Towards Lattice QCD Studies of High Moments of Parton Distribution Functions

September 2016
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WHAT IS THE PROBLEM OF OPERATOR MIXING IN LATTICE QCD CALCULATIONS?
Operators with different angular momentum do not mix in the continuum. Not true for lattice operators with different angular momentum do not mix in the continuum.

Cubic group with only 10 irreps

Full rotational group with infinite number of irreps

LQCD on (Hyper)Cubic lattices
Operators with different angular momentum do not mix in the continuum. Less symmetries, less constraints.

Not true for lattice operators with different angular momentum do not.

A PROBLEM...

Full rotational group with infinite number of irreps

Cubic group with only 10 irreps

LQCD on (HYPER)CUBIC lattices

POWER DIVERGENCE
What quantities have been suffering from this problem?
Higher moments

\[
\langle n! | x \rangle b_u \xi | x \rangle \prod \int = \varepsilon^{n!} \langle u | x \rangle
\]

\[
\{ \varphi \eta | d | \varphi \eta | d \} \bar{d} \varepsilon^{n!} \langle u | x \rangle \bar{Z} = \varepsilon^{n!} \langle s, d | \varphi \eta | \eta \rangle^{n!} \langle s, d | \bar{Z} \}
\]

Moments of parton distributions functions

Spin identification?

Lattice to be built on the

Continuum states up

\[
\langle 0 | (0) \sigma (\sigma \sigma) | 0 \rangle = (\sigma) C
\]

Excited states spectroscopy

Two Examples
Higher moments?

\[
\left( x \right) b_{u,x} \int = x b \left\langle u x \right\rangle
\]

Moments of parton distributions functions

Spin identification?

Continuum states up to

\[
\left\langle 0 \right| (0) \left\langle \tau \right| \left| n \right> = \left( \tau \right) \mathcal{C}
\]

Excited states spectroscopy

TWO EXAMPLES

Moments of parton distributions functions

Lattice

To be built on the

Continuum states up

\[
\left\langle 0 \right| (0) \left\langle \tau \right| \left| n \right> = \left( \tau \right) \mathcal{C}
\]

Excited states spectroscopy
Can this problem be circumvented anyway?
How to build $O$?

\[ \langle W', \chi V \rangle = \chi V \sum_{W, J} W, J, \chi V \]  

Smear out the fields:  

\[ (x) \tilde{\chi} \to (x) \chi \]

\[ (x) \tilde{\varphi} \to (x) \varphi \]  

\[ \langle \mathcal{O} \rangle \]

\textbf{Overlap function}

\[ \langle 0 | \mathcal{O} | u \rangle \langle u | 1 \mathcal{O} | 0 \rangle \sum_{W, z, \varphi} u \varphi \tilde{\varphi} \]  

\[ = \langle \mathcal{O} \rangle \]

Excited states spectroscopy by hadron spectrum collaboration:

\textbf{A PROPOSAL THAT IS SHOWN TO WORK}
In principle the most rigorous method to determine the spin of a state is to perform the extraction of the spectrum for each lattice irrep at successively finer lattices and hence at increasing computational cost. Secondly, the continuum spectrum, classified according to the continuum quantum numbers, exhibits a high degree of degeneracy; when classified according to the pattern of degeneracies emerging as seen in Figure 3 and subsequent figures.

The sensitivity of extracted spectral quantities to the spin of a state is to perform the extraction of the correlator matrix at timeslice 5 in the reconstructed matrix is compared to the original data. This procedure gives a guide to the minimal increase in statistical noise prevents further improvement.

The results for the overlap functions: A PROPOSAL THAT IS SHOWN TO WORK.
Built-in angular momentum "memory" function

Smear out the modes of the fields

THE LESSONS

The results for the overlap functions:

A PROPOSAL THAT IS SHOWN TO WORK

The sensitivity of extracted spectral quantities to

The increase in statistical noise prevents further

The reconstruction of the correlator matrix at times

The procedure has been successfully applied to

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The results for the overlap functions:
WHAT IS THE THEORETICAL EXPLANATION FOR THIS OBSERVATION?
A toy model:

\[ (\mathbf{u}) W^{\mathcal{L}, \Lambda} (v\mathbf{u} + \mathbf{x}) \phi (\mathbf{x}) \phi \sum_{\mathcal{N} \gg |\mathbf{u}|} \frac{3\mathcal{N}^{\mathcal{L}, \Lambda}}{\varepsilon} = (\Delta, q, \mathbf{x}) W^{\mathcal{L}, \theta} \]

