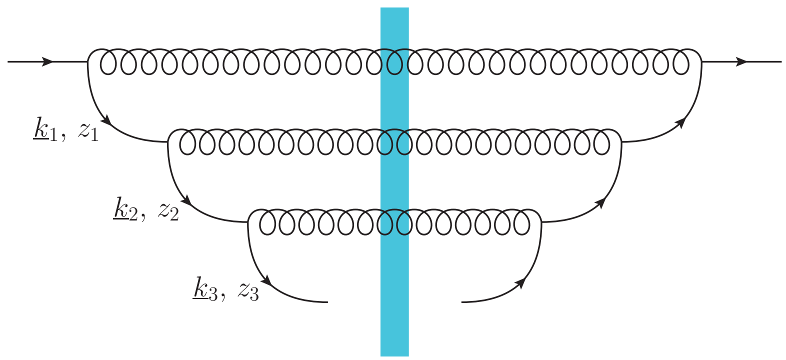
$$\frac{\underline{k}_1^2}{z_1} \ll \frac{\underline{k}_2^2}{z_2} \ll \frac{\underline{k}_3^2}{z_3} \ll \ldots \qquad\qquad z_1\, \underline{x}_1^2 \gg z_2\, \underline{x}_2^2 \gg z_3\, \underline{x}_3^2 \gg \ldots$$

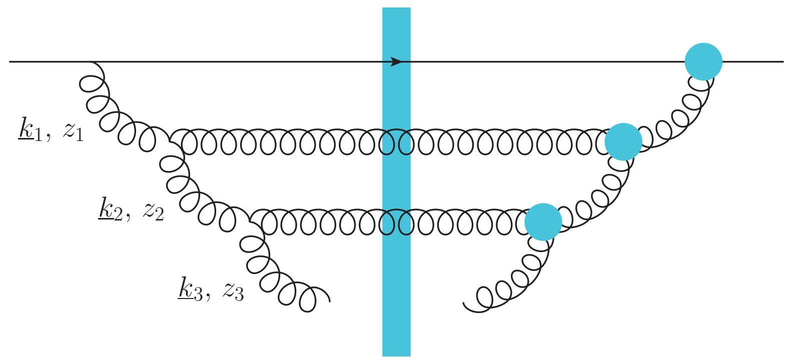
we would get integrals like

$$\int\limits_{1/(z_n\, s)}^{x^2_{n-1,\perp}\, z_{n-1}/z_n} \frac{dx^2_{n,\,\perp}}{x^2_{n,\,\perp}}$$

also generating logs of energy.

Helicity Evolution: Ladders





- To summarize, the above ladder diagrams are parametrically of the order

$$\frac{1}{s}\,\alpha_s^3\ \ln^6 s$$

- Note two features:
  - 1/s suppression due to non-eikonal exchange
  - two logs of energy per each power of the coupling!

Resummation Parameter

- For helicity evolution the resummation parameter is different from BFKL, BK or JIMWLK, which resum powers of leading logarithms (LLA)

$$\alpha_s \ \ln(1/x)$$

- Helicity evolution resummation parameter is double-logarithmic (DLA):

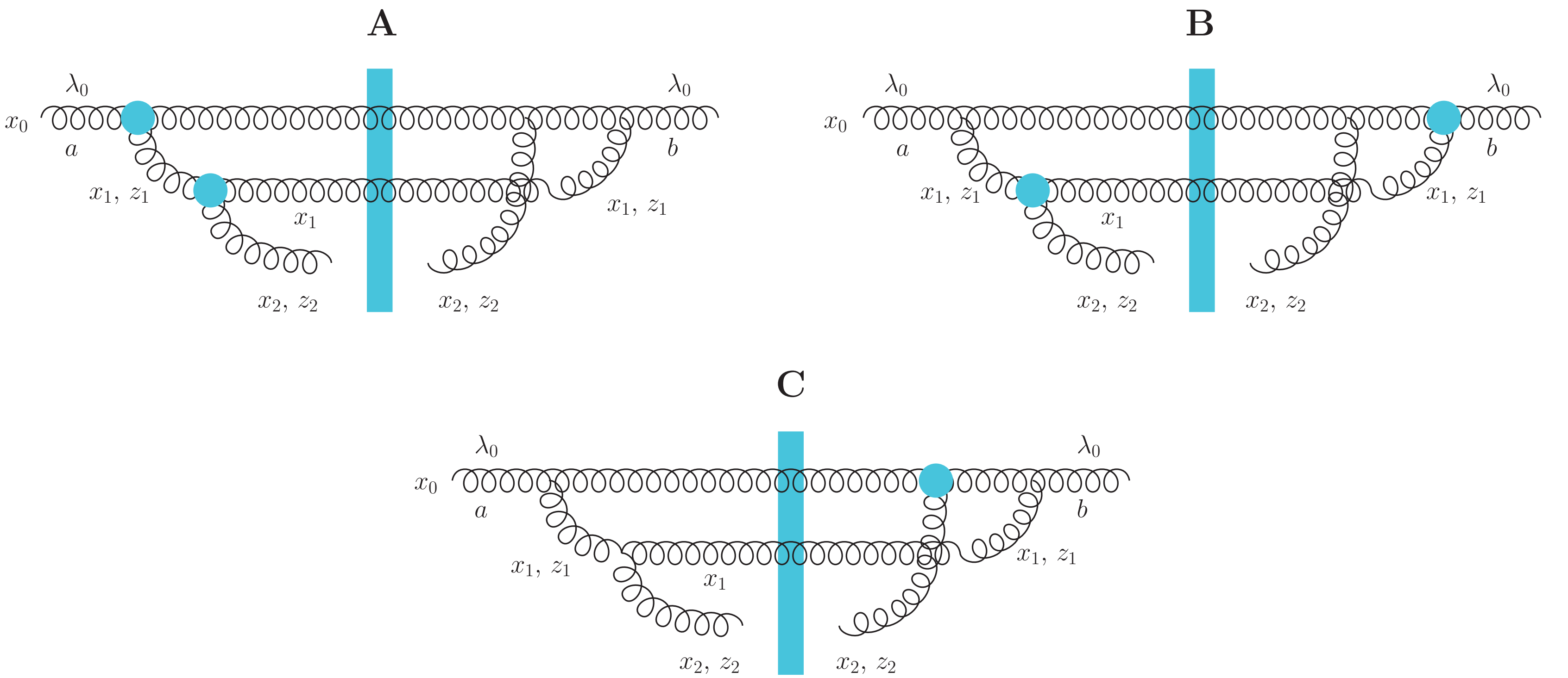
$$\alpha_s \ \ln^2 \frac{1}{x}$$

- The second logarithm of x arises due to transverse momentum integration being logarithmic both in UV and IR.

- This was known before: Kirschner and Lipatov ’83; Kirschner ’84; Bartels, Ermolaev, Ryskin ‘95, ‘96; Griffiths and Ross ’99; Itakura et al ’03; Bartels and Lublinsky ‘03.

Non-Ladder Diagrams

- Ladder diagrams are not the whole story. The non-ladder diagrams below are also leading-order (that is, DLA).

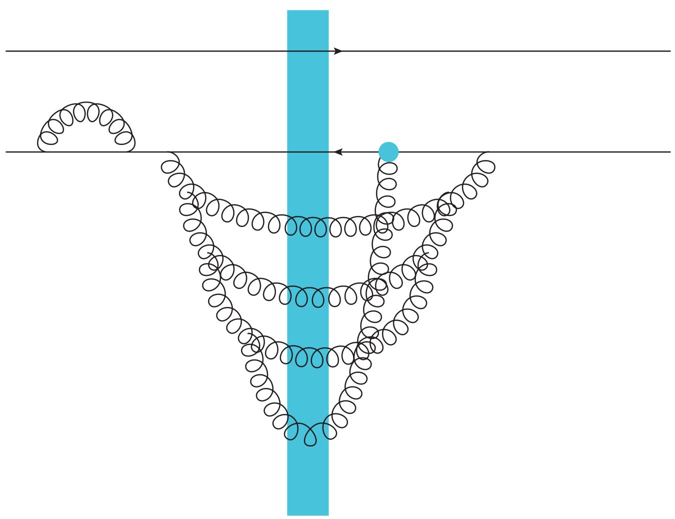


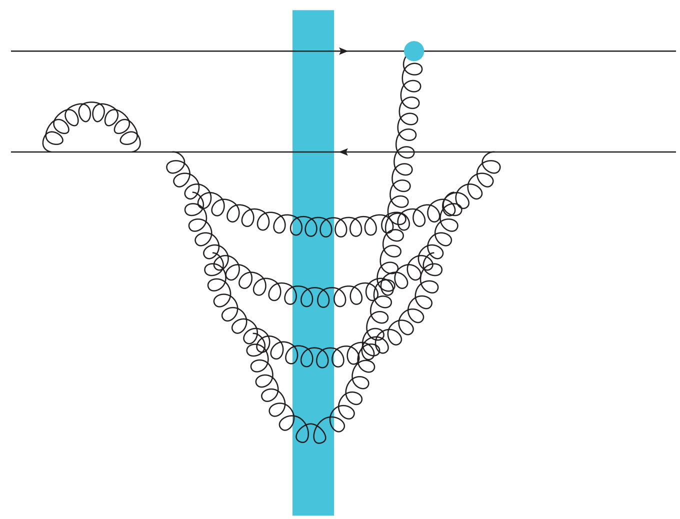
- Non-ladder soft quark emissions cancel for flavor-singlet observables we are primarily interested in. Non-ladder gluons do not cancel.

Virtual Corrections

- In addition, virtual corrections from the unpolarized LLA evolution have UV divergences, which cancel between real and virtual diagrams. Here the corrections are not cancelled, but are regulated by the cms energy.

