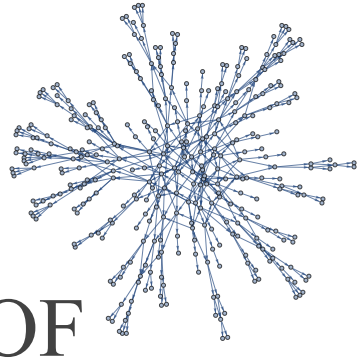
SPIN 2016
UNIVERSITY OF ILLINOIS-URBANA CHAMPAIGN
SEPTEMBER 2016



TOWARDS LATTICE QCD STUDIES OF HIGH MOMENTS OF PARTON DISTRIBUTION FUNCTIONS

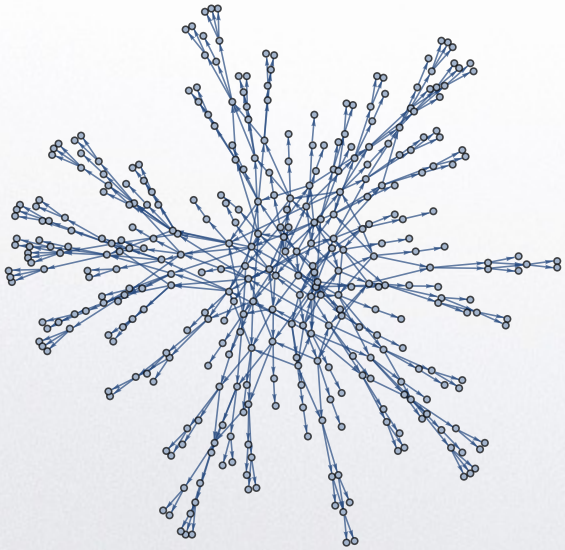


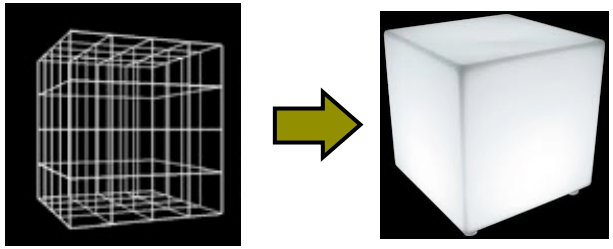
Image by M. G. Endres

Zohreh Davoudi
MIT

1) ZD and M. J. Savage, PRD 86, 054505 (2012)
2) J-W. Chen, ZD, M. G. Endres, W. Detmold, J. Negele, A. V. Pochinsky, P. Shanahan, (MITPDF collaboration), in progress.

WHAT IS THE PROBLEM OF OPERATOR MIXING IN LATTICE QCD CALCULATIONS?

A PROBLEM…



LQCD on (HYPER)CUBIC lattices

Full rotational group with infinite number of irreps

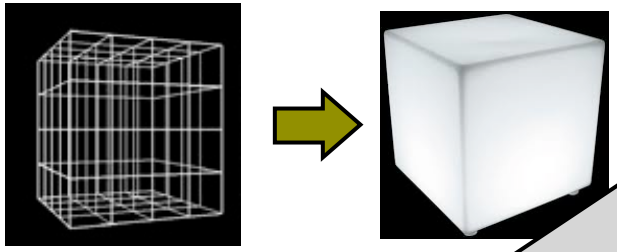
Cubic group with only 10 irreps

L	
0	A_1^+
1	T_1^-
2	$E^+ \oplus T_2^+$
3	$A_2^- \oplus T_1^- \oplus T_2^-$
4	$A_1^+ \oplus T_1^+ \oplus T_2^+$
5	$E^- \oplus T_1^- \oplus T_1^- \oplus T_2^-$

Operators with different angular momentum do not mix in the continuum.

Not true for lattice operators → Less symmetries, less constraints.

A PROBLEM…



LQCD on (HYPER)CU

Full rotational group with irreps

Cub 10 irreps

L	
0	A^+
1	T
2	$E^+ \oplus$
3	$A_2^- \oplus T_1^- \oplus$
4	$A_1^+ \oplus T_1^+ \oplus T_2^+$
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POWER DIVERGENCE

ators with different angular momentum do not mix in the continuum.

Not true for lattice operators → Less symmetries, less constraints.

WHAT QUANTITIES HAVE BEEN SUFFERING FROM THIS PROBLEM?

TWO EXAMPLES

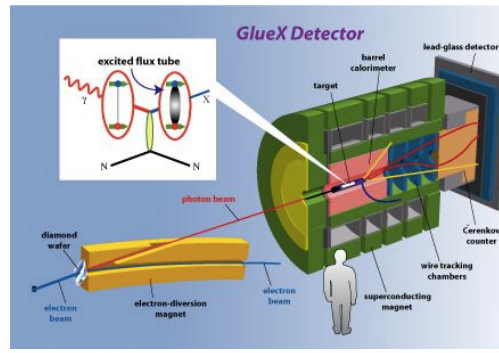
1 Excited states spectroscopy

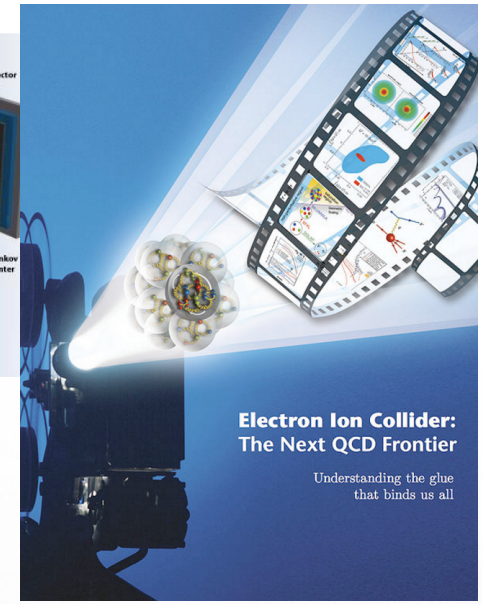
$$C(t) = \langle 0 | \mathcal{O}^\dagger(t)\, \mathcal{O}(0) | 0 \rangle$$

Continuum states up to $\mathcal{O}(a^n)$.

To be built on the lattice

Spin identification?



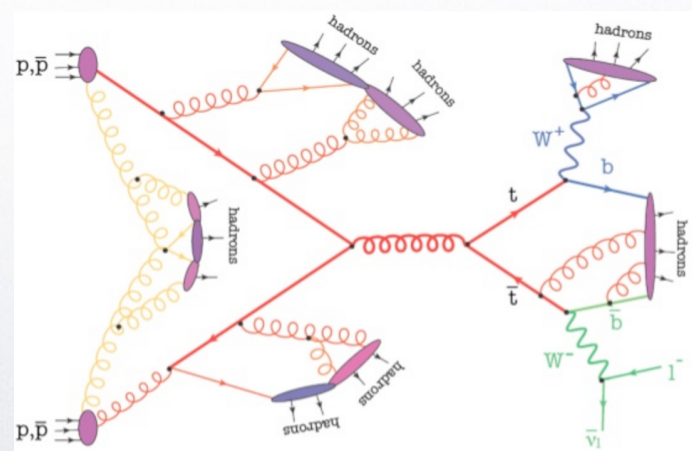


2 Moments of parton distributions functions

$$\langle p, s | \mathcal{O}_{\mu_1 \mu_2 \ldots \mu_n} | p, s \rangle |_{\mu^2} = 2 \langle x^n \rangle_{q,\mu^2} p^{\{\mu_1} p^{\mu_2} \ldots p^{\mu_n\}}$$

$$\langle x^n \rangle_{q,\mu^2} = \int dx x^n q(x; \mu^2)$$

Higher moments?



Picture by Juan Rojo, CERN.

TWO EXAMPLES

1 Excited states spectroscopy

$$C(t) = \langle 0 | \mathcal{O}^\dagger(t) \mathcal{O}(0) | 0 \rangle$$

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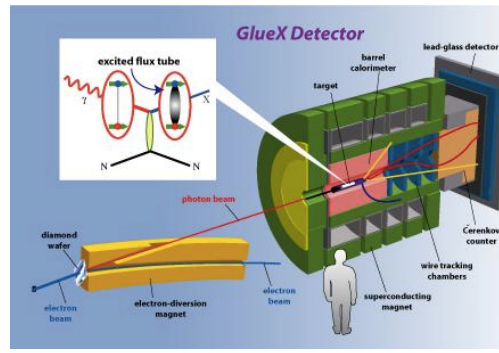
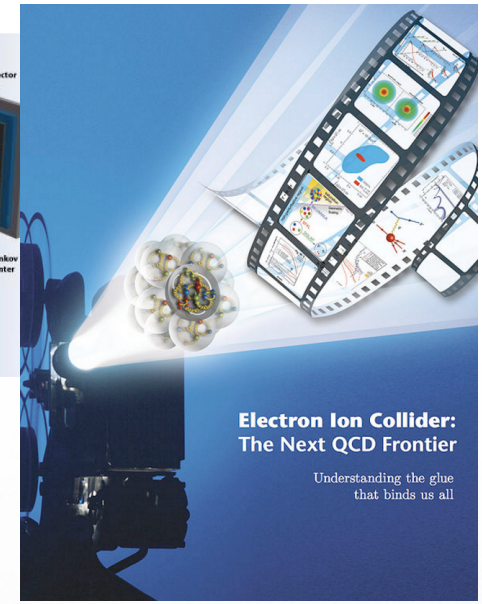
To be built on the lattice

Spin identification?





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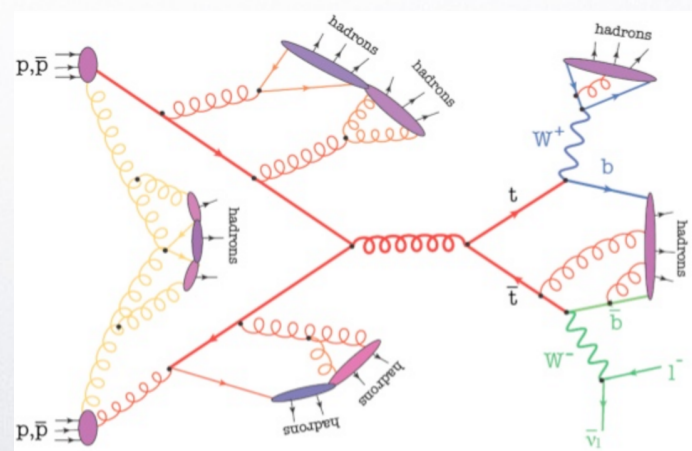


Picture by Juan Rojo, CERN.