A theoretical investigation of the proper operator
A THEORETICAL INVESTIGATION OF THE
PROPER OPERATOR

Relevant scales:

\[
\frac{d}{\lambda} \frac{V_{H_{a}d}}{\lambda} \gg \frac{V}{\lambda} \gg a
\]
HOW DOES THE OPERATOR SCALE TOWARDS THE CONTINUUM LIMIT?
The number of derivatives

\[ (\mathbf{x})_{\mathcal{O}}^{\mathbf{p} \mathbf{V}} (N)_{0, \mathcal{I}} \mathcal{O} = (N \mathbf{q} : \mathbf{x})_{0, \mathcal{I}} \theta \]

The number of derivatives

\[ (\mathbf{u}) \mathcal{W} \mathcal{A} (\mathbf{x}) \phi_{\mathcal{A}} (\Delta \cdot \mathbf{u} \mathbf{u}) (\mathbf{x}) \phi \frac{i y}{l} \sum_{\mathbf{u}} \sum_{N > |\mathbf{u}|} \frac{\varepsilon N \nu \bar{\nu}}{\xi} = (N \mathbf{q} : \mathbf{x}) \mathcal{W} \mathcal{I} \theta \]

A derivative expansion

\[ (\mathbf{u}) \mathcal{W} \mathcal{A} (\mathbf{u} \mathbf{u} + \mathbf{x}) \phi (\mathbf{x}) \phi \sum_{N > |\mathbf{u}|} \frac{\varepsilon N \nu \bar{\nu}}{\xi} = (N \mathbf{q} : \mathbf{x}) \mathcal{W} \mathcal{I} \theta \]
Violates rotational invariance.

Operators with $L=1$, $M=0$:

Example

What is the operator basis $O$?
Mixing occurs already in the classical operator $\mathcal{L}$.

Example

Coefficients of operator $\mathcal{L} = T$

\[
\begin{align*}
\frac{\epsilon V}{\epsilon \Delta}(\mathcal{L})_0 &+ (\mathcal{O}: x)_{\frac{(N)}{\epsilon \Delta}(\mathcal{L})} + (\mathcal{O}: x)_{\frac{(N)}{\epsilon \Delta}(\mathcal{L})} + (\mathcal{O}: x)_{\frac{(N)}{\epsilon \Delta}(\mathcal{L})} + (\mathcal{O}: x)_{\frac{(N)}{\epsilon \Delta}(\mathcal{L})} = \left( N \right)_{0^{0}0^{0}0^{0}}(\epsilon \Delta)_{0^{0}}
\end{align*}
\]
Reducing the pixelation of the lattice

\[
\zeta(N, \nu, x, \theta) \propto \theta
\]
So an $I = 3$ operator is recovered.

\[ \infty \xleftarrow{\text{as}} N \]

\[ 0 \xleftarrow{\mathcal{C}} (\mathcal{N})^{0, T \mid 3 \mid (p \lor \mathcal{R} \land R)} \]

\[ 3 \neq I \text{ for } 0 \xleftarrow{\mathcal{C}} (\mathcal{N})^{0, T \mid 3 \mid (p \lor \mathcal{R} \land R)} \]

\[ 3 = I \text{ is finite for } (\mathcal{N})^{0, T \mid 3 \mid (p \lor \mathcal{R} \land R)} \]

