Zurich, 10th October 2016

The anomalous magnetic moment of the muon: theoretical determination of hadronic contributions

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Work in collaboration with Luigi Cappiello Oscar Catà and David Greynat

The hadronic light by light contribution to the (g-2)_(muon} with holographic models of QCD.P.R.D83:093006,2011. : arXiv:1009.1161**C69 (2010) 315**

and work in progress

Outline

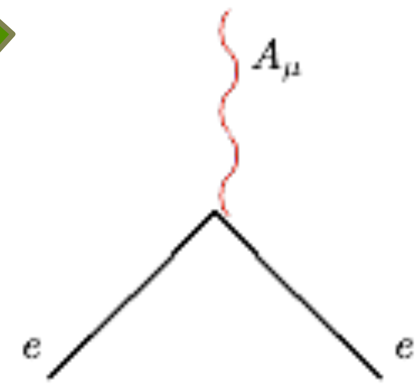
- g-2
- summary of experimental results
- status of hadronic contributions
- light by light contributions, theoretical models
- Three-point functions $\pi^0 \to \gamma^*\gamma^*$, hadronic contribution (light by light)
- Melnikov-Vainshtein model leading log approx
- Our work in progress

Precision physics=>solid theory

Two crucial precise measurements

Lamb shifts
(hydrogen level splittings)

$g_e \, \mu_B \, H$
1.00118(3)



- Accurate theoretical calculation required by 0.5% measurement in 1946=> QED and Shwinger calculation

Linear response of a charged lepton to an external electromagnetic field

$$\langle \ell; p'|J_\rho(0)|\ell; p\rangle \equiv \bar{\mathrm{u}}(p')\Gamma_\rho(p',p)\mathrm{u}(p)$$

$$= \bar{\mathrm{u}}(p')\left[F_1(k^2)\gamma_\rho + \frac{i}{2m_\ell}F_2(k^2)\sigma_{\rho\nu}k^\nu - F_3(k^2)\gamma_5\sigma_{\rho\nu}k^\nu + F_4(k^2)(k^2\gamma_\rho - 2m_\ell k_\rho)\gamma_5\right]\mathrm{u}(p)$$

(Lorentz invariance + conservation of the electromagnetic current J_ρ)

$F_1(k^2) \rightarrow$ Dirac form factor, $F_1(0) = 1$

$F_2(k^2) \rightarrow$ Pauli form factor $\rightarrow F_2(0) = a_\ell$

$F_3(k^2) \rightarrow \not{P}, \not{T}$, electric dipole moment $\rightarrow F_3(0) = d_\ell/e_\ell$

$F_4(k^2) \rightarrow \not{P}$, anapole moment

$$G_E(k^2) = F_1(k^2) + \frac{k^2}{4m_\ell^2}F_2(k^2),\ G_M(k^2) = F_1(k^2) + F_2(k^2)$$

$$\boldsymbol{\mu}_\ell = g_\ell\left(\frac{e_\ell}{2m_\ell c}\right)\mathrm{S},\ \mathrm{S} = \hbar\frac{\boldsymbol{\sigma}}{2} \qquad g_\ell = g_\ell^{\mathrm{Dirac}} \times G_M(0)$$

At tree level, $F_1 = 1,\ F_2 = F_3 = F_4 = 0,\ g_\ell = g_\ell^{\mathrm{Dirac}} \equiv 2$

The *anomalous* magnetic moment a_ℓ is induced at loop level $\left(a_\ell \equiv \frac{g_\ell - g_\ell^{\mathrm{Dirac}}}{g_\ell^{\mathrm{Dirac}}}\right)$

a_ℓ probes the contributions of quantum loops from SM and BSM degrees of freedom

Response of a charged lepton to an external (and static) electromagnetic field

For a relativistic, point-like spin 1/2 particle, described by the Dirac equation with the minimal coupling prescription, one has

$$i\hbar \frac{\partial \psi}{\partial t} = \left[c\boldsymbol{\alpha} \cdot \left(-i\hbar \boldsymbol{\nabla} - \frac{e_\ell}{c} \boldsymbol{\mathcal{A}} \right) + \beta m_\ell c^2 + e_\ell \mathcal{A}_0 \right] \psi$$

In the non relativistic limit, this reduces to the Pauli equation for the two-component spinor φ describing the large components of the Dirac spinor ψ,