CAN THIS PROBLEM BE CIRCUMVENTED ANYWAY?

Meyer and Teper, Nucl. Phys. B658, 113 (2003), R. W. Johnson, Phys. Rev. D 66, 074502 (2002). W. Detmold and C.D. Lin, Phys. Rev. D 73, 014501 (2006). C. Dawson, G. Martinelli, G. C. Rossi, C. T. Sachrajda, S. Sharpe, M. Talevi, and M. Testa, Nucl. Phys. B514, 313 (1998).

A PROPOSAL THAT IS SHOWN TO WORK

Excited states spectroscopy by hadron spectrum collaboration:

$$C_{ij}(t) = \sum_n \frac{1}{2E_n} \langle 0 | \mathcal{O}_i{}^\dagger | n \rangle \underbrace{\langle n | \mathcal{O}_j | 0 \rangle}_{Z_i^n} e^{-E_n t}$$

Overlap function

How to build $\mathcal{O}_i$?

- Smear out the fields: $\begin{cases} \psi(x) \to \widetilde{\psi}(x) \\ U(x) \to \widetilde{U}(x) \end{cases}$
- Subduce it from a continuum angular momentum J:

$$\mathcal{O}^{[J]}_{\Lambda,\lambda} \equiv \sum_M \mathcal{S}^{J,M}_{\Lambda,\lambda} \mathcal{O}^{J,M}$$

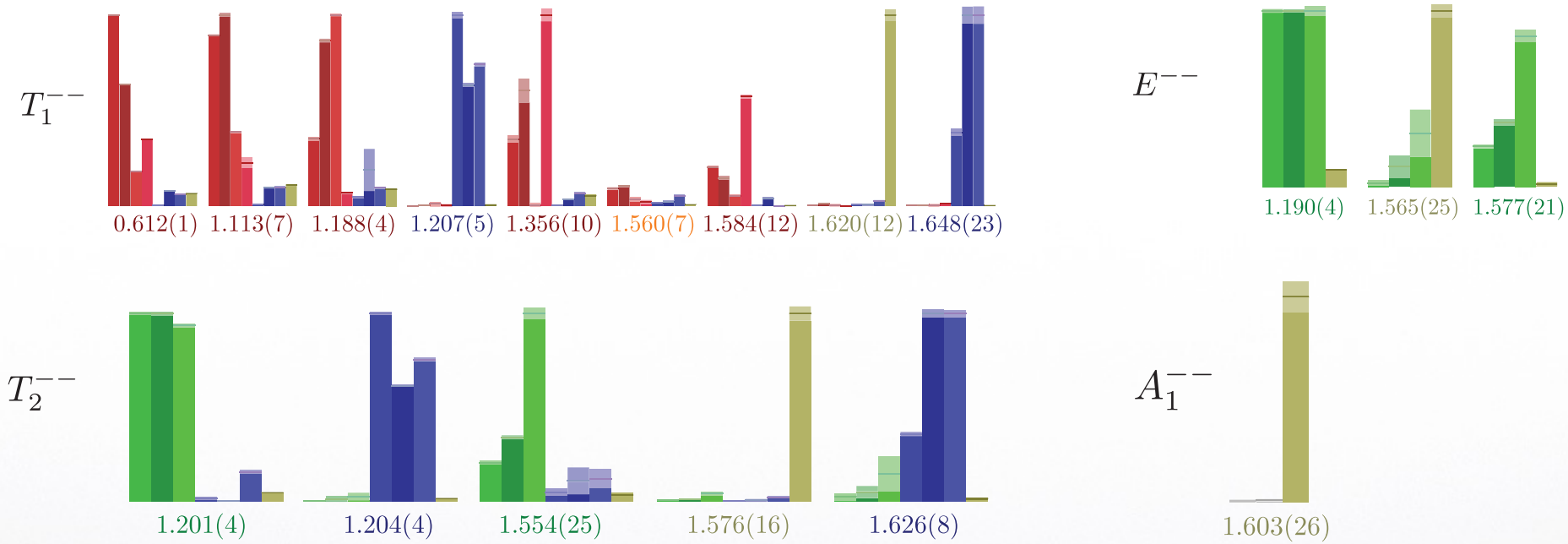
$\mathcal{S}^{J,M}_{\Lambda,\lambda} = \langle \Lambda, \lambda | J, M \rangle$

$\mathcal{O}^{J,M} \equiv (\Gamma \times D^{n_D})^{J,M}$

J. J. Dudek, R. G. Edwards, M. J. Peardon, D. G. Richards, and C. E. Thomas, Phys.Rev.Lett., 103, 262001 (2009); Phys. Rev. D82, 034508 (2010),

A PROPOSAL THAT IS SHOWN TO WORK

The results for the overlap functions:



* *J. J. Dudek, R. G. Edwards, M. J. Peardon, D. G. Richards, C. E. Thomas, Phys.Rev.Lett., 103, 262001 (2009).*

A PROPOSAL THAT IS SHOWN TO WORK

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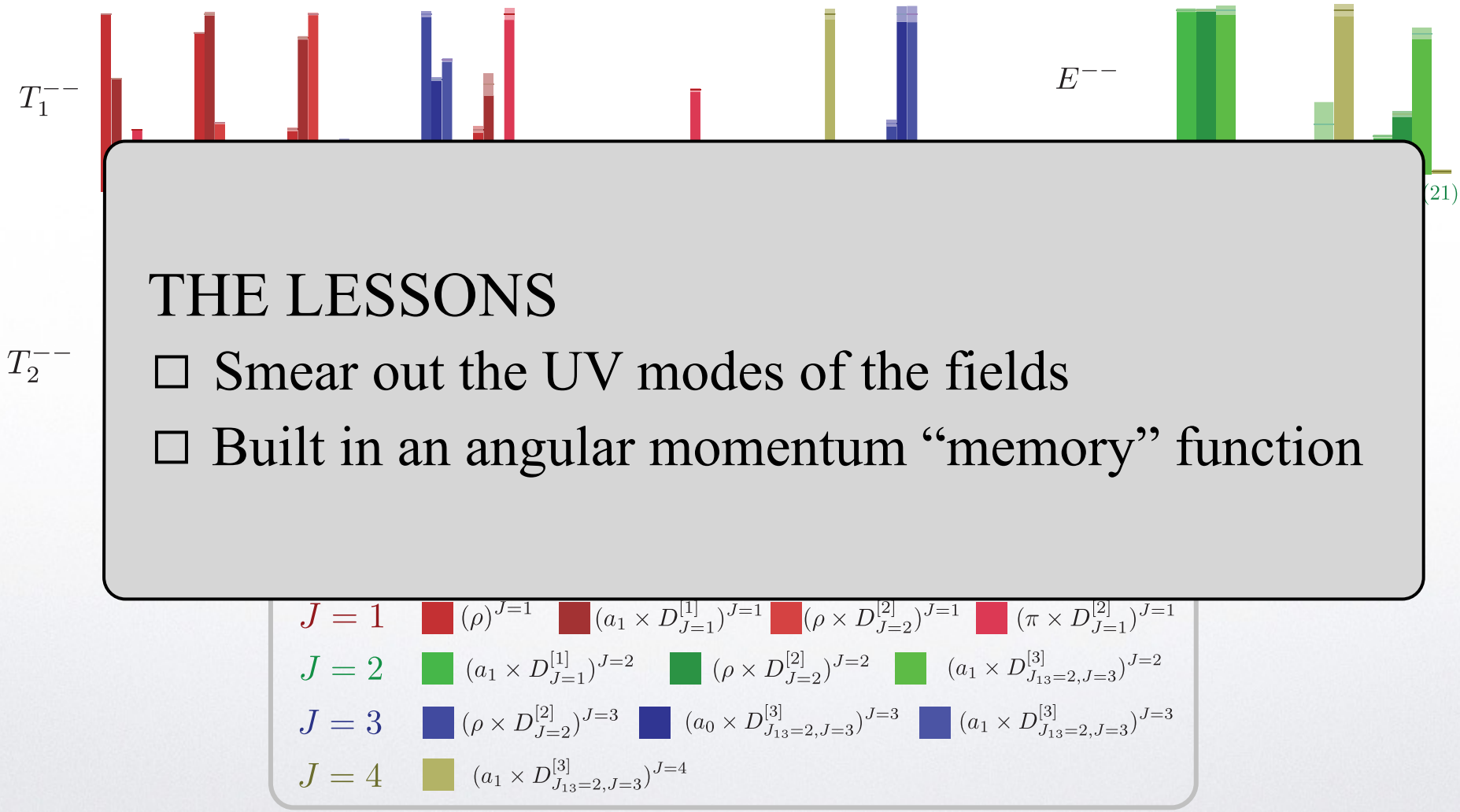


THE LESSONS

- ☐ Smear out the UV modes of the fields
- ☐ Built in an angular momentum "memory" function

