Strange Form Factors of the Proton
INTRODUCTION

- Goal: Determine the strange quark contributions to the nucleon form factors
- Experimental effort to measure precisely these contributions. (HAPEX, G0, A4...)
- Theoretical work estimate these contributions
- Lattice QCD calculations for more than 15 years now
- Give a flavor of the numerical algorithms involved
- Present some recent results from the LHPC/NME/JLab collaborations
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LATTICE QCD

In continuous Euclidian space:

\[
\mathcal{Z} = \int \mathcal{D}q \mathcal{D}\bar{q} \mathcal{D}A_\mu \ e^{-S[\bar{q}, q, A_\mu]}
\]

\[
\langle \mathcal{O} \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}q \mathcal{D}\bar{q} \mathcal{D}A_\mu \ \mathcal{O}(\bar{q}, q, A_\mu) \ e^{-S[\bar{q}, q, A_\mu]}
\]

Lattice regulator:

Gauge sector:

\[
U_\mu(x) = e^{-iaA_\mu(x+\frac{\mu}{2})}
\]

Fermion sector:

\[
S_f = \bar{\Psi} D \Psi
\]

\(\Psi\) is now a vector whose components leave on the sites of the lattice

D is the Dirac matrix which is large and sparse
NUCLEON FORM FACTOR

Connected

Disconnected

Strange quark: disconnected only
DISCONNECTED CONTRIBUTIONS

\[ \text{Tr}(D^{-1} \Gamma) \]

D: is the Dirac matrix
\[ \Gamma \]: is a spin matrix

D: In lattice QCD is a large space matrix
STOCHASTIC TRACE

\[ \text{Tr}(A) \approx \langle \eta^\dagger A\eta \rangle \]

\( \eta \) is a random vector with components that satisfy

\[ \langle \eta_i^* \eta_j \rangle = \delta_{ij} \quad \text{and} \quad \langle \eta_i \eta_j \rangle = 0 \]


For complex \( \mathbb{Z}_N \) noise (\( N>2 \)):

\[ \text{Var} (\text{Tr}(A)) = \| \tilde{A} \|_F^2 = \left( \| A \|_F^2 - \sum_{i=1}^{L} |A_{i,i}|^2 \right) \sim \frac{1}{N_{\text{samples}}} \]

W. M. Wilcox, (1999), arXiv:hep-lat/9911013
\[ \text{Var}(\text{Tr}(A)) = \| \tilde{A} \|_F^2 = \left( \| A \|_F^2 - \sum_{i=1}^{L} |A_{i,i}|^2 \right) \]

splitting the trace in two reduces variance
\[ \text{Var}(\text{Tr}(A)) = \| \tilde{A} \|_F^2 = \left( \| A \|_F^2 - \sum_{i=1}^{L} |A_{i,i}|^2 \right) \]

Further subdivision reduces variance
\[ Var(Tr(A)) = \| \tilde{A} \|_F^2 = \left( \| A \|_F^2 - \sum_{i=1}^{L} |A_{i,i}|^2 \right) \]

\( D^{-1}_{ij} \) is the quark propagator from point \( i \) to point \( j \).

\( i \) and \( j \) points that are close contribute most to the variance.
HIERARCHICAL PROBING

- Decompose (color) the lattice points according to distance
- Estimate the trace of each subdomain stochastically
- Choose a nested coloring so that one can continuously improve the approximation

Hierarchical probing variance reduction

single gauge field conf.  
a=0.08fm  
quark mass tuned to  
physical strange quark mass

circles mark the color closing points
Disconnected contribution to nucleon form factors

dynamical 2 + 1 flavors of Clover fermions

32^3 \times 96 lattice of dimensions (3.6 fm)^3 \times (10.9 fm)
a=0.115\text{fm}, pion mass 317 \text{MeV}

z-expansion fit:

\[ G(Q^2) = \sum_{k}^{k_{\text{max}}} a_k z^k, \quad z = \frac{\sqrt{t_{\text{cut}} + Q^2} - \sqrt{t_{\text{cut}}}}{\sqrt{t_{\text{cut}} + Q^2} + \sqrt{t_{\text{cut}}}}, \]

strange magnetic moment

\[
\begin{align*}
(r_E^2)^s &= -0.0067(10)(17)(15) \text{ fm}^2, \\
(r_M^2)^s &= -0.018(6)(5)(5) \text{ fm}^2, \\
\mu^s &= -0.022(4)(4)(6) \mu_N,
\end{align*}
\]

extrapolation to physical point


Comparison with experiments

Experiment: forward-angle parity-violating elastic $e$-p scattering

$$G_E^s + \eta G_M^s$$

$$\eta = A Q^2, \quad A = 0.94$$

Prediction: very hard for such experiments to measure a non-zero result
Hierarchical probing works because of the exponential decay of the matrix elements of the Dirac matrix inverse. Light quark masses result slower decay. Largest contributions to the trace are from low modes. Compute those exactly: Deflation. Stochastically estimate the rest. Re-examined this idea, model the effectiveness and combine it with hierarchical probing.


HP WITH DEFLATION

\[ Tr(A^{-1}) = Tr(A_D^{-1}) + Tr(A_R^{-1}) = Tr(V\Sigma^{-1}U^\dagger) + Tr(A^{-1} - V\Sigma^{-1}U^\dagger) \]

\( V, U \)  singular vectors of A
\( \Sigma \)  the singular values

Estimate stochastically the second term

One can understand analytically and model the effectiveness of deflation

Conclusion: Results depend on the spectrum of singular values of A. Deflation is not always effective.

Lattice: $32^3 \times 64$, 2+1 flavor Clover-Wilson ensemble $a = 0.081$ fm and $m_\pi = 312$ MeV

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Hierarchical Probing vectors

\[ \text{Speed Up} = \frac{V_{stoc}}{V_{hp}(s) \times s} \]

where \( V_{stoc} \) is variance of a stochastic estimator,

\( V_{hp}(s) \) is variance of HP + Deflation for the s-th probing vector.

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Gambhir, for the NME/LHPC collaboration (Lattice 2016)
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CONCLUSIONS

- Progress has been made in the numerical methods needed for disconnected diagram calculations
- Further improvements are on the way
  - Multi-level algorithms
- Presented results on disconnected contributions to vector form factors
- Corresponding results for axial vector are forthcoming (J. Green Lattice 2016)
- Lattice QCD results represent the most accurate theoretical determinations of sea quark effects