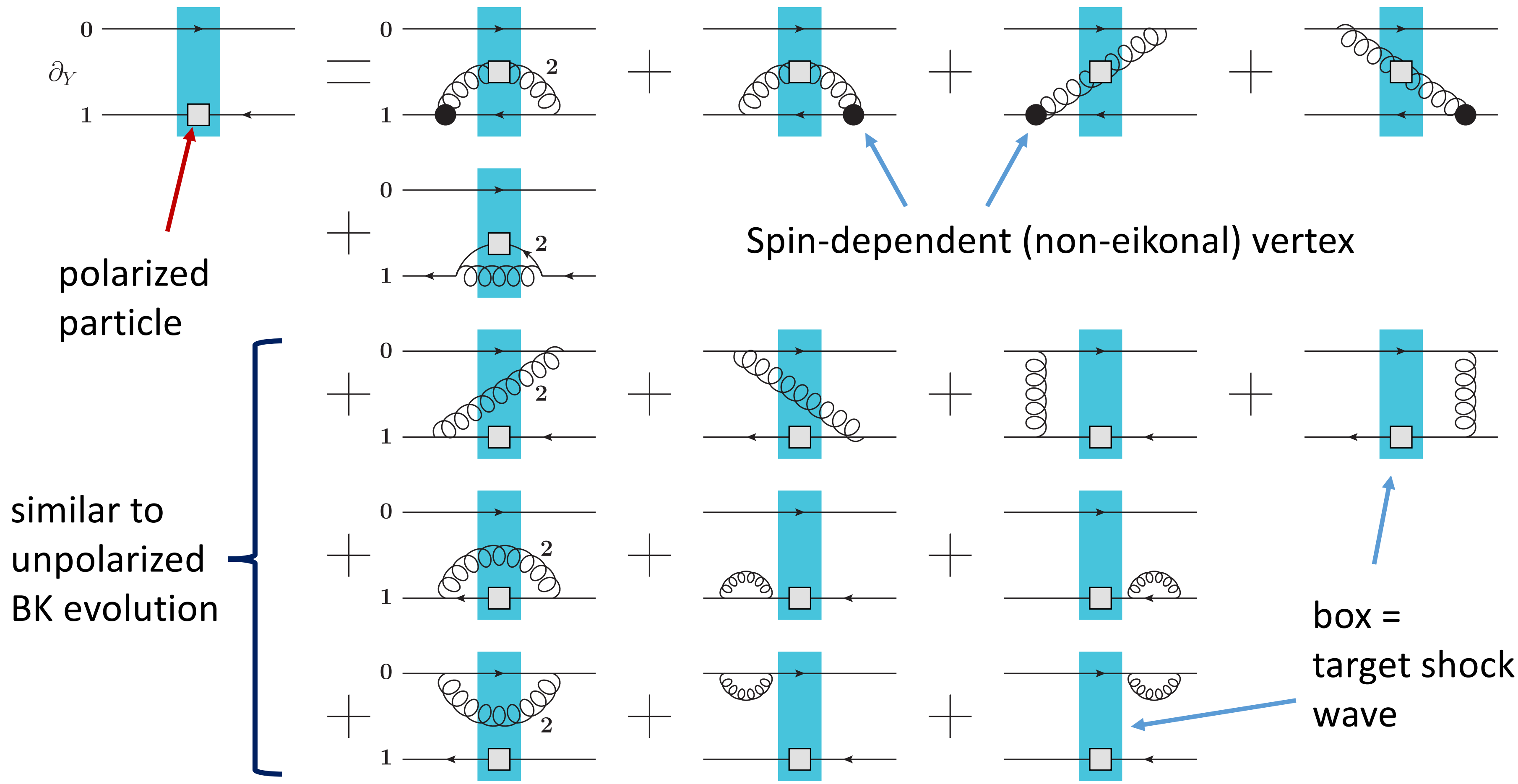
- Helicity evolution thus also contains the following types of graphs:



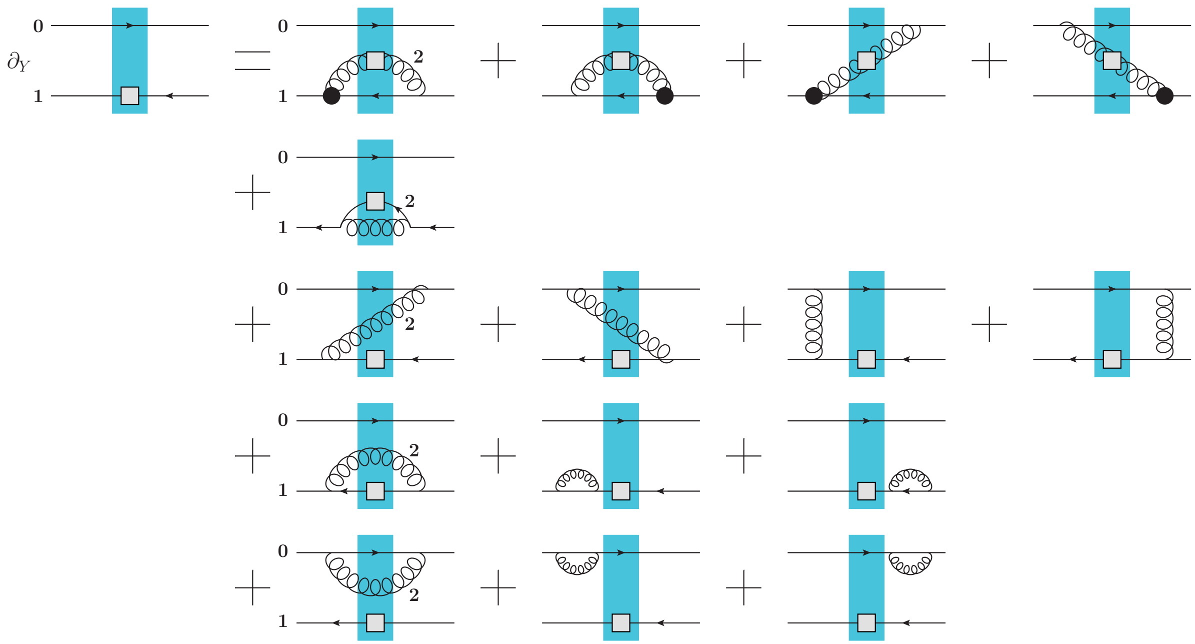


Evolution for Polarized Quark Dipole

One can construct an evolution equation for the polarized dipole:



Evolution for Polarized Quark Dipole



$$\langle\langle \ldots \rangle\rangle = \frac{1}{z\, s} \langle \ldots \rangle$$

$$\rho'^{\,2} = \frac{1}{z'\, s}$$

$$\frac{1}{N_c} \left\langle\!\!\left\langle \mathrm{tr}\left[V_0^{unp}\, V_1^{pol\,\dagger}\right] \right\rangle\!\!\right\rangle (z) = \frac{1}{N_c} \left\langle\!\!\left\langle \mathrm{tr}\left[V_0^{unp}\, V_1^{pol\,\dagger}\right] \right\rangle\!\!\right\rangle_0 (z) + \frac{\alpha_s}{2\pi^2} \int\limits_{z_i}^{z} \frac{dz'}{z'} \int\limits_{\rho'^2} \frac{d^2 x_2}{x_{21}^2}$$

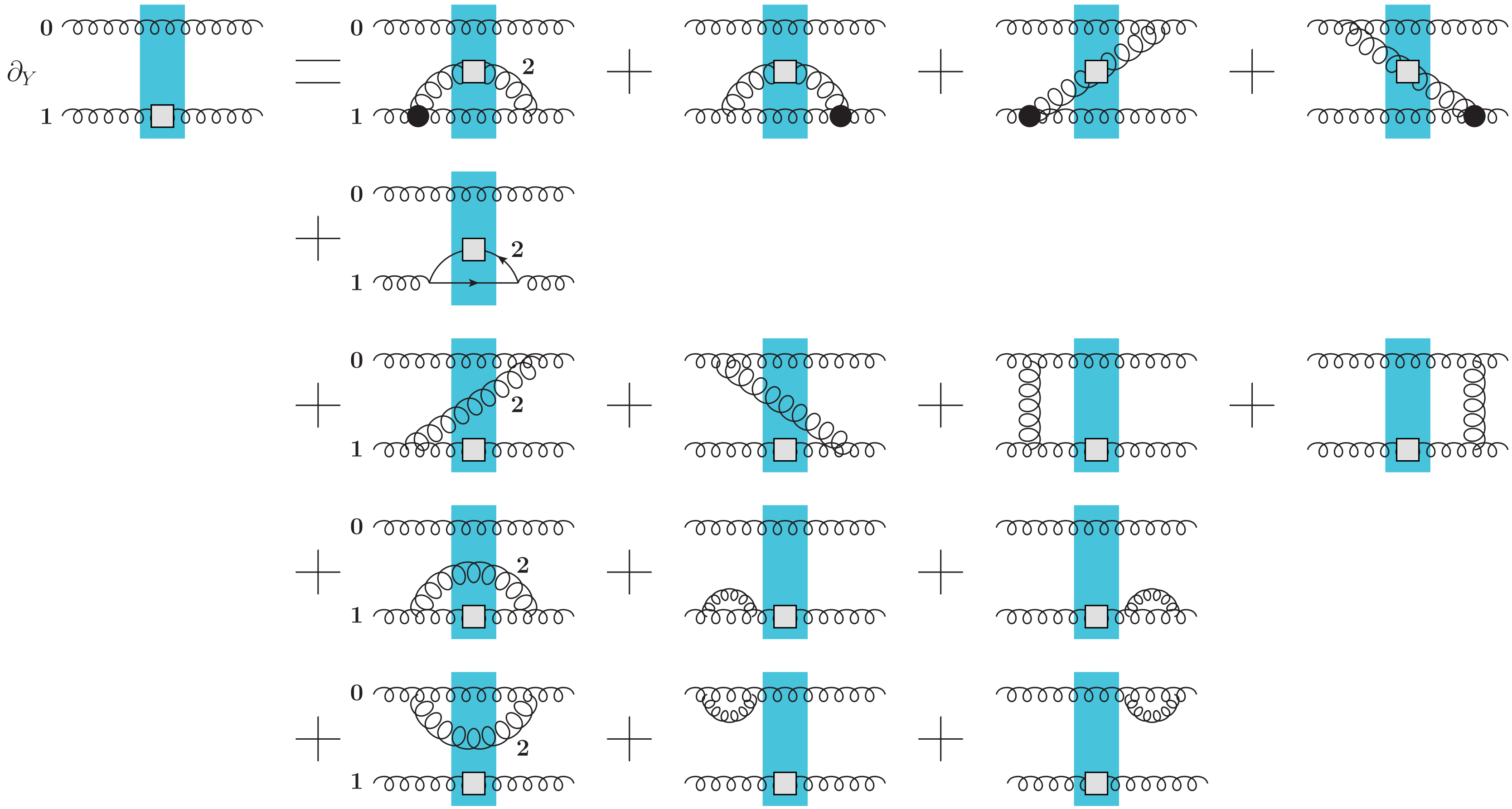
$$\times \left\{ \theta(x_{10} - x_{21})\, \frac{2}{N_c} \left\langle\!\!\left\langle \mathrm{tr}\left[t^b\, V_0^{unp}\, t^a\, V_1^{unp\,\dagger}\right]\, U_2^{pol\,ba} \right\rangle\!\!\right\rangle (z') \right.$$

$$+ \theta(x_{10}^2 z - x_{21}^2 z')\, \frac{1}{N_c} \left\langle\!\!\left\langle \mathrm{tr}\left[t^b\, V_0^{unp}\, t^a\, V_2^{pol\,\dagger}\right]\, U_1^{unp\,ba} \right\rangle\!\!\right\rangle (z')$$

$$\left. + \theta(x_{10} - x_{21})\, \frac{1}{N_c} \left[\left\langle\!\!\left\langle \mathrm{tr}\left[V_0^{unp}\, V_2^{unp\,\dagger}\right]\, \mathrm{tr}\left[V_2^{unp}\, V_1^{pol\,\dagger}\right] \right\rangle\!\!\right\rangle (z') - N_c \left\langle\!\!\left\langle \mathrm{tr}\left[V_0^{unp}\, V_1^{pol\,\dagger}\right] \right\rangle\!\!\right\rangle (z') \right] \right\}$$

Equation does not close!