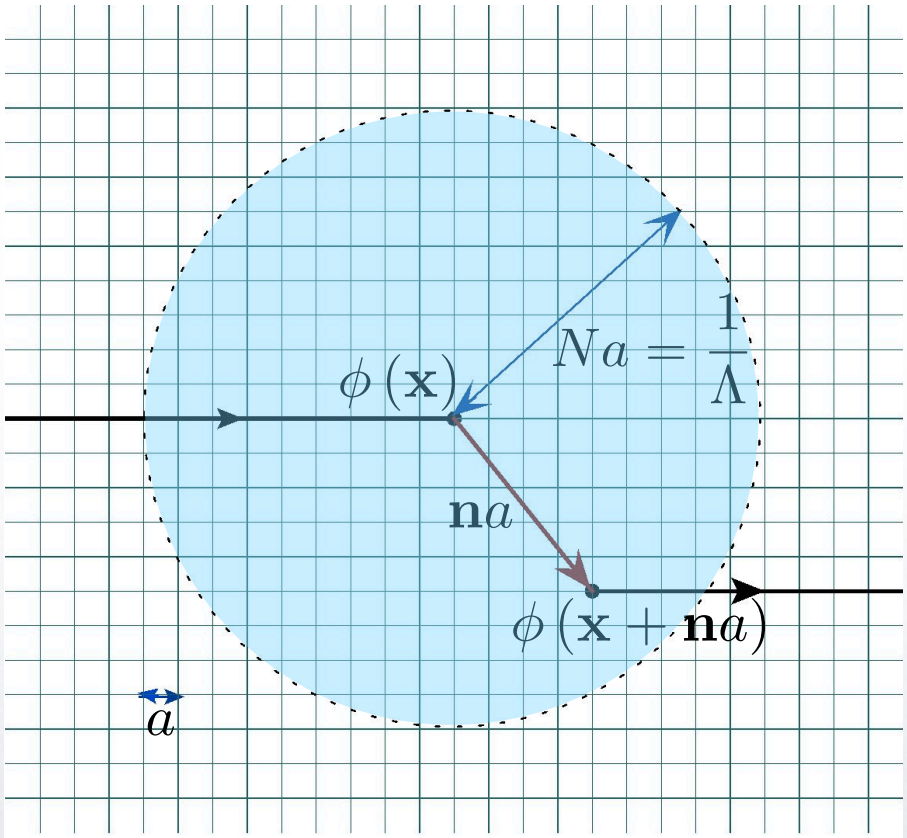
* *J. J. Dudek, R. G. Edwards, M. J. Peardon, D. G. Richards, and C. E. Thomas, Phys.Rev.Lett., 103, 262001 (2009).*

WHAT IS THE THEORETICAL EXPLANATION FOR THIS OBSERVATION?

A THEORETICAL INVESTIGATION OF THE PROPER OPERATOR

A toy model:

$$\hat{\theta}_{L,M}\left(\mathbf{x}; a, N\right) = \frac{3}{4\pi N^3} \sum_{\mathbf{n}}^{|\mathbf{n}|\leq N} \phi\left(\mathbf{x}\right)\phi\left(\mathbf{x}+\mathbf{n}a\right)\; Y_{L,M}\left(\hat{\mathbf{n}}\right)$$

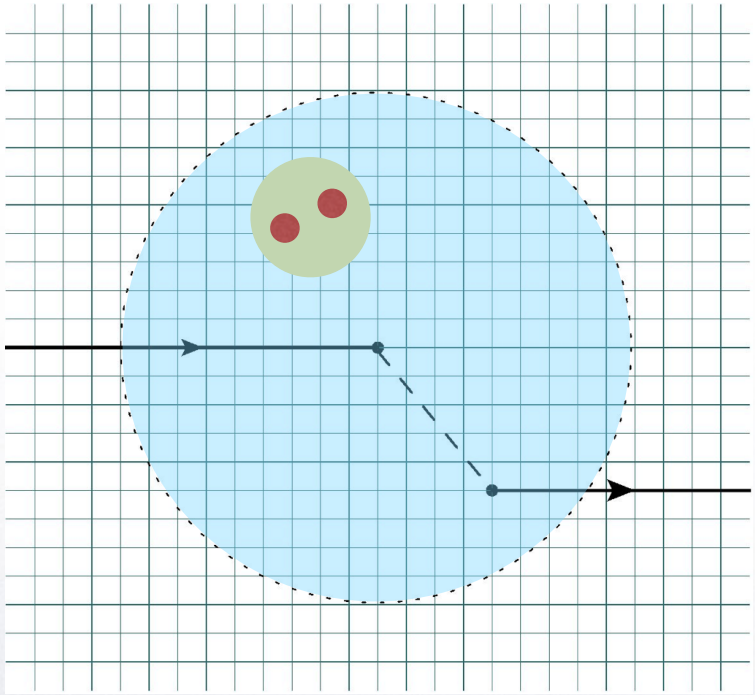


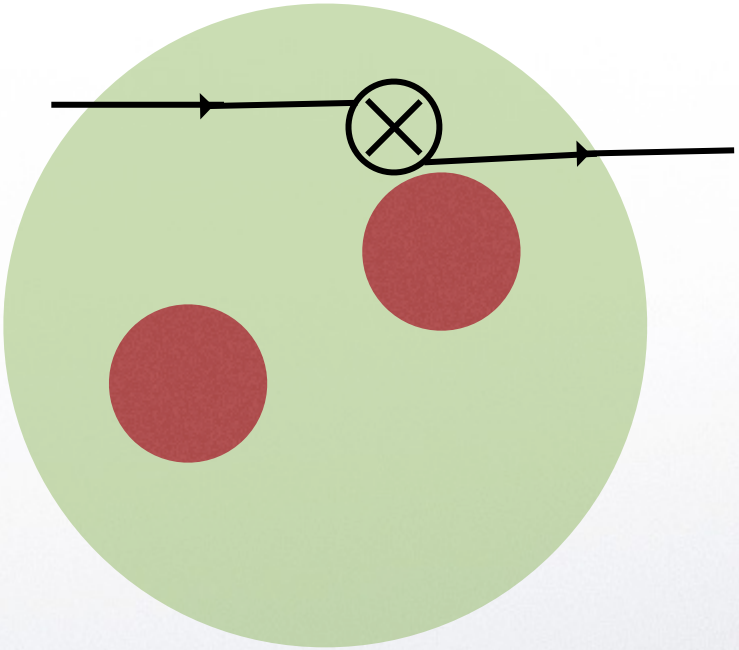
ZD and M. J. Savage, PRD 86, 054505 (2012)

A THEORETICAL INVESTIGATION OF THE PROPER OPERATOR

Relevant scales:

$$a \ll \frac{1}{\Lambda} \ll \frac{1}{\Lambda_{Hadron}} (\frac{1}{p})$$





HOW DOES THE OPERATOR SCALE TOWARDS THE CONTINUUM LIMIT?

A derivative expansion

$$\hat{\theta}_{L,M}(\mathbf{x}; a, N) = \frac{3}{4\pi N^3} \sum_{\mathbf{n}}^{|\mathbf{n}|\leq N} \phi(\mathbf{x})\, \phi(\mathbf{x} + \mathbf{n}a)\; Y_{L,M}(\hat{\mathbf{n}})$$

$$\hat{\theta}_{L,M}(\mathbf{x}; a, N) = \frac{3}{4\pi N^3} \sum_{\mathbf{n}}^{|\mathbf{n}|\leq N} \sum_{k} \frac{1}{k!}\, \phi(\mathbf{x})\, (a\mathbf{n}\cdot\nabla)^k\, \phi(\mathbf{x})\; Y_{L,M}(\hat{\mathbf{n}})$$

$$\hat{\theta}_{L,0}(\mathbf{x}; a, N) = \sum_{L',d} \frac{C^{(d)}_{L0;L'0}(N)}{\Lambda^d} \mathcal{O}^{(d)}_{z^{L'}}(\mathbf{x})$$

The number of derivatives

The operator scale

L' **number of free** z **indices**

ZD and M. J. Savage, PRD 86, 054505 (2012)

What is the operator basis $\mathcal{O}^{(d)}_{z^{L'}}(\mathbf{x})$?

Example

Operators with L=1, M=0

$$\mathcal{O}^{(1)}_{z}(\mathbf{x}) = \phi(\mathbf{x})\nabla_z\phi(\mathbf{x})$$

$$\mathcal{O}^{(3)}_{z}(\mathbf{x}) = \phi(\mathbf{x})\nabla^2\nabla_z\phi(\mathbf{x})$$

$$\mathcal{O}^{(5)}_{z}(\mathbf{x}) = \phi(\mathbf{x})\left(\nabla^2\right)^2\nabla_z\phi(\mathbf{x})$$

$$\mathcal{O}^{(5,RV)}_{z}(\mathbf{x}) = \phi(\mathbf{x})\sum_j \nabla_j^4\nabla_z\phi(\mathbf{x})$$

Violates rotational invariance.

Mixing occurs already in the classical operator:

Example

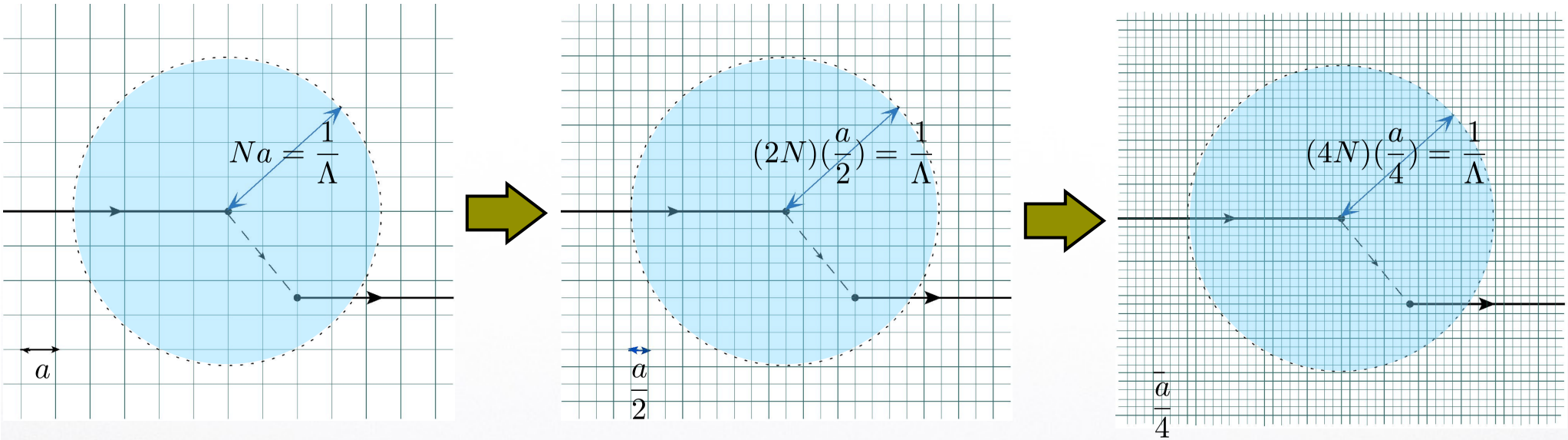
$$\hat{\theta}_{3,0}(\mathbf{x};a,N) = \frac{C^{(1)}_{30;10}(N)}{\Lambda}\mathcal{O}^{(1)}_{z}(\mathbf{x};a) + \frac{C^{(3)}_{30;10}(N)}{\Lambda^3}\mathcal{O}^{(3)}_{z}(\mathbf{x};a) + \frac{C^{(5)}_{30;10}(N)}{\Lambda^5}\mathcal{O}^{(5)}_{z}(\mathbf{x};a) +$$