A good operator is a classically scaling one.
Coefficients as a function of $N$?
Coefficients as a function of $N$?
The numerical values of the coefficients as a function of the \( N \) are shown in the plots. It is clear that while the coefficients in eq. (4), at the classical level, as a function of the \( n \) and \( L \), reach a finite value for large \( N \). The plots also indicate that the coefficients decrease as \( N \) increases.
For large $N$

\[
\left(\Delta_{\varphi^p}\right) O + z^{\varphi^p} O z^p \varphi + z^{\varphi^p} O z^p \varphi + (\varphi_{\text{eff}}) O z^p \varphi + (\varphi_{\text{eff}}) O z^p \varphi + (\varphi_{\text{eff}}) O z^p \varphi + (\varphi_{\text{eff}}) O z^p \varphi
\]

\[
\left(\frac{\varphi V}{z \Delta}\right) O + (x) z^{\varphi^p} O z^p \varphi
\]

\[
+ (x) z^{\varphi} O z^p \varphi + (x) z^{\varphi} O z^p \varphi + (x) z^{\varphi} O z^p \varphi + (x) z^{\varphi} O z^p \varphi + (x) z^{\varphi} O z^p \varphi + (x) z^{\varphi} O z^p \varphi
\]

\[
+ (x) z^{\varphi} O z^p \varphi + (x) z^{\varphi} O z^p \varphi + (x) z^{\varphi} O z^p \varphi + (x) z^{\varphi} O z^p \varphi + (x) z^{\varphi} O z^p \varphi + (x) z^{\varphi} O z^p \varphi + (x) z^{\varphi} O z^p \varphi
\]

\[
\varphi = (N, \varphi)^0 \varphi \theta^0 \varphi
\]

For large $N$

**CLASSICAL SCALING**
For large $N$
What about quantum corrections, in particular in QCD?
The operator in QCD

Quantum operator

Tree-Level operator

Extended links

Tadpoles

Quantum operator

Tree-Level operator

Spin/Flavor

Link

Differences:

Quantum Scaling in QCD

MORE ON THE OPERATOR:
Continuum operator: Radial path vs. other paths?

\[
\sum_{N \geq |u|} \frac{3 N \nu A}{3} = \delta \Lambda
\]

MORE ON THE OPERATOR: QUANTUM SCALING IN QCD

Extended links on the grid requires identifying the path.

Explicitly rotational invariant gauge link

Identifying the path: Extended links on the grid requires
MORE ON THE OPERATOR: QUANTUM SCALING IN QCD

Closest to the radial path

\[ k^p p^p_0 = V_g(k) + O(gk^2a^2) \]

Both RI and RI violating terms

Lattice operator: Closest to the radial path

Identifying the path: Extended links on the grid requires

Quantum Scaling in QCD

More on the Operator:
Operator renormalization at one-loop order:

\[ \frac{N}{s} \sim \frac{N^2}{s} \sim V^2 \rho s \sim N \]

All RI violating corrections:

\[ a \to 0 \quad a \to 0 \]

RI violating corrections:

\[ \frac{6 N}{s} \sim \frac{6}{V^2} \rho s \sim N \]

RI corrections for Wilson fermions:

\[ \frac{N}{s} \sim \frac{N}{s} \]

Zero external momentum.
EVALUATING PDFS WITH THE PROPOSED OPERATOR: IN PROGRESS!
Matrix elements of an operator:

For our proposal:

\[
\frac{\langle 0 | \mathcal{A} | n \rangle}{\langle 1 | \mathcal{A} | n \rangle} = N
\]
Generating configurations with a multi-scale algorithm.

Matrix elements of an operator:

For our proposal:

We will implement...
TO CONCLUDE...
SUMMARY

- The proposed operator on the lattice approaches the continuum operator in a smooth way with corrections that scale at most by $a$. Tadpole improvement and gauge field smearing are essential for recovering rotational invariance in lattice gauge theories.

- No power divergences survive! The spectrum of excited states and higher moments of hadron distribution functions are calculable from lattice QCD.

- Re-normalization of the operator to match to non-perturbative results.

- Restoration of SO(4) from hyper-cubic symmetry.

- Future Work

  - Are other smearing profiles potentially more useful?
  - Can the operator be further improved towards the continuum limit?
  - Restoration of SO(4) from hyper-cubic symmetry.
  - Renormalization of the operator to match to non-perturbative results.
  - Comparison with other methods and proposals, e.g., Detmold and Lin; Ji, Monahan and Orginos.