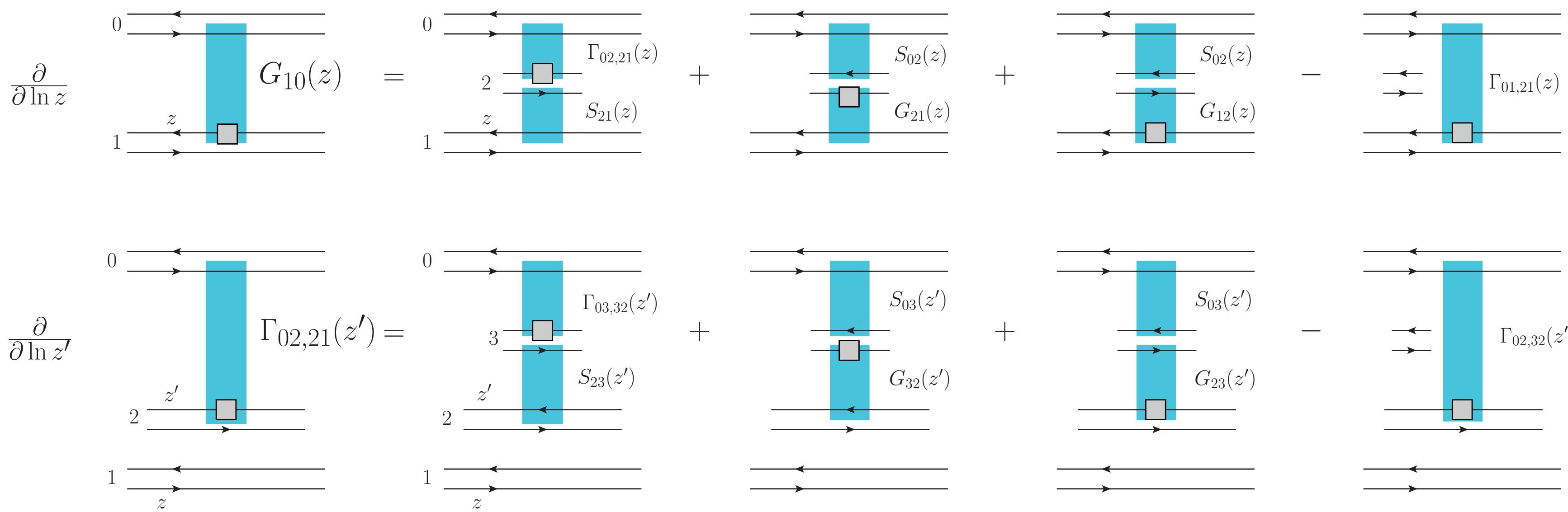
Polarized Gluon Dipole Evolution



Note that at our sub-eikonal level, gluon dipole is a product of two quark dipoles color-wise, but these 'quark' dipoles evolve differently from the polarized dipole made of actual quarks.

Polarized Dipole Evolution in the Large-N$_c$ Limit

In the large-N$_c$ limit the equations close, leading to a closed system of 2 equations:

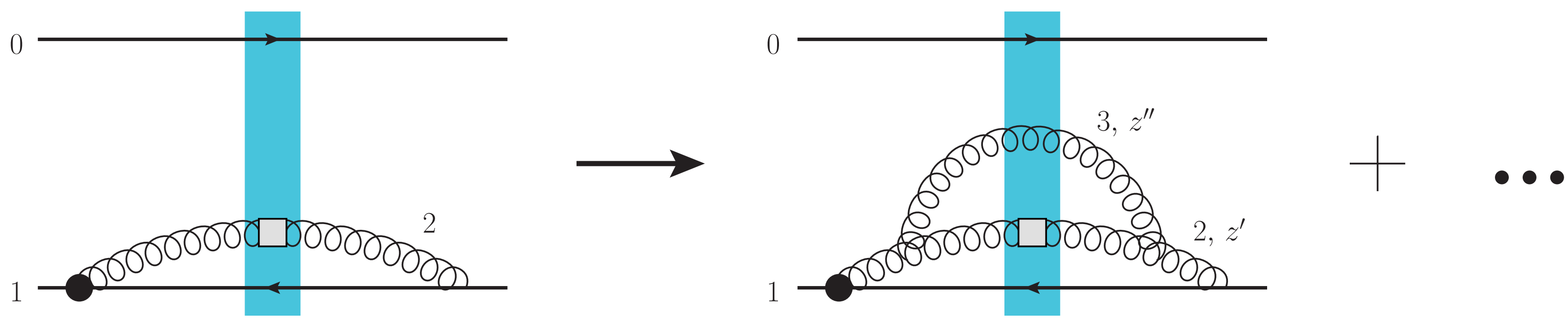


$$G_{10}(z) = G_{10}^{(0)}(z) + \frac{\alpha_s\, N_c}{2\pi} \int\limits_{z_i}^{z} \frac{dz'}{z'} \int\limits_{\rho'^2}^{x_{10}^2} \frac{dx_{21}^2}{x_{21}^2} \left[2\,\Gamma_{02,\,21}(z')\, S_{21}(z') + 2\, G_{21}(z')\, S_{02}(z') + G_{12}(z')\, S_{02}(z') - \Gamma_{01,\,21}(z') \right]$$

$$\Gamma_{02,\,21}(z') = \Gamma_{02,\,21}^{(0)}(z') + \frac{\alpha_s\, N_c}{2\pi} \int\limits_{z_i}^{z'} \frac{dz''}{z''} \int\limits_{\rho''^2}^{\min\{x_{02}^2, x_{21}^2\, z'/z''\}} \frac{dx_{32}^2}{x_{32}^2} \left[2\,\Gamma_{03,\,32}(z'')\, S_{23}(z'') + 2\, G_{32}(z'')\, S_{03}(z'') + G_{23}(z'')\, S_{03}(z'') - \Gamma_{02,\,32}(z'') \right]$$

You friendly “neighborhood” dipole

- There is a new object in the evolution equation – **the neighbor dipole**.
- This is specific for the DLA evolution. Gluon emission may happen in one dipole, but, due to transverse distance ordering, may ’know’ about another dipole:

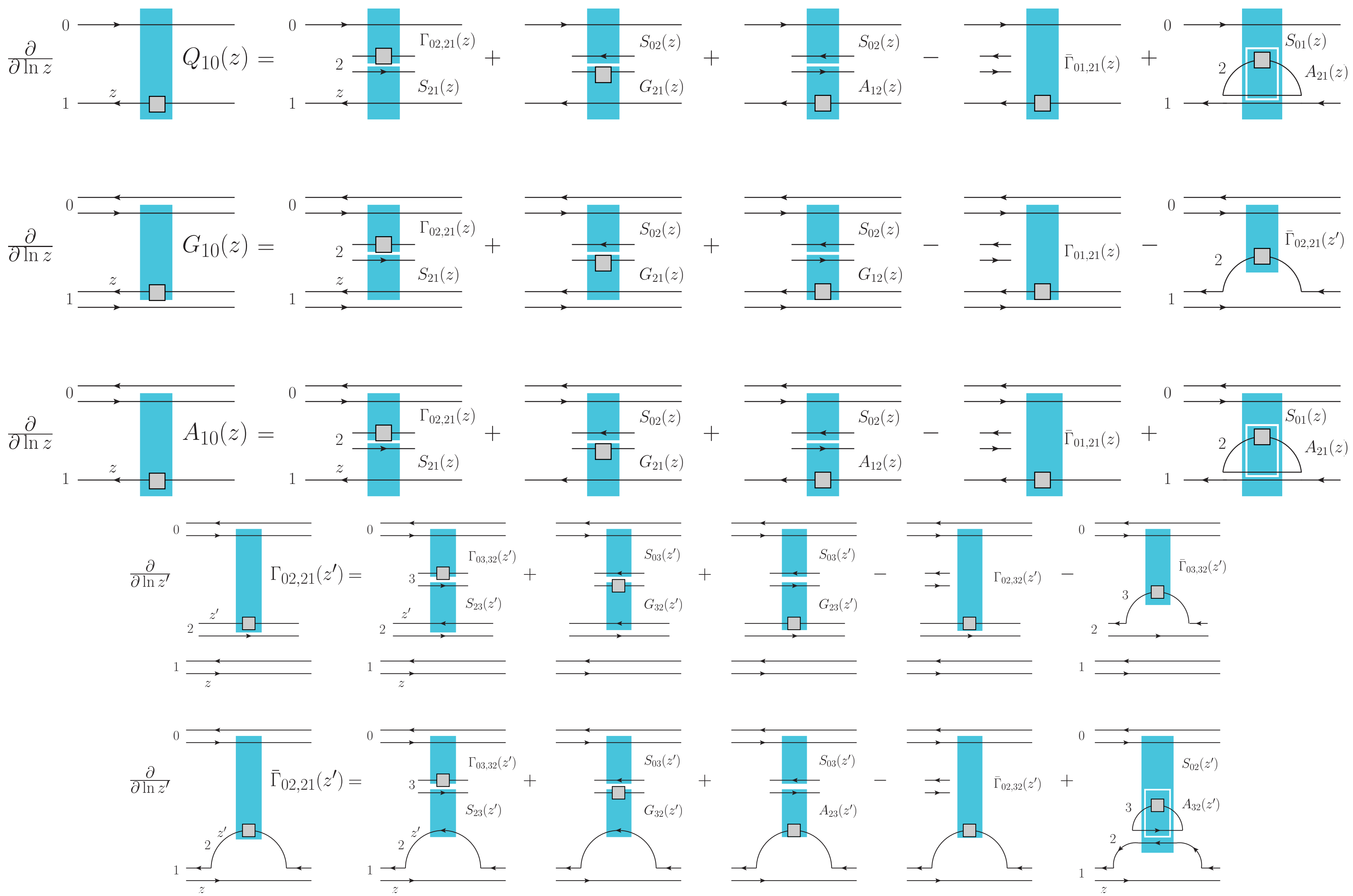


$$x_{21}^2 \, z' \gg x_{32}^2 \, z''$$

- We denote the evolution in the neighbor dipole 02 by $\Gamma_{02,\,21}(z')$

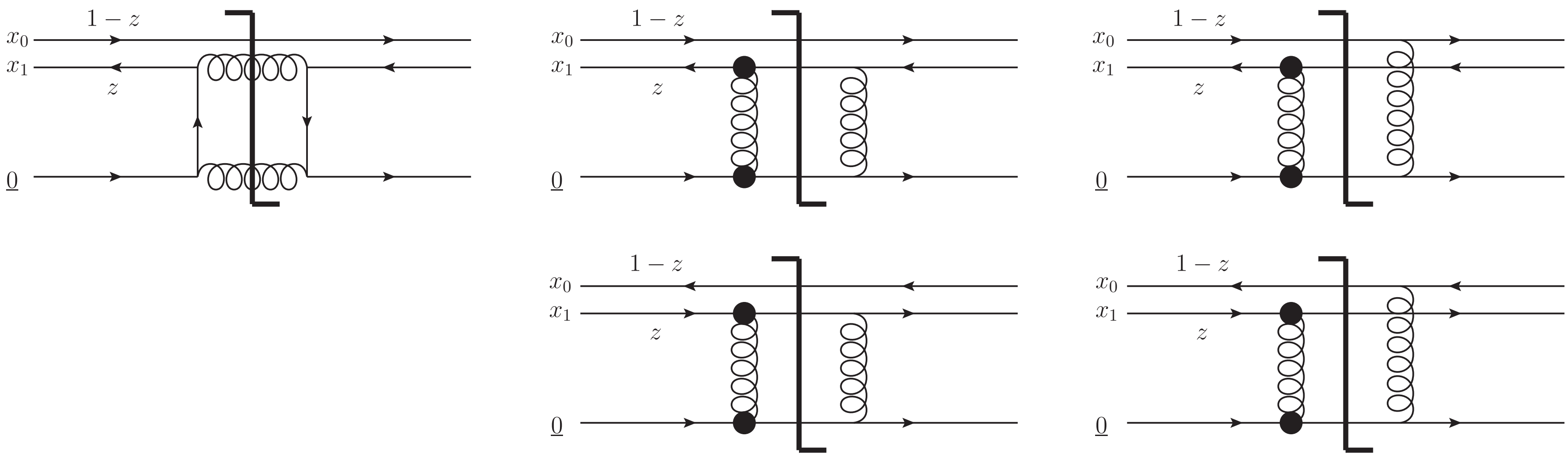
Polarized Dipole Evolution in the Large-N_c&N_f Limit

In the large-N_c&N_f limit the equations close too, leading to a closed system of 5 equations:



Initial Conditions

- Initial conditions for all our evolution equations should be given by Born-level interactions ("dressed" by multiple rescatterings in the saturation case):



$$G^{(0)}(x_{10}^2, z) = \frac{\alpha_s^2 C_F}{N_c} \pi \left[C_F \ln \frac{zs}{\Lambda^2} - 2 \, \ln(zs \, x_{10}^2) \right]$$

Small x Asymptotics of the Quark Helicity Distribution

Yu.K., D. Pitonyak, M. Sievert, in preparation

Prior Results

- Small-x DLA evolution for the g_1 structure function was first considered by Bartels, Ermolaev and Ryskin (BER) in ‘96.

- Including the mixing of quark and gluon ladders, they obtained

$$\Delta\Sigma \sim g_1 \sim \left(\frac{1}{x}\right)^{z_s \sqrt{\frac{\alpha_S N_C}{2\pi}}}$$

with z_s = 3.45 for 4 quark flavors and z_S=3.66 for pure glue.

$$S_q(Q^2) = \frac{1}{2} \int\limits_0^1 dx \, \Delta\Sigma(x, Q^2)$$

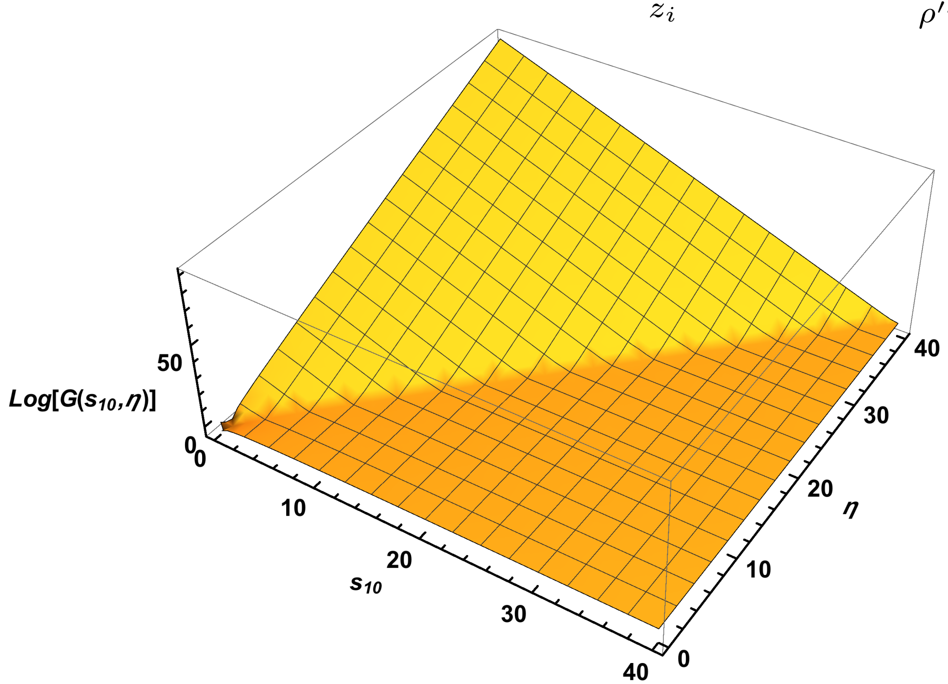
- The power is large: it becomes larger than 1 for the realistic strong coupling of the order of α_s = 0.2 − 0.3, resulting in polarized PDFs which actually grow with decreasing x fast enough for the integral of the PDFs over the low-x region to be (potentially) large (infinite).

Solution of the large-N_C Equations

- We found a numerical solution of the large-N_c DLA evolution equations (linearized, without saturation corrections):

$$G_{01}(z) = G_{01}^{(0)}(z) + \frac{\alpha_s N_c}{2\pi} \int\limits_{z_i}^{z} \frac{dz'}{z'} \int\limits_{\rho'^2}^{x_{10}^2} \frac{dx_{21}^2}{x_{21}^2} \left[\Gamma_{02,\,21}(z') + 3\,G_{21}(z')\right],$$

$$\Gamma_{02,\,21}(z') = \Gamma_{02,\,21}^{(0)}(z') + \frac{\alpha_s N_c}{2\pi} \int\limits_{z_i}^{z'} \frac{dz''}{z''} \int\limits_{\rho''^2}^{\min\{x_{02}^2, x_{21}^2\, z'/z''\}} \frac{dx_{32}^2}{x_{32}^2} \left[\Gamma_{03,\,32}(z'') + 3\,G_{23}(z'')\right]$$



$$\eta = \sqrt{\frac{\alpha_s N_c}{2\pi}}\ \ln \frac{zs}{\Lambda^2}$$

$$s_{10} = \sqrt{\frac{\alpha_s N_c}{2\pi}}\ \ln \frac{1}{x_{10}^2\,\Lambda^2}$$

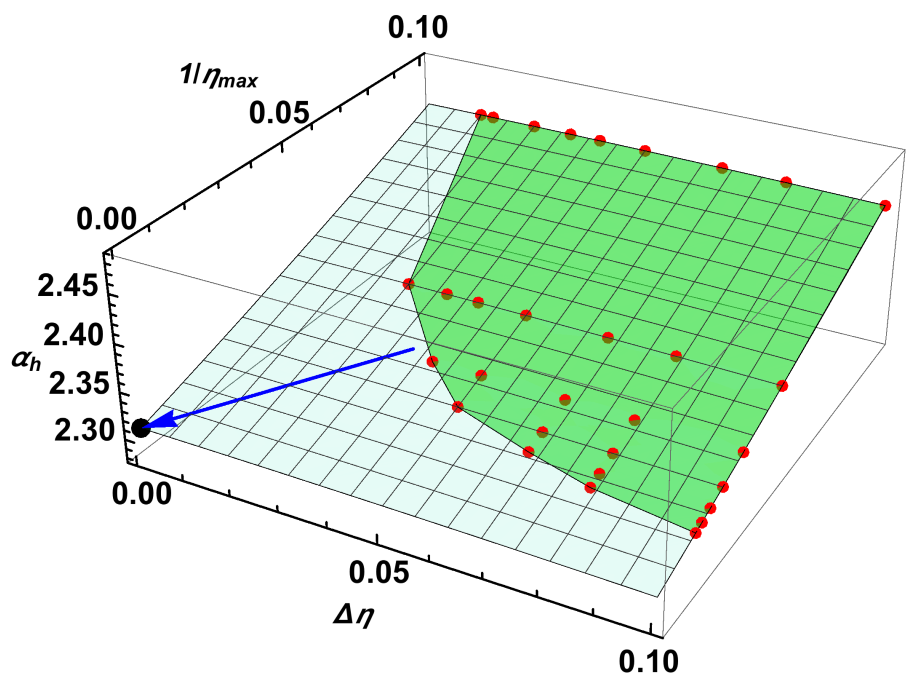
Numerical Solution

- We discretized the equations and solved them iteratively:

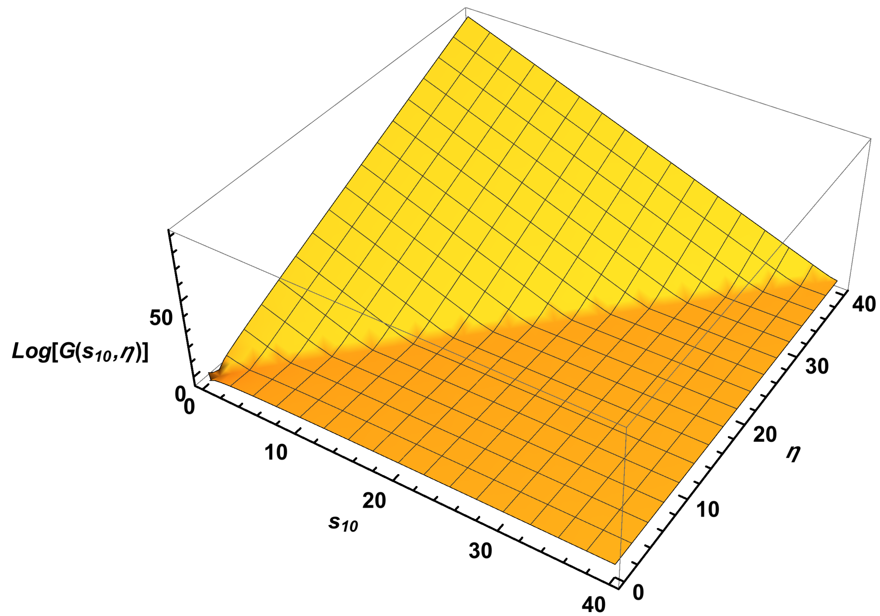
$$G_{ij} = G_{ij}^{(0)} + \Delta\eta\,\Delta s \sum_{j'=i}^{j-1} \sum_{i'=i}^{j'} \left[\Gamma_{ii'j'} + 3\,G_{i'j'}\right],$$

$$\Gamma_{ikj} = \Gamma_{ikj}^{(0)} + \Delta\eta\,\Delta s \sum_{j'=i}^{j-1} \sum_{i'=\max\{i,k+j'-j\}}^{j'} \left[\Gamma_{ii'j'} + 3\,G_{i'j'}\right]$$

- We then extrapolated the intercept to the continuum:



Solution of the large-N_C Equations

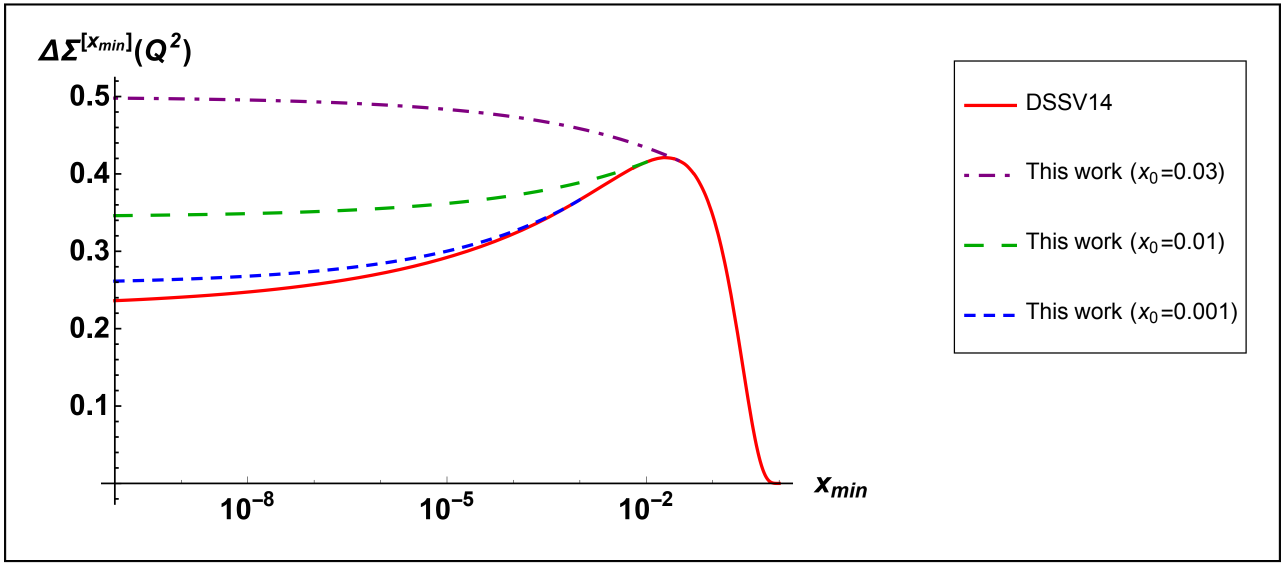


- The resulting small-x asymptotics is (about 35% smaller than BER's 3.66 any-N_C pure glue):

$$g_1^S(x,Q^2) \sim \Delta q^S(x,Q^2) \sim g_{1L}^S(x,k_T^2) \sim \left(\frac{1}{x}\right)^{\alpha_h} \approx \left(\frac{1}{x}\right)^{2.31\,\sqrt{\frac{\alpha_s N_c}{2\pi}}}$$

Impact on proton spin

- We have attached a $\Delta\tilde{\Sigma}(x, Q^2) = N\, x^{-\alpha_h}$ curve to the existing hPDF's fits at some ad hoc small value of x labeled x_0 .
- Defining $\Delta\Sigma^{[x_{min}]}(Q^2) \equiv \int_{x_{min}}^{1} dx\, \Delta\Sigma(x, Q^2)$ we plot it for x_0=0.03, 0.01, 0.001:



- We observe a moderate to significant enhancement of quark spin.
- More detailed phenomenology is needed in the future.

Conclusions

- We have constructed new DLA evolution equations for the polarized dipole operator, which allow us to find the small-x asymptotics of the quark helicity TMDs and PDFs and of the g_1 structure function.
- Like the B-JIMWLK hierarchy, our equations do not close in general. They close in the large-N_C and large-N_C&N_f limits.
- Solution of the flavor singlet evolution equations at large-N_C appears to give

$$g_1^S(x,Q^2) \sim \Delta q^S(x,Q^2) \sim g_{1L}^S(x,k_T^2) \sim \left(\frac{1}{x}\right)^{\alpha_h} \approx \left(\frac{1}{x}\right)^{2.31\sqrt{\frac{\alpha_s N_c}{2\pi}}}$$

  which may potentially generate a solid amount of spin at small-x.
  (D. Pitonyak, M. Sievert, YK, in preparation)
- Future work may involve including running coupling and saturation corrections + solving the large-N_C&N_f equations. All are likely to slightly lower the intercept.

Backup Slides

Flavor Non-Singlet Observables

- In the flavor non-singlet case, all helicity observables again depend on the polarized dipole amplitude:

$$g_1^{NS}(x,Q^2) = \frac{N_c}{2\,\pi^2\alpha_{EM}} \int\limits_{z_i}^{1} \frac{dz}{z^2(1-z)} \int dx_{01}^2 \left[\frac{1}{2}\sum_{\lambda\sigma\sigma'} |\psi^T_{\lambda\sigma\sigma'}|^2_{(x_{01}^2,z)} + \sum_{\sigma\sigma'} |\psi^L_{\sigma\sigma'}|^2_{(x_{01}^2,z)}\right] G^{NS}(x_{01}^2,z),$$

$$\Delta q^{NS}(x,Q^2) = \frac{N_c}{2\pi^3} \int\limits_{z_i}^{1} \frac{dz}{z} \int\limits_{\frac{1}{zs}}^{\frac{1}{zQ^2}} \frac{dx_{01}^2}{x_{01}^2}\, G^{NS}(x_{01}^2,z),$$

$$g_{1L}^{NS}(x,k_T^2) = \frac{8\,N_c}{(2\pi)^6} \int\limits_{z_i}^{1} \frac{dz}{z} \int d^2x_{01}\, d^2x_{0'1}\, e^{-i\underline{k}\cdot(\underline{x}_{01}-\underline{x}_{0'1})}\, \frac{\underline{x}_{01}\cdot\underline{x}_{0'1}}{x_{01}^2 x_{0'1}^2}\, G^{NS}(x_{01}^2,z)$$

- Polarized dipole amplitude is different (difference instead of sum):

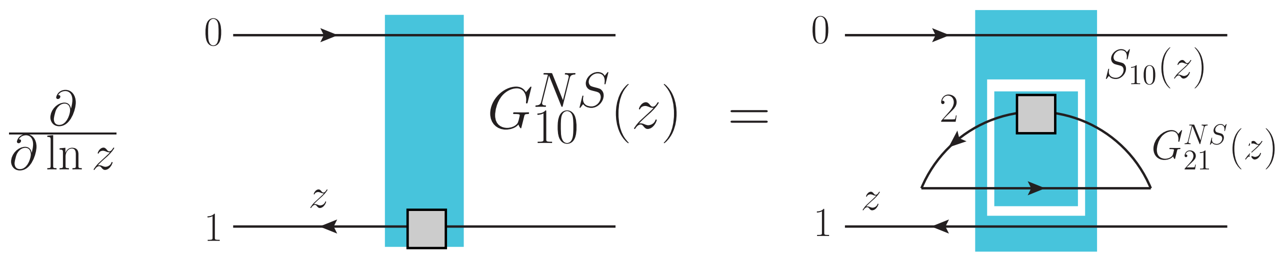
$$G_{10}^{NS}(z) \equiv \frac{1}{2N_c}\left\langle\!\!\left\langle \mathrm{tr}\left[V_{\underline{0}} V_{\underline{1}}^{pol\,\dagger}\right] - \mathrm{tr}\left[V_{\underline{1}}^{pol} V_{\underline{0}}^{\dagger}\right] \right\rangle\!\!\right\rangle(z)$$

- This is related to the definition

$$\Delta q^{NS}(x,Q^2) \equiv \Delta q^f(x,Q^2) - \Delta\bar{q}^f(x,Q^2)$$

Flavor Non-Singlet Evolution

- Evolution equations end up being much simpler in the non-singlet case:



$$G_{10}^{NS}(z) = G_{10}^{NS\,(0)}(z) + \frac{\alpha_s N_c}{4\pi} \int\limits_{\frac{\Lambda^2}{s}}^{z} \frac{dz'}{z'} \int\limits_{\frac{1}{z' s}}^{x_{10}^2 \frac{z}{z'}} \frac{dx_{21}^2}{x_{21}^2} \, S_{10}(z') \, G_{21}^{NS}(z')$$

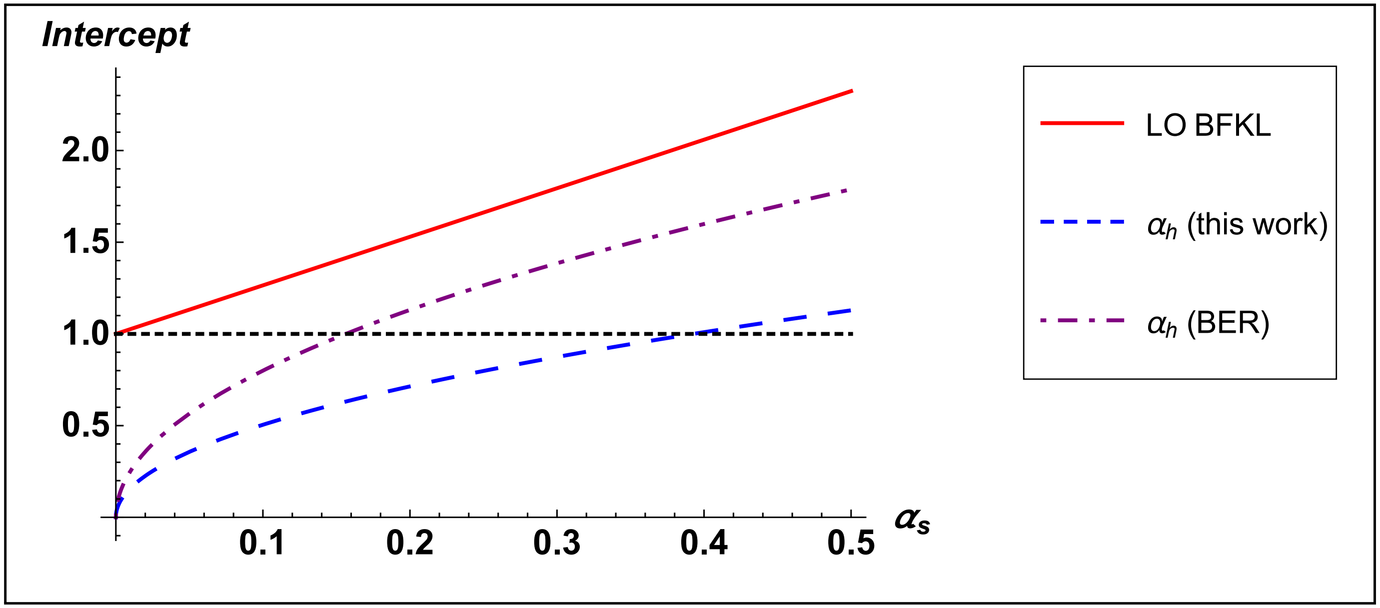
- Analytical solution (in the DLA case, S=1) leads to

$$g_1^{NS}(x, Q^2) \sim \Delta q^{NS}(x, Q^2) \sim g_{1L}^{NS}(x, k_T^2) \sim \left(\frac{1}{x}\right)^{\alpha_h^{NS}} \approx \left(\frac{1}{x}\right)^{\sqrt{\frac{\alpha_s N_c}{\pi}}}$$

- The resulting intercept is smaller than the flavor-singlet intercept.

Intercepts

Here we plot our (flavor-singlet) helicity intercept as a function of the coupling. We show BER result and LO BFKL for comparison.



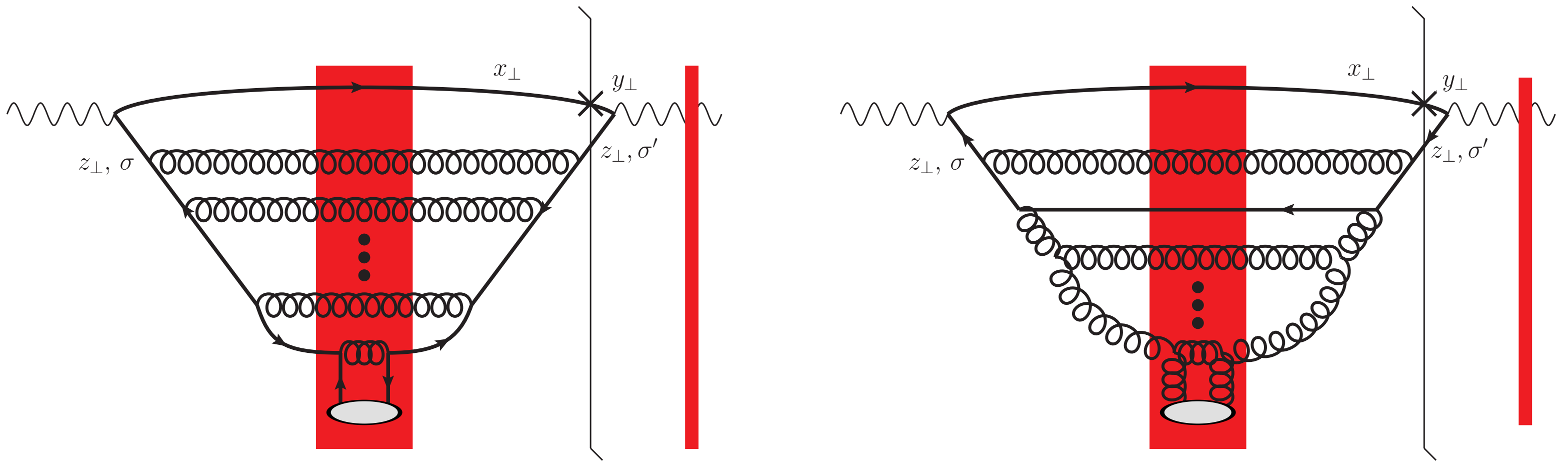
Intercepts

- We can summarize some LO intercepts, including the ones we found, in the following table:

Observable	Evolution	Intercept	$Q^2 = 3$ GeV^2 $\alpha_s = 0.343$	$Q^2 = 10$ GeV^2 $\alpha_s = 0.249$	$Q^2 = 87$ GeV^2 $\alpha_s = 0.18$
Unpolarized flavor singlet structure function F_2	LO BFKL Pomeron	$1 + \frac{\alpha_s N_c}{\pi} 4 \ln 2$	1.908	1.659	1.477
Unpolarized flavor non-singlet structure function F_2	Reggeon	$\sqrt{\frac{2\,\alpha_s C_F}{\pi}}$	0.540	0.460	0.391
Flavor singlet structure function g_1^S	us (Pure Glue)	$2.31 \sqrt{\frac{\alpha_s N_c}{2\pi}}$	0.936	0.797	0.678
	BER (Pure Glue)	$3.66 \sqrt{\frac{\alpha_s N_c}{2\pi}}$	1.481	1.262	1.073
	BER ($N_f = 4$)	$3.45 \sqrt{\frac{\alpha_s N_c}{2\pi}}$	1.400	1.190	1.011
Flavor non-singlet structure function g_1^{NS}	BER and us (large-N_c)	$\sqrt{\frac{\alpha_s N_c}{\pi}}$	0.572	0.488	0.415

Small-x Quark Helicity TMD Evolution: Ladders

A part of this evolution equation comes from ladder diagrams:



Interestingly the quark and non-eikonal gluon ladders mix (see the right panel), resulting in a more complicated evolution equation:

$$W^{pol}_{\underline{1}\,\underline{0}}(z) = W^{(0)\,pol}_{\underline{1}\,\underline{0}}(z) + \frac{\alpha_s}{2\,\pi} \int\limits_{z_i}^{z} \frac{dz'}{z'} \int\limits_{\rho'^2}^{x^2_{01} z/z'} \frac{dx^2_{21}}{x^2_{21}} \, M \, W^{pol}_{\underline{2}\,\underline{1}}(z')$$

$$M \equiv \begin{pmatrix} C_F & 2\,C_F \\ -N_f & 4\,N_c \end{pmatrix} \qquad W^{pol}_{\underline{x}\underline{y}} = \begin{pmatrix} \frac{1}{N_c} \left\langle\!\!\left\langle \mathrm{tr}[V^{pol}_{\underline{x}}] + \mathrm{tr}[V^{pol\,\dagger}_{\underline{x}}] \right\rangle\!\!\right\rangle \\ \frac{1}{N_c^2-1} \left\langle\!\!\left\langle \mathrm{Tr}[U^{pol}_{\underline{x}}] + \mathrm{Tr}[U^{pol\,\dagger}_{\underline{x}}] \right\rangle\!\!\right\rangle \end{pmatrix} \begin{matrix} \text{quarks} \\ \text{gluons} \end{matrix}$$

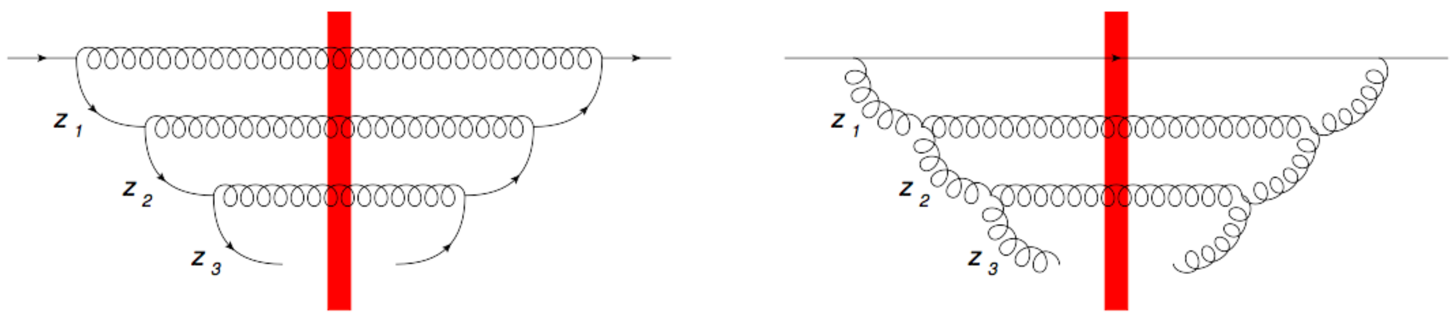
Ballpark Estimate: Ladders

- Summing up mixing quark and gluon ladders yields

$$\Delta\Sigma \sim \left(\frac{1}{x}\right)^{\omega_+} \qquad\qquad S_q(Q^2) = \frac{1}{2}\int\limits_0^1 dx\, \Delta\Sigma(x, Q^2)$$

with

$$\omega_+ = \sqrt{\frac{\alpha_s}{2\pi N_c}}\sqrt{9\,N_c^2 - 1 + \sqrt{(1+7\,N_c^2)^2 + 16\,N_c\,N_f\,(1-N_c^2)}}$$



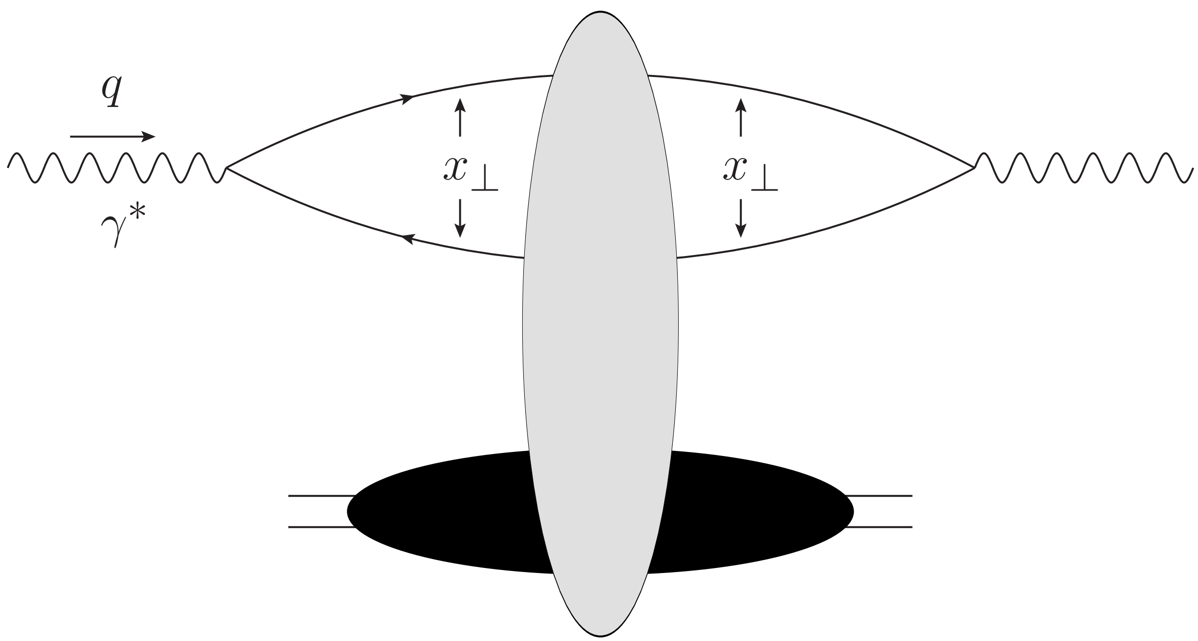
- The numbers are encouraging (α_s=0.3, N_c=N_f=3): $\Delta\Sigma \sim \left(\frac{1}{x}\right)^{1.46}$

- But: need to include the non-ladder graphs.

Unpolarized DIS: Small-x Evolution

Dipole picture of DIS

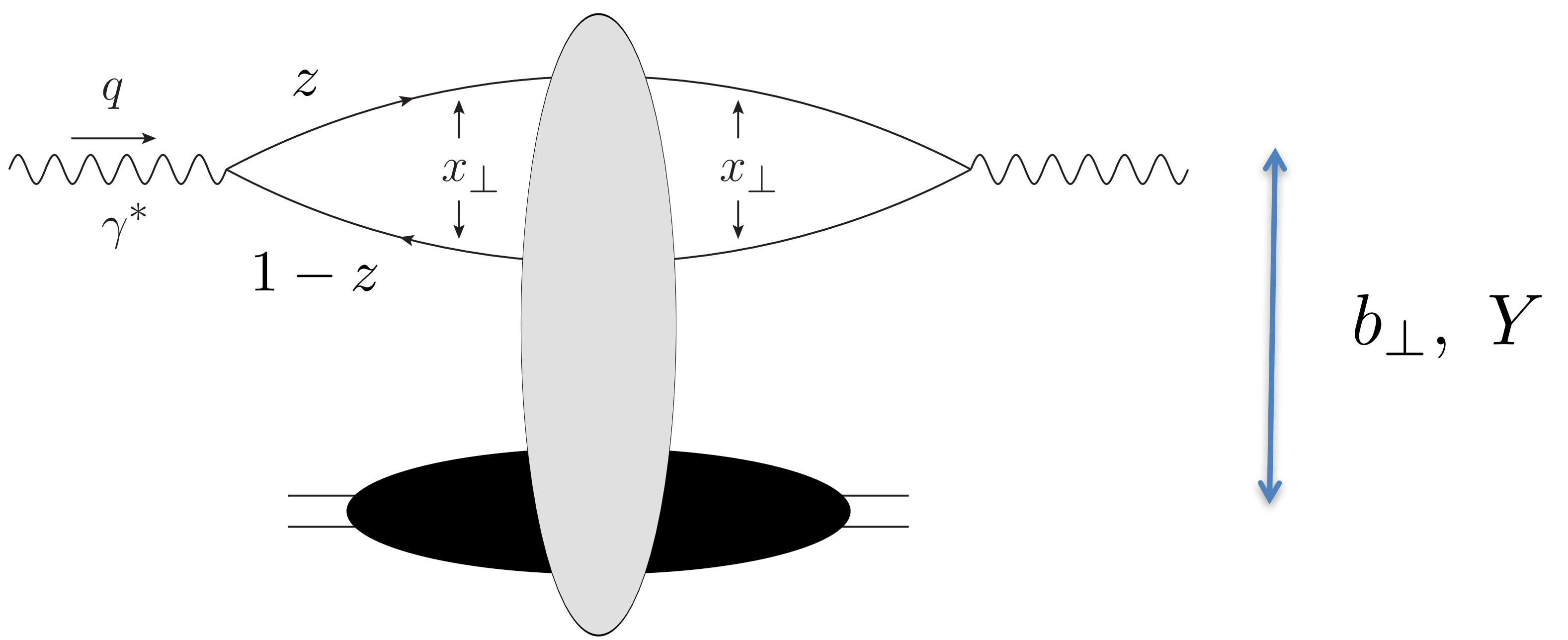
- In the dipole picture of DIS the virtual photon splits into a quark-antiquark pair, which then interacts with the target.
- The total DIS cross section and structure functions are calculated via:



Dipole Amplitude

- The total DIS cross section is expressed in terms of the (Im part of the) forward quark dipole amplitude N:

$$\sigma_{tot}^{\gamma^* A} = \int \frac{d^2 x_\perp}{2\,\pi}\, d^2 b_\perp \int\limits_0^1 \frac{dz}{z\,(1-z)}\ |\Psi^{\gamma^* \to q \bar{q}}(\vec{x}_\perp, z)|^2\ N(\vec{x}_\perp, \vec{b}_\perp, Y)$$



Dipole Amplitude

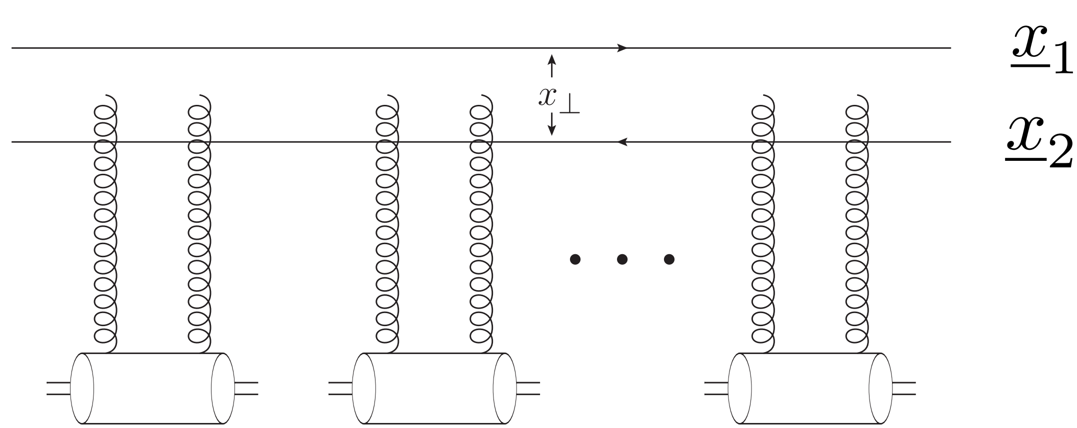
- The quark dipole amplitude is defined by

$$N(\underline{x}_1, \underline{x}_2) = 1 - \frac{1}{N_c} \left\langle \mathrm{tr}\left[V(\underline{x}_1)\, V^\dagger(\underline{x}_2)\right]\right\rangle$$

- Here we use the Wilson lines along the light-cone direction

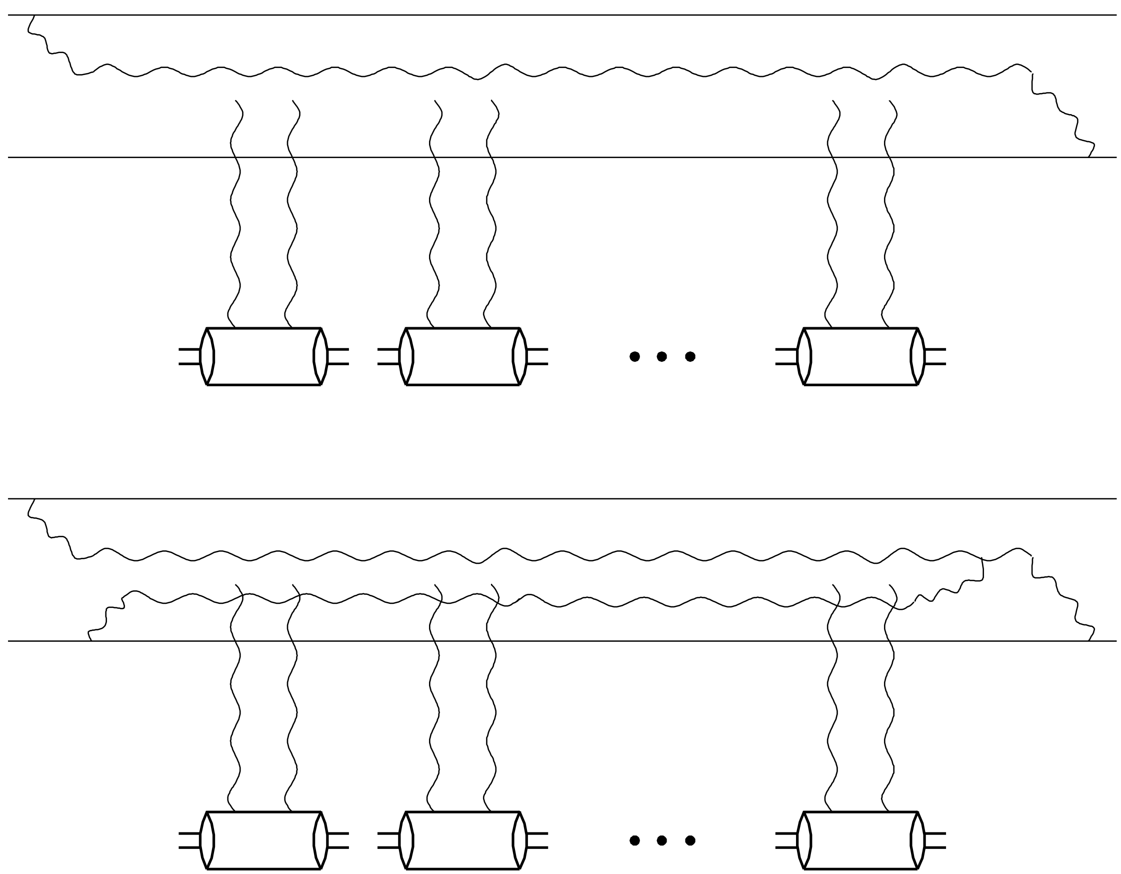
$$V(\underline{x}) = \mathrm{P}\exp\left[i\,g \int\limits_{-\infty}^{\infty} dx^+\, A^-(x^+, x^- = 0, \underline{x})\right]$$

- In the classical Glauber-Mueller/McLerran-Venugopalan approach the dipole amplitude resums multiple rescatterings:



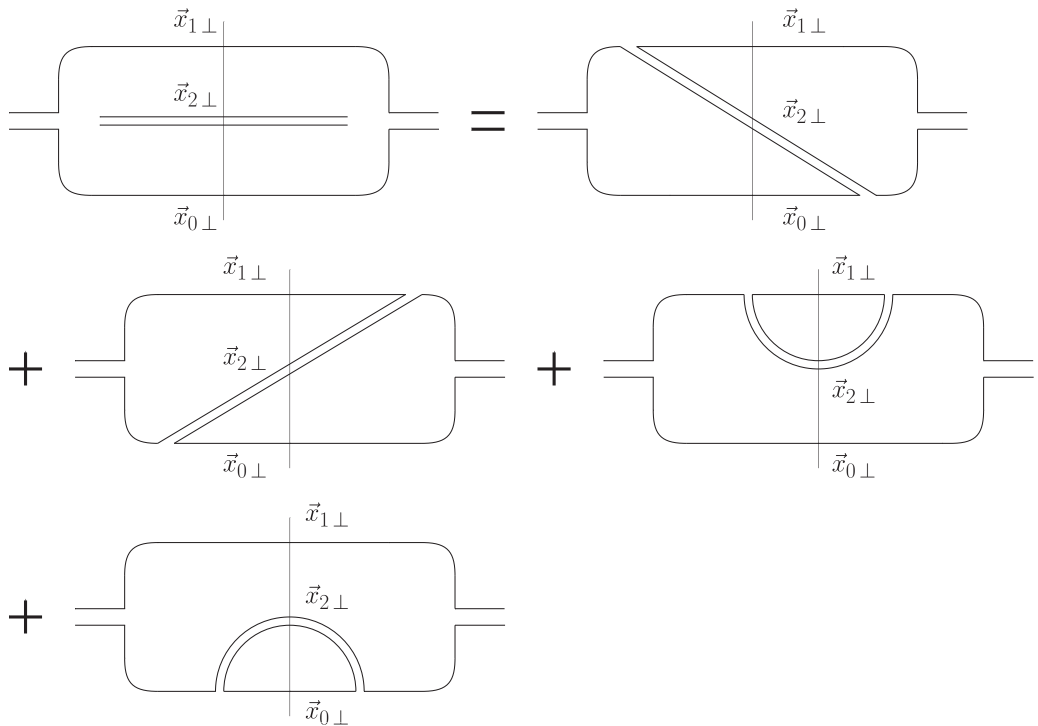
Dipole Amplitude

- The energy dependence comes in through nonlinear small-x BK/JIMWLK evolution, which resums the long-lived s-channel gluon corrections:



$$\alpha_s \, \ln \frac{1}{x} \sim \alpha_s \, Y \sim 1$$

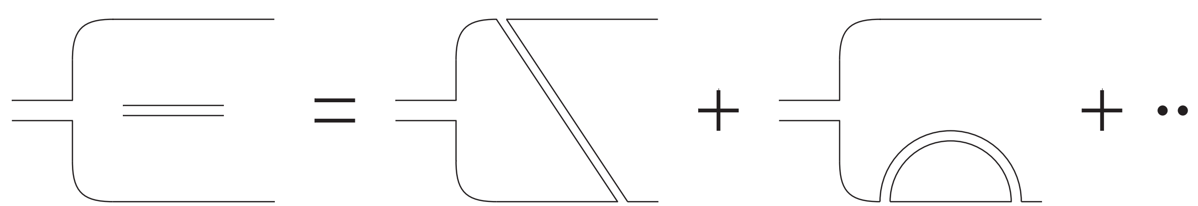
Notation (Large-N_C)



Real emissions in the amplitude squared

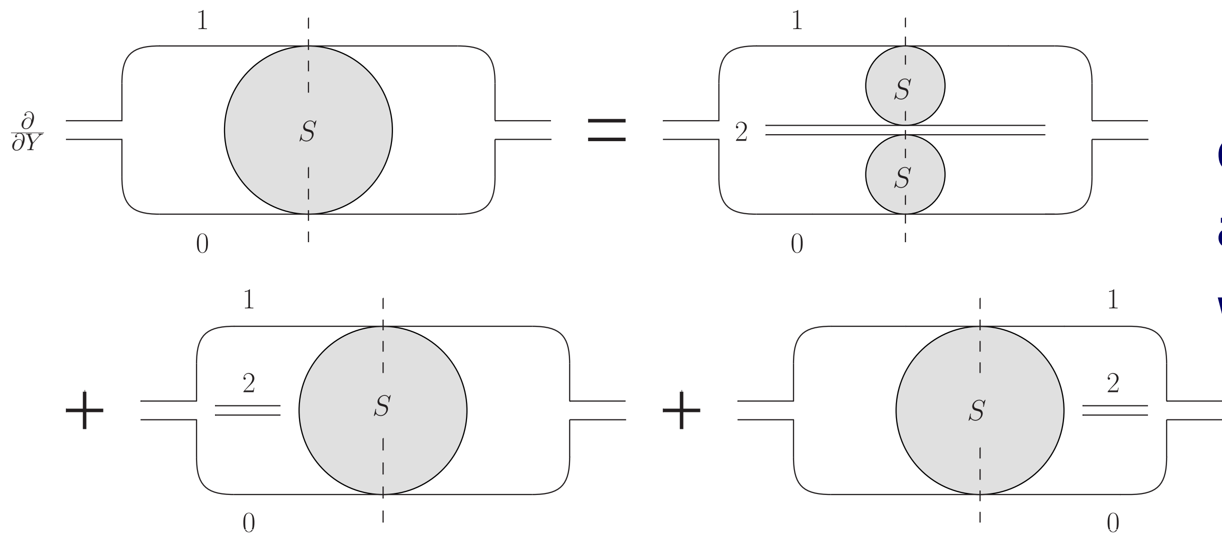
(dashed line – all Glauber-Mueller exchanges at light-cone time =0)

Virtual corrections in the amplitude (wave function)



Nonlinear Evolution

To sum up the gluon cascade at large-N_C we write the following equation for the dipole S-matrix:



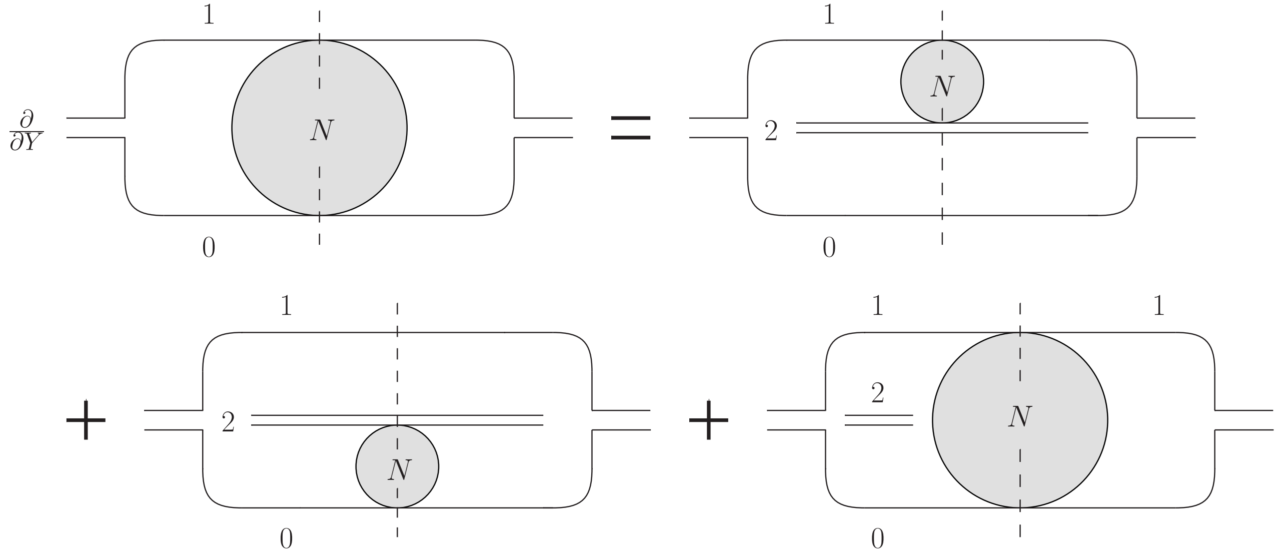
dashed line = all interactions with the target

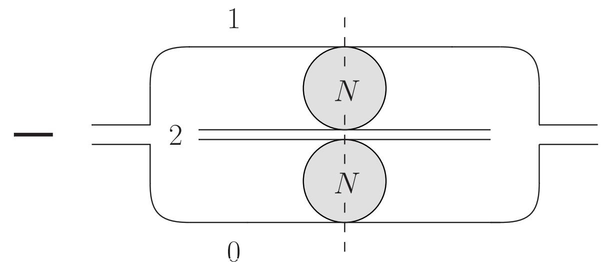
$$\partial_Y S_{\mathbf{x}_0,\mathbf{x}_1}(Y) = \frac{\alpha_s N_c}{2\pi^2} \int d^2 x_2 \frac{x_{01}^2}{x_{02}^2 \, x_{21}^2} \left[S_{\mathbf{x}_0,\mathbf{x}_2}(Y) \, S_{\mathbf{x}_2,\mathbf{x}_1}(Y) - S_{\mathbf{x}_0,\mathbf{x}_1}(Y) \right]$$

Remembering that S= 1-N we can rewrite this equation in terms of the dipole scattering amplitude N.

Nonlinear evolution at large N_c

As N=1-S we write





dashed line =
all interactions
with the target

$$\partial_Y N_{\mathbf{x}_0,\mathbf{x}_1}(Y) = \frac{\alpha_s N_c}{2\pi^2} \int d^2x_2 \, \frac{x_{01}^2}{x_{02}^2 \, x_{21}^2} \, \left[N_{\mathbf{x}_0,\mathbf{x}_2}(Y) + N_{\mathbf{x}_2,\mathbf{x}_1}(Y) - N_{\mathbf{x}_0,\mathbf{x}_1}(Y) - N_{\mathbf{x}_0,\mathbf{x}_2}(Y) \, N_{\mathbf{x}_2,\mathbf{x}_1}(Y) \right]$$

Balitsky '96, Yu.K. '99

Linear terms = BFKL equation.

What About Spin?

- Spin dependence is energy-suppressed and is thus sub-leading in the small-x asymptotics of total cross sections and unpolarized structure functions. It is often neglected.
- The small-x asymtotics of g_1 structure function was studied in the double-logarithmic approximation (DLA) by Bartels, Ermolaev and Ryskin (BER) in 1995-1996, using the technique developed by Kirschner and Lipatov in 1983.
- DLA resums powers of $\alpha_s \ln^2 \frac{1}{x}$
- BER obtained a steep rise of g_1 with decreasing x. Can we see this in our formalism?