$$\frac{C^{(5;RV)}_{30;10}(N)}{\Lambda^5}\mathcal{O}^{(5;RV)}_{z}(\mathbf{x};a) + \frac{C^{(3)}_{30;30}(N)}{\Lambda^3}\mathcal{O}^{(3)}_{zzz}(\mathbf{x};a) + \frac{C^{(5)}_{30;30}(N)}{\Lambda^5}\mathcal{O}^{(5)}_{zzz}(\mathbf{x};a) +$$

$$\frac{C^{(5)}_{30;50}(N)}{\Lambda^5}\mathcal{O}^{(5)}_{zzzzz}(\mathbf{x};a) + \mathcal{O}\left(\frac{\nabla_z^7}{\Lambda^7}\right)$$

Coefficients of $L = 3$ operator

Reducing the pixelation of the lattice



$$\hat{\theta}_{3,0}\left(\mathbf{x}; a, N\right)?$$

A good operator if

$$C^{(d)}_{30;L'0}(N) \quad \text{is finite for} \quad L' = 3$$

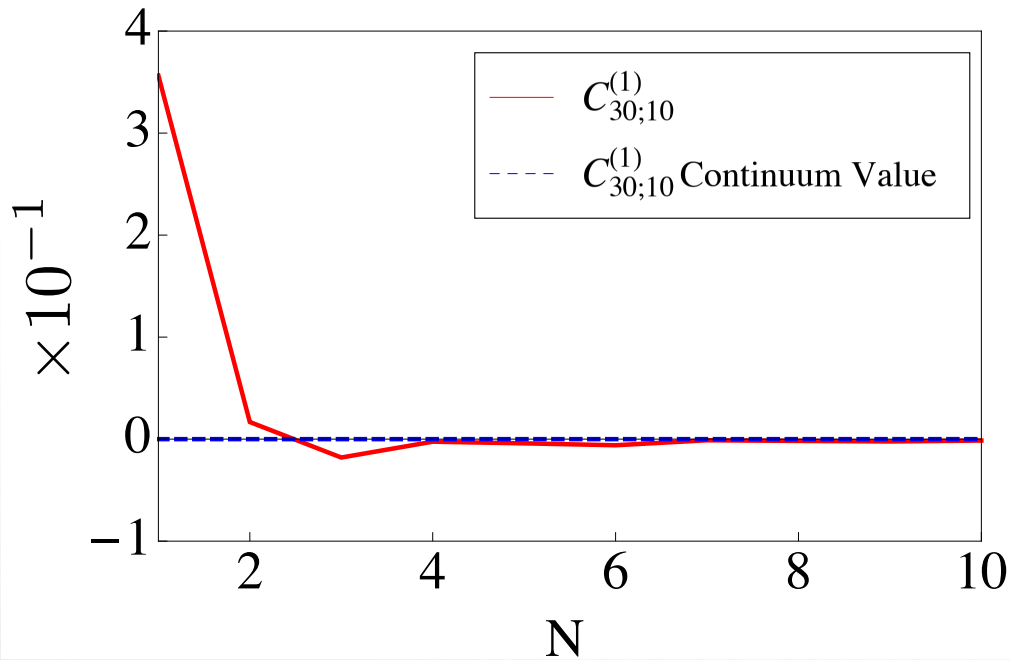
$$C^{(d)}_{30;L'0}(N) \to 0 \quad \text{for} \quad L' \neq 3$$

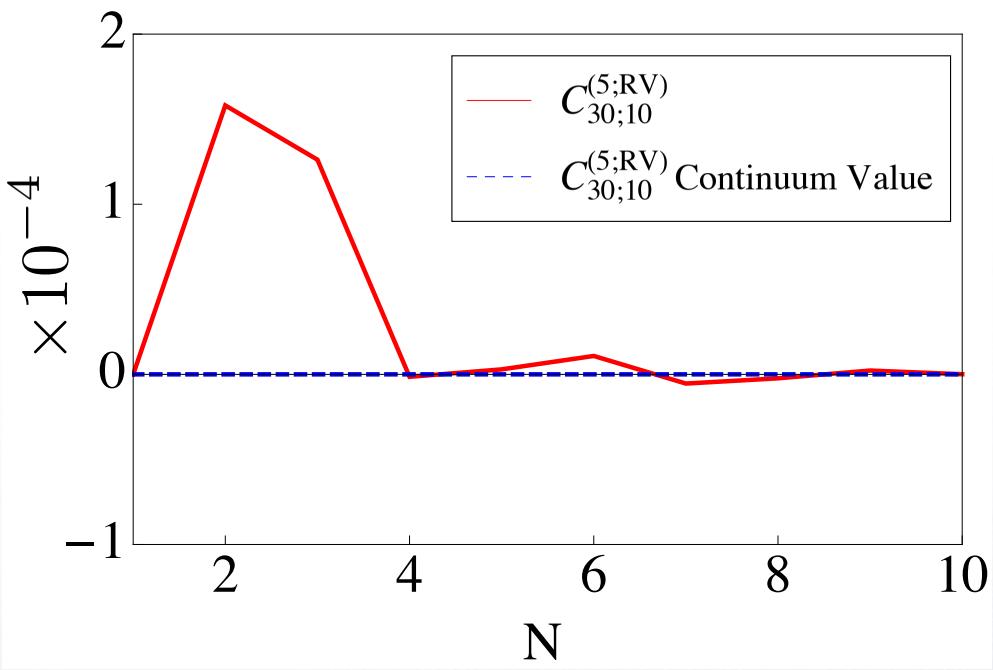
$$C^{(d;RV)}_{30;L'0}(N) \to 0$$

as $N \to \infty$.

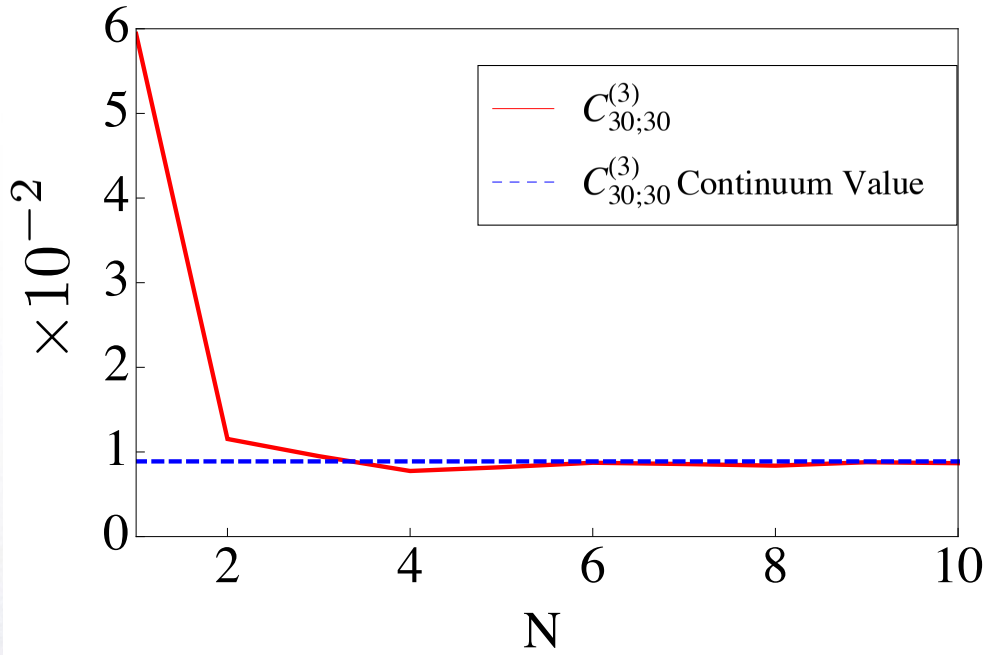
So an $L = 3$ operator is recovered!

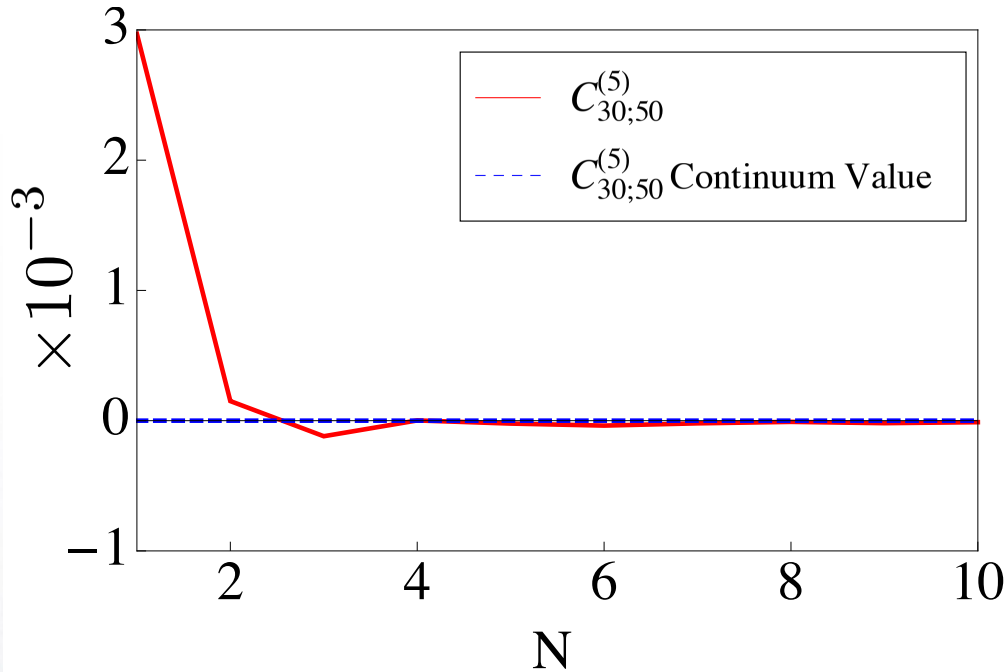
Coefficients as a function of N?