TO CONCLUDE...
THANK YOU!
MORE ON CLASSICAL SCALING OF THE PROPOSED OPERATOR
\[ \mathcal{V} \mathcal{V}^p = \frac{z^N}{I} \mathcal{O}^p \]

\[ \cdots, I + T, T = p \quad \text{with} \quad \left( \frac{z^N}{I} \right)^p \mathcal{O} = \mathcal{O}^{0 T, 0, 0, 0}_{\mathcal{N}^p} \]

\[ \cdots, I + T, T = p \quad \text{and} \quad \zeta \neq I \quad \text{with} \quad \left( \frac{z^N}{I} \right)^p \mathcal{O} = \mathcal{O}^{0 T, 0, 0, 0}_{\mathcal{N}^p} \]

\[ \cdots, \zeta = p \quad \text{with} \quad \left( \frac{z^N}{I} \right)^p \mathcal{O} + \frac{i(\bar{v} + p)}{1 - \zeta^p} \frac{\nu}{I} \overline{\mathcal{V}^p} = \mathcal{O}^{0 T, 0, 0, 0}_{\mathcal{N}^p} \]

Analytically
No short distance fluctuations

Classical operator

Why

\[ \frac{zN}{\ell} \sim \mathcal{O} \]

Corrections?

\[ \mathcal{Z} = \frac{zN}{\ell} \text{ Universal Scaling} \]
EFFECT ON PROPOSED OPERATOR
PERTURBATION THEORY AND THEIR
TADPOLES IN LATTICE QCD
They are harmless in the continuum!

\[ |x|/s^2 \sim \]

\[ |x|/s^2 \]

Vanishes!

MORE ON THE OPERATOR: QUANTUM SCALING IN QCD

\[ \Box \] Tadpoles of the continuum operator

\[ \Box \] Tadpoles
Tadpole improvement

What to do?

Perturbative LQCD is poorly convergent!

Non-vanishing!

Tadpoles of the lattice operator

Quantum scaling in QCD

More on the operator:
Even worse for \( \hat{L}, \hat{M} \) and \( \hat{N} \):
Break-down of rotational invariance at $O(N)_c$:

\[ (u_x + x', x) \left\{ \begin{array}{l} \frac{1}{V} \mathcal{N} \frac{1}{V} n \\ \prod \end{array} \right\} \leftarrow (u_x + x', x) \left\{ \begin{array}{l} \frac{1}{V} \mathcal{N} \frac{1}{V} n \\ \prod \end{array} \right\} \]

Tadpole improvement is crucial.

Different $A_1's$:

3 tadpoles of the first kind

$\left\{ \begin{array}{l} (0', 0) \\ (3', 0) \\ (2', 1) \end{array} \right\}$

\[ 6 = \begin{array}{l} (u_x) \end{array} \]

Example

Lesso
ONE-LOOP LATTICE QCD PERTURBATION

The Operator Theory and The Convergence Of
Operator renormalization at one-loop order:

\[ \mathcal{N}/s^N \sim \left( \frac{s}{N} \right)^2 \]

Corrections for Wilson fermions:

RV corrections:

RI corrections for Wilson fermions:

Continuum operator (\( T = 0, 1 \)):

\[ s^N \sim \mathcal{N}/s^N \]

Operator renormalization at one-loop order:

Quantum Scaling in QCD

More on the Operator:
Operator renormalization at one-loop order:

\[
\frac{N}{s^o} \sim s^o, \quad \log s^o, \quad s^o \sim 0 = T
\]

Continuum operator (Cons.)

\[
\sum_{m=1}^{\infty} \frac{s^o}{m^2} \sim I = T
\]

\[
RV \text{ corrections for Wilson fermions:}
\]

\[
RL \text{ corrections for Wilson fermions:}
\]

More on the Operator:

Quantum Scaling in QCD
Operator renormalization at one-loop order:

- Continuum operator
- RI corrections for Wilson fermions
- RV corrections

\[ \frac{N}{s} \sim s \log N \sim 0 = T, 1 = T \]

**CAUTION**

Zero external momentum

More on the operator: Quantum Scaling in QCD
All RI violating corrections $\to 0$ as $a \to 0$

\[ 0 \sim \alpha_s A_g^2 \sim \frac{\alpha_s}{N_c} \]

$\alpha_s = 1$

**WHY?**

RI violating corrections:

**SOLUTION**

$O(\alpha_s)$ RI violation

**UV modes of the gauge fields!**

Smear them over $aN_g = \frac{1}{A_g}$

**MORE ON THE OPERATOR:**

Quantum scaling in QCD