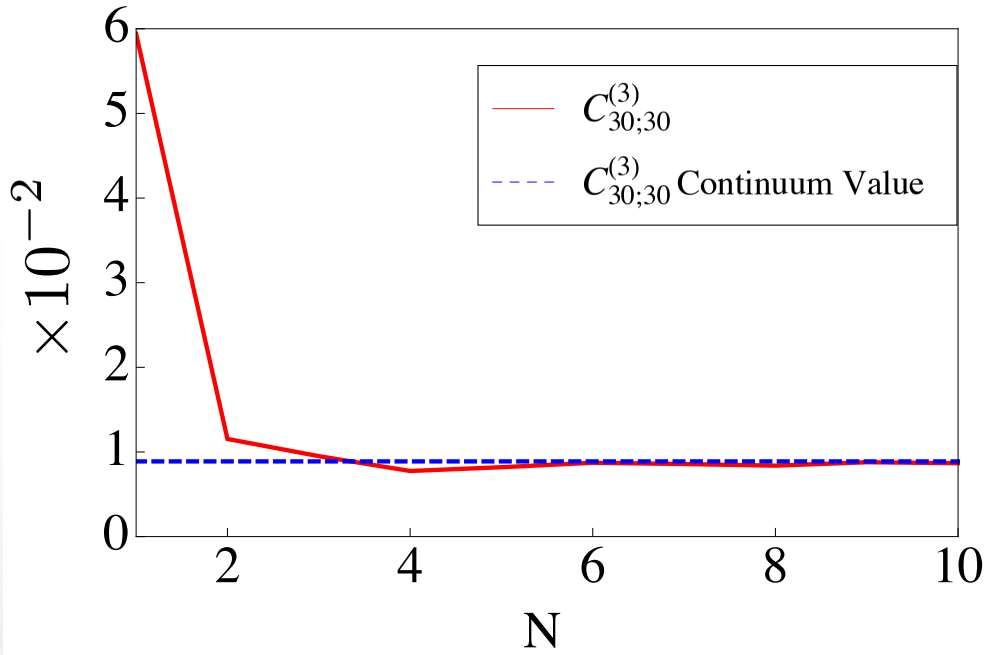


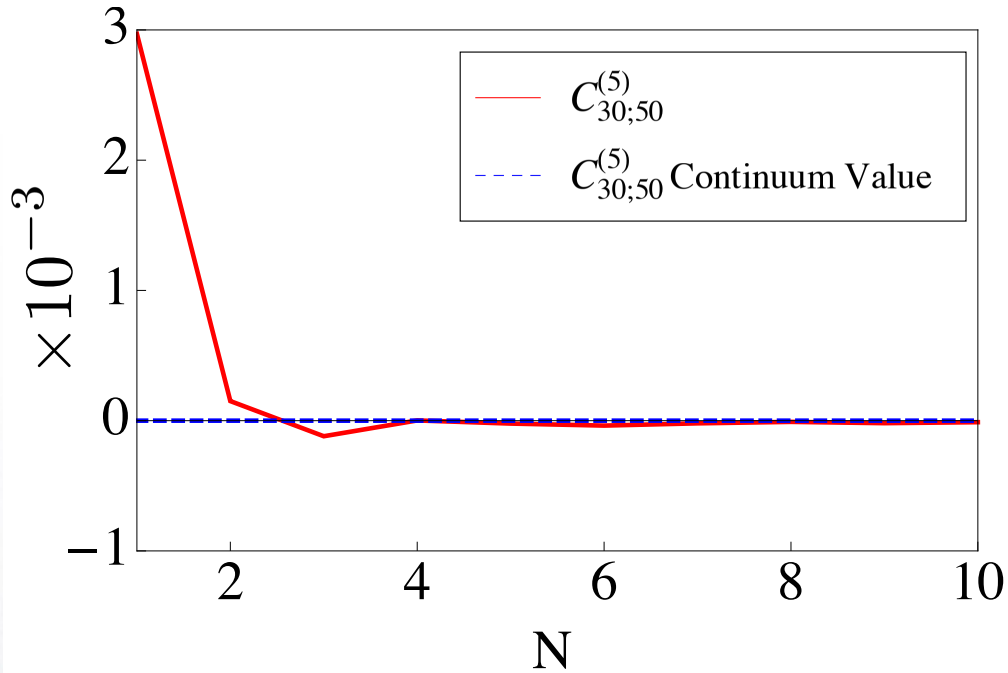
Coefficients as a function of N?





Coefficients as a function of N?





For large N

$$\Lambda^3 \hat{\theta}_{3,0}(\mathbf{x}; a, N) = \alpha_1 \frac{\Lambda^2}{N^2} \mathcal{O}_z^{(1)}(\mathbf{x}) + \alpha_2 \frac{1}{N^2} \mathcal{O}_z^{(3)}(\mathbf{x}) + \alpha_3 \frac{1}{\Lambda^2 N^2} \mathcal{O}_z^{(5)}(\mathbf{x}) +$$

$$\alpha_4 \frac{1}{\Lambda^2 N^2} \mathcal{O}_z^{(5;RV)}(\mathbf{x}) + \alpha_5 \mathcal{O}_{zzz}^{(3)}(\mathbf{x}) + \alpha_6 \frac{1}{\Lambda^2} \mathcal{O}_{zzz}^{(5)}(\mathbf{x}) +$$

$$\alpha_7 \frac{1}{\Lambda^2 N^2} \mathcal{O}_{zzzzz}^{(5)}(\mathbf{x}) + \mathcal{O}\left(\frac{\nabla_z^7}{\Lambda^4}\right)$$

$N = 1$ ⇒

$$\alpha_1 \frac{1}{a^2} \mathcal{O}_z^{(1)} + \alpha_2 \mathcal{O}_z^{(3)} + \alpha_3 a^2 \mathcal{O}_z^{(5)} + \alpha_4 a^2 \mathcal{O}_z^{(5;RV)} +$$

$$\alpha_5 \mathcal{O}_{zzz}^{(3)} + \alpha_6 a^2 \mathcal{O}_{zzz}^{(5)} + \alpha_7 a^2 \mathcal{O}_{zzzzz}^{(5)} + \mathcal{O}\left(a^4 \nabla_z^7\right)$$

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$N = 1$

$$\alpha_1 \frac{1}{a^2} \mathcal{O}_z^{(1)} + \alpha_2 \mathcal{O}_z^{(3)} + \alpha_3 \quad \alpha_4 a^2 \mathcal{O}_z^{(5;RV)} +$$

$$\alpha_5 \mathcal{O}_{zzz}^{(3)} + \alpha_6 a^2 \quad {}_7 a^2 \mathcal{O}_{zzzzz}^{(5)} + \mathcal{O}\left(a^4 \nabla_z^7\right)$$

POWER DIVERGENCE

WHAT ABOUT QUANTUM CORRECTIONS, IN PARTICULAR IN QCD?

MORE ON THE OPERATOR: QUANTUM SCALING IN QCD

The operator in QCD

Differences:

- Link
- Spin/Flavor

$$\hat{\theta}_{L,M}\left(\mathbf{x};a,N\right)=\frac{3}{4\pi N^3}\sum_{\mathbf{n}}^{|\mathbf{n}|\leq N}\overline{\psi}\left(\mathbf{x}\right)\underbrace{U\left(\mathbf{x},\mathbf{x}+\mathbf{n}a\right)}\psi\left(\mathbf{x}+\mathbf{n}a\right)\;Y_{L,M}\left(\hat{\mathbf{n}}\right)$$

$$U\left(\mathbf{x},\mathbf{x}+\mathbf{n}a\right)=1+ig\int_{\mathbf{x}}^{\mathbf{x}+\mathbf{n}a}\mathbf{A}\left(z\right)\cdot d\mathbf{z}+\mathcal{O}\left(g^2\right)$$

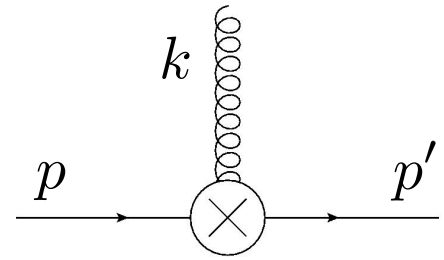
Tree-level operator ⇒ A $J=L$ operator with $1/N^2$ corrections

Quantum operator ⇒ Two complications:
- Tadpoles
- Extended links

ZD and M. J. Savage, PRD 86, 054505 (2012)

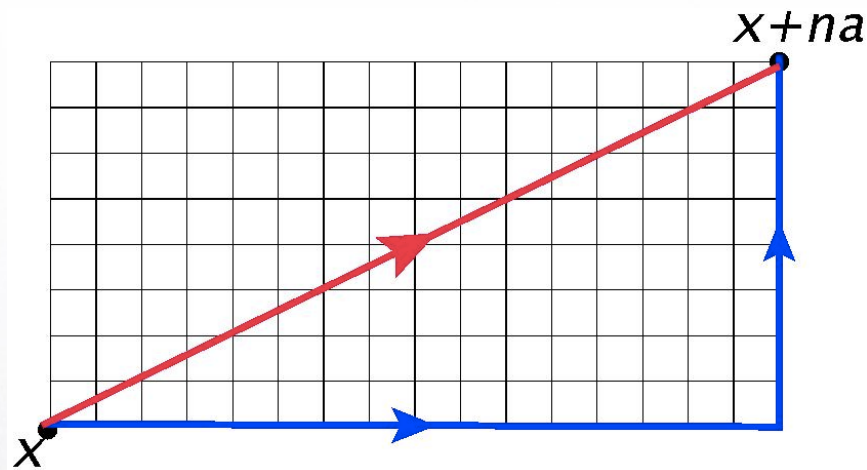
MORE ON THE OPERATOR: QUANTUM SCALING IN QCD

Extended links on the grid requires identifying the path:



- Continuum operator: Radial path vs. other paths?

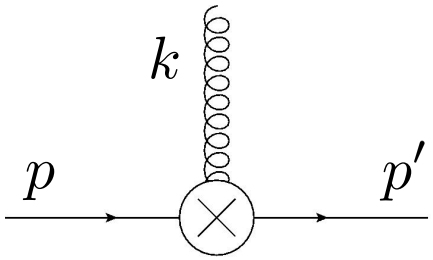
Explicitly rotational invariant gauge link



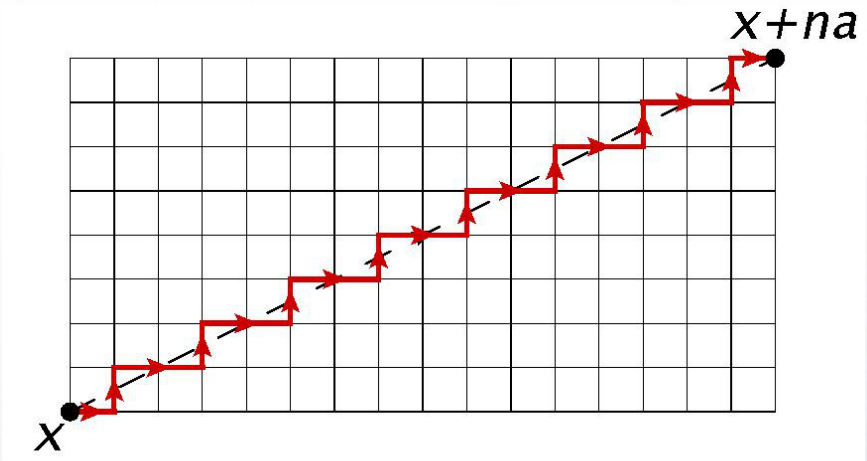
$$V_g^\lambda = \frac{3}{4\pi N^3} \sum_{\mathbf{n}}^{|\mathbf{n}|\leq N} g a n^\lambda \frac{1}{(\mathbf{p}-\mathbf{p}')\cdot \mathbf{n}a} \left(e^{i(\mathbf{k}+\mathbf{p}')\cdot \mathbf{n}a} - e^{i\mathbf{p}'\cdot \mathbf{n}a} \right) \delta^4 (p - p' - k) \; Y_{L,M}(\hat{\mathbf{n}})$$

MORE ON THE OPERATOR: QUANTUM SCALING IN QCD

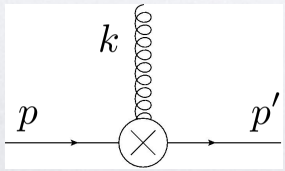
Extended links on the grid requires identifying the path:



- Lattice operator: Closest to the radial path

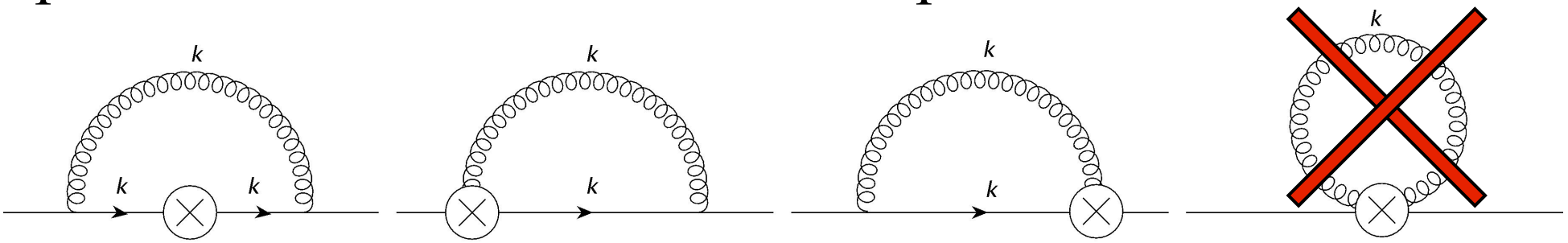


Both RI and RI violating terms



$$= V_g^{\lambda}(k) + \mathcal{O}(g\ k^2 a^2)$$

Operator renormalization at one-loop order: zero external momentum



RI corrections for Wilson fermions: $\sim \alpha_s / N$

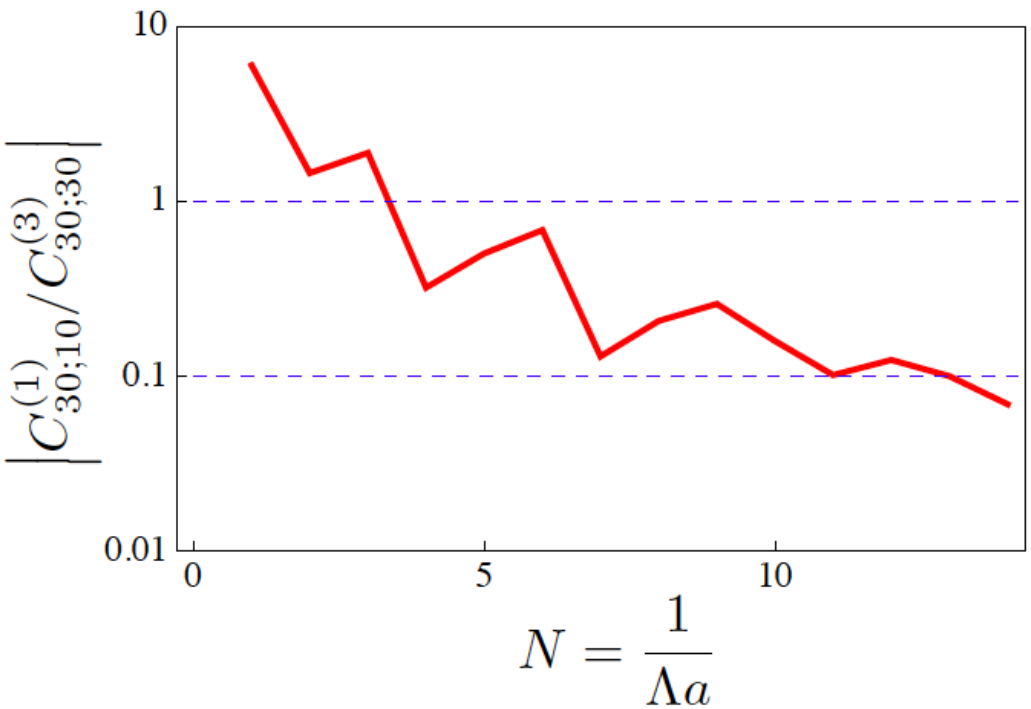
RI violating corrections: $\sim \alpha_s a^2 \Lambda_g^2 \sim \dfrac{\alpha_s}{N_g^2}$

All RI violating corrections $\to 0$ as $a \to 0$

EVALUATING PDFS WITH THE PROPOSED OPERATOR: IN PROGRESS!

WE* WILL IMPLEMENT…

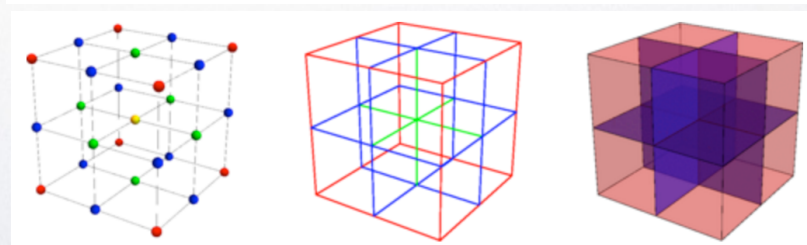
Matrix elements of an $L = 3$ operator:



ZD and M. J. Savage, PRD 86, 054505 (2012)

For our proposal:

a	Λ	N^2
0.08 fm	– $\sim$ 2 GeV	– 1
0.06 fm	– $\sim$ 2 GeV	– 2
0.05 fm	– $\sim$ 2 GeV	– 4
0.04 fm	$\sim$ 5 GeV $\sim$ 2 GeV	1 6
0.03 fm	$\sim$ 5 GeV $\sim$ 2 GeV	2 11
0.02 fm	$\sim$ 5 GeV $\sim$ 2 GeV	4 25



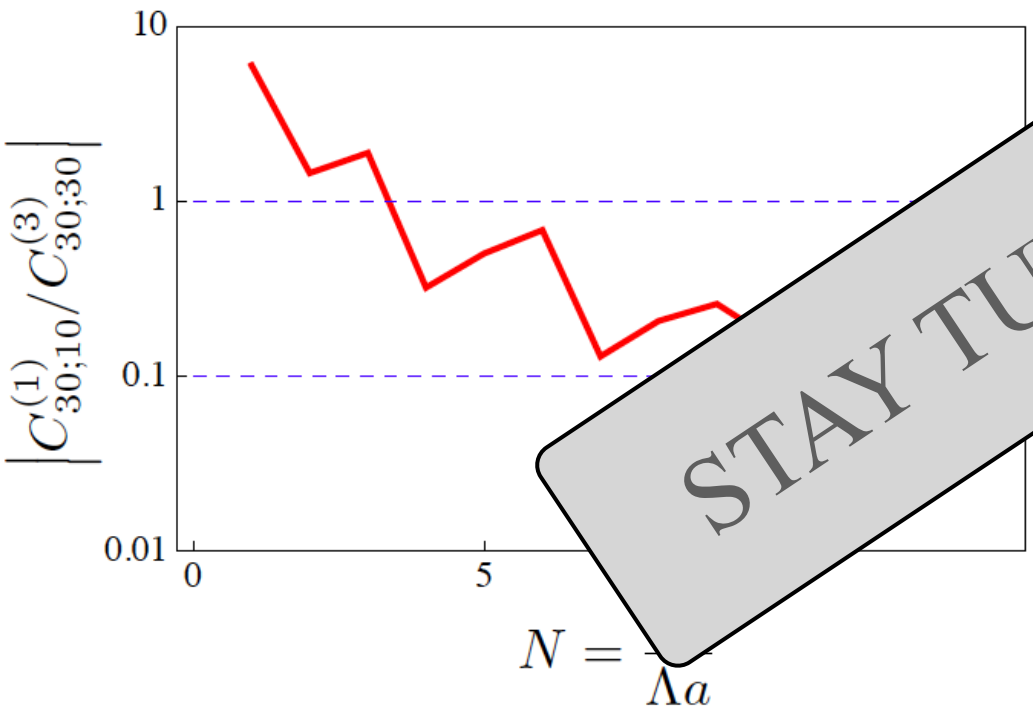
Generating configurations with a multi-scale algorithm.**

* *J-W. Chen, ZD, M. G. Endres, W. Detmold, J. Negele, A. V. Pochinsky, P. Shanahan.*

** *M. G. Endres, R. C. Brower, W. Detmold, K. Orginos, and A. V. Pochinsky, Phys. Rev. D 92, 114516, M. G. Endres and W. Detmold, arXiv:1605.09650 [hep-lat].*

WE* WILL IMPLEMENT…

Matrix elements of an $L = 3$ operator:



STAY TUNED!

ZD and M. J. Savage, PRD 86, 054505 (2012)

For our proposal:

USQCD

a	μ	N^2
0.08 fm	– $\sim$ 2 GeV	– 1
	– $\sim$ 2 GeV	– 2
	– $\sim$ 2 GeV	– 4
04 fm	$\sim$ 5 GeV $\sim$ 2 GeV	1 6
0.03 fm	$\sim$ 5 GeV $\sim$ 2 GeV	2 11
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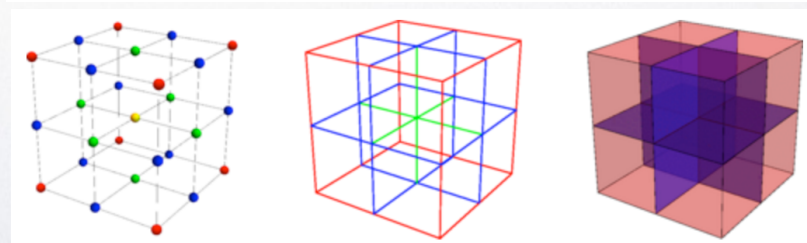

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TO CONCLUDE…

TO CONCLUDE…

SUMMARY

- The proposed operator on the lattice approaches the continuum operator in a smooth way with corrections that scale at most by a^2. Tadpole improvement and gauge field smearing are essential for recovering rotational invariance in lattice gauge theories.

- No power divergences survive! The spectrum of excited states and higher moments of hadron distribution functions are calculable from lattice QCD.

FUTURE WORK

- Are other smearing profiles potentially more useful?

- Can the operator be further improved towards the continuum limit?

- Restoration of SO(4) from hyper-cubic symmetry.

- Renormaliztion of the operator to match to nonperturbative results.

- Comparison with other methods and proposals, e.g., Detmold and Lin; Ji; Monahan and Orginos.

THANK YOU!

BACKUP SLIDES

MORE ON CLASSICAL SCALING OF THE PROPOSED OPERATOR

Analytically

$$C^{(d)}_{30;30} = \frac{15}{4}\sqrt{\frac{7}{\pi}}\,\frac{d^2-1}{(d+4)!} + \mathcal{O}\left(\frac{1}{N^2}\right) \qquad \text{with} \qquad d = 3, 5, \ldots$$

$$C^{(d)}_{30;L0} = \mathcal{O}\left(\frac{1}{N^2}\right) \qquad \text{with} \qquad L \neq 3 \qquad \text{and} \qquad d = L, L+1, \ldots$$

$$C^{(d;RV)}_{30;L0} = \mathcal{O}\left(\frac{1}{N^2}\right) \qquad \text{with} \qquad d = L, L+1, \ldots$$

UNIVERSAL $\frac{1}{N^2} = a^2\Lambda^2$ CORRECTIONS!

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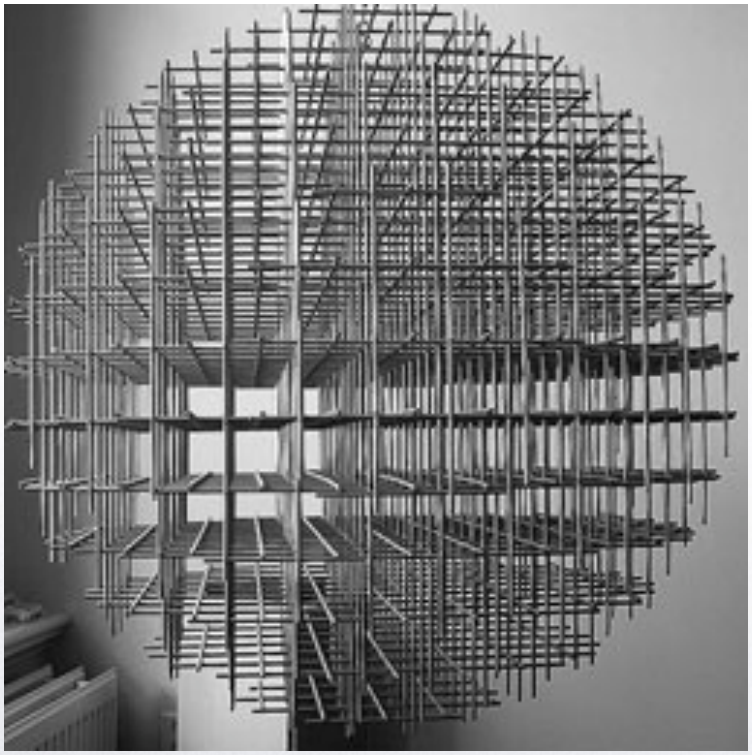
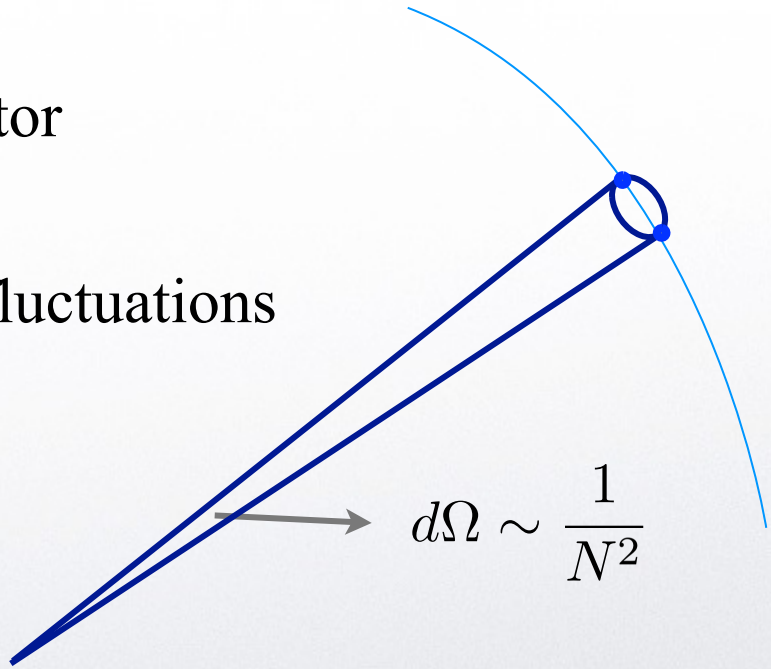


Why $\frac{1}{N^2}$?

Classical operator

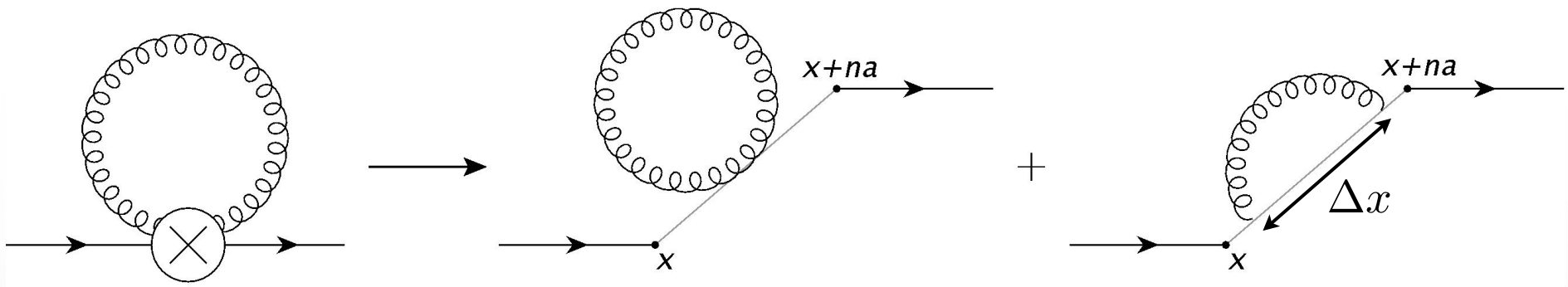
No short distance fluctuations



TADPOLES IN LATTICE QCD PERTURBATION THEORY AND THEIR EFFECT ON PROPOSED OPERATOR

Tadpoles

- Tadpoles of the *continuum* operator



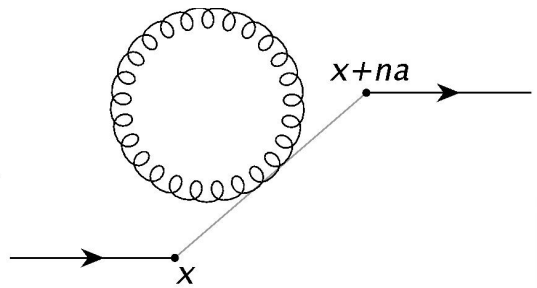
Vanishes!

$\sim \alpha_s / |\mathbf{\Delta x}|^2$

They are harmless in the continuum!

Tadpoles

- Tadpoles of the *lattice* operator



$$\sim \alpha_s a^2 \left(\frac{\pi}{a}\right)^2$$

Non-vanishing!

Perturbative LQCD is poorly convergent!

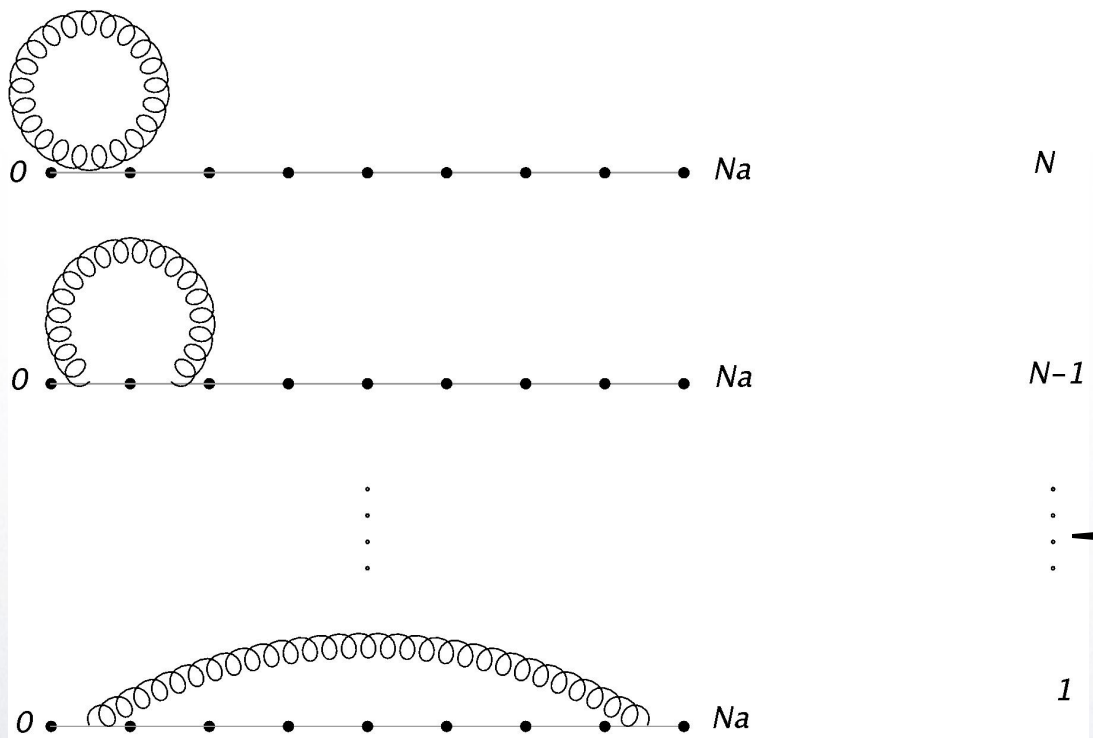
What to do? Tadpole improvement

$$U(x, x + a\hat{\mu}) \to \frac{1}{u_0} U(x, x + a\hat{\mu}) \qquad \text{with} \qquad u_0 \equiv \left\langle \frac{1}{3} \mathrm{Tr}\left(U_{plaq}\right) \right\rangle^{1/4}$$

* *G. P. Lepage and P. B. Mackenzie, Phys. Rev., D48, 2250 (1993)*

MORE ON THE OPERATOR: QUANTUM SCALING IN QCD

Even worse for $\hat{\theta}_{L,M}(\mathbf{x}; a, N)$!



$$\mathcal{O}(N\alpha_s)$$

$$\sum_{m=1}^{N-1}(N-m)\frac{\alpha_s}{m^2} = \mathcal{O}(N\alpha_s)$$

A CLOSER LOOK

Break-down of rotational invariance at $\mathcal{O}\left(N\alpha_s\right)$!

Example

$$\mathbf{n}^2 = 9 : \begin{cases} (2,2,1) \Rightarrow \text{5 tadpoles of the first kind} \\ (3,0,0) \Rightarrow \text{3 tadpoles of the first kind} \end{cases}$$

Different $A_1{}'s$

LESSON

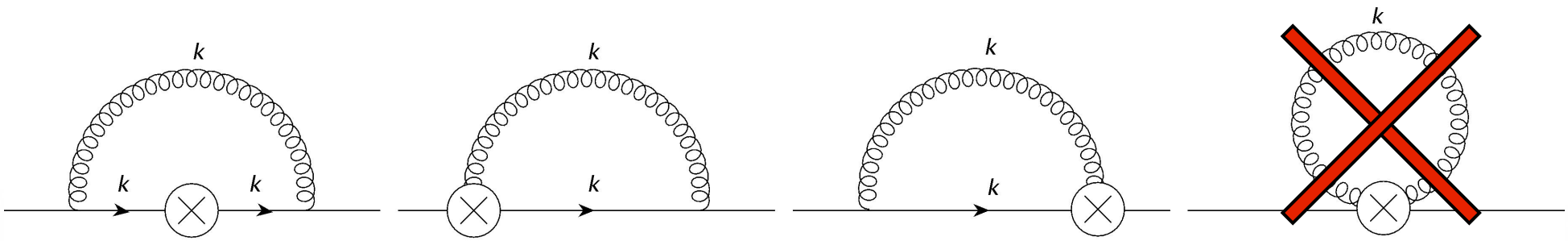
Tadpole improvement is crucial.

$$U_{A_1^i}\left(x, x + a\mathbf{n}\right) \rightarrow \frac{1}{u_{A_1^i}} U_{A_1^i}\left(x, x + a\mathbf{n}\right)$$

ONE-LOOP LATTICE QCD PERTURBATION THEORY AND THE CONVERGENCE OF THE OPERATOR

Operator renormalization at one-loop order:

zero external momentum



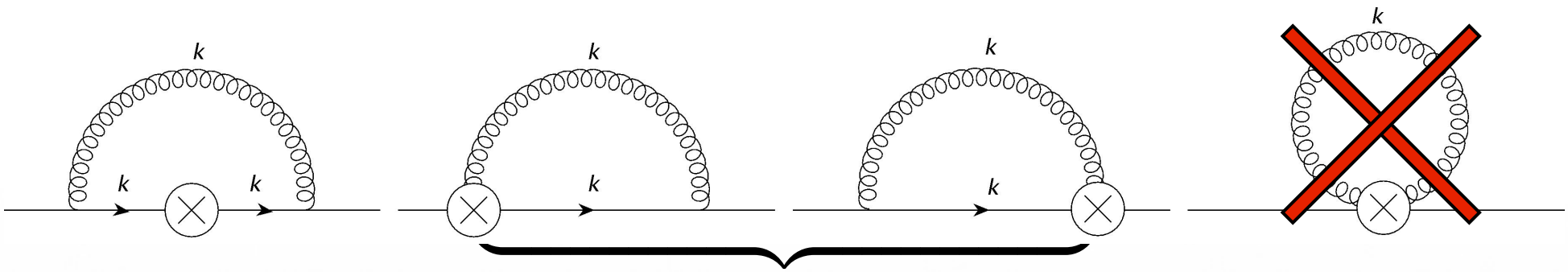
Continuum operator $(L = 0, 1)$: $\sim \alpha_s$

RI corrections for Wilson fermions: $\sim \alpha_s / N$

RV corrections: $\sim \alpha_s / N^2$

Operator renormalization at one-loop order:

zero external momentum



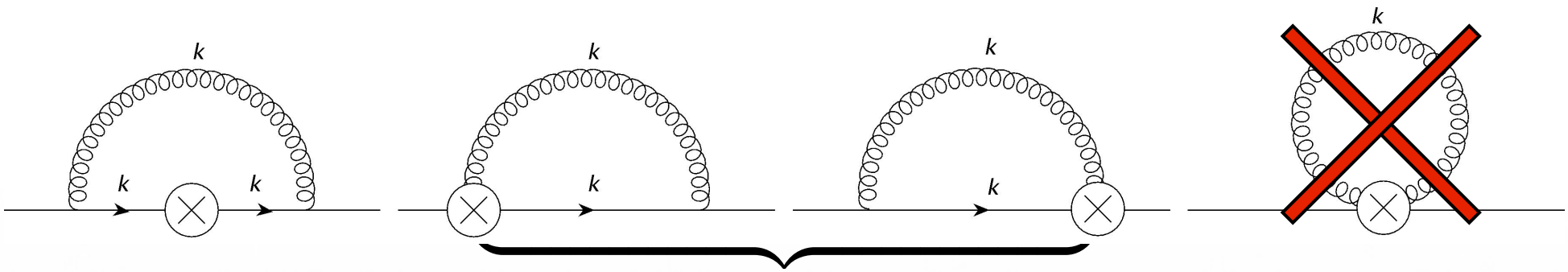
Continuum operator $(L = 0, 1)$: $\begin{cases} L = 0 \sim \alpha_s, \alpha_s \log N \\ L = 1 \sim \alpha_s m_q \end{cases}$

RI corrections for Wilson fermions: $\sim \alpha_s / N$

RV corrections: $\sim \alpha_s$

Operator renormalization at one-loop order:

zero external momentum



Continuum operator $(L = 0, 1)$: $\begin{cases} L = 0 \sim \alpha_s, \alpha_s \log N \\ L = 1 \sim \alpha_s m_q \end{cases}$

RI corrections for Wilson fermions: $\sim \alpha_s / N$

RV corrections: $\sim \alpha_s$ CAUTION

$\mathcal{O}(\alpha_s)$ RI violation

WHY? → UV modes of the gauge fields!

SOLUTION → Smear them over $aN_g = \dfrac{1}{\Lambda_g}$

⇒ RI violating corrections: $\sim \alpha_s a^2 \Lambda_g^2 \sim \dfrac{\alpha_s}{N_g^2}$

All RI violating corrections $\to 0$ as $a \to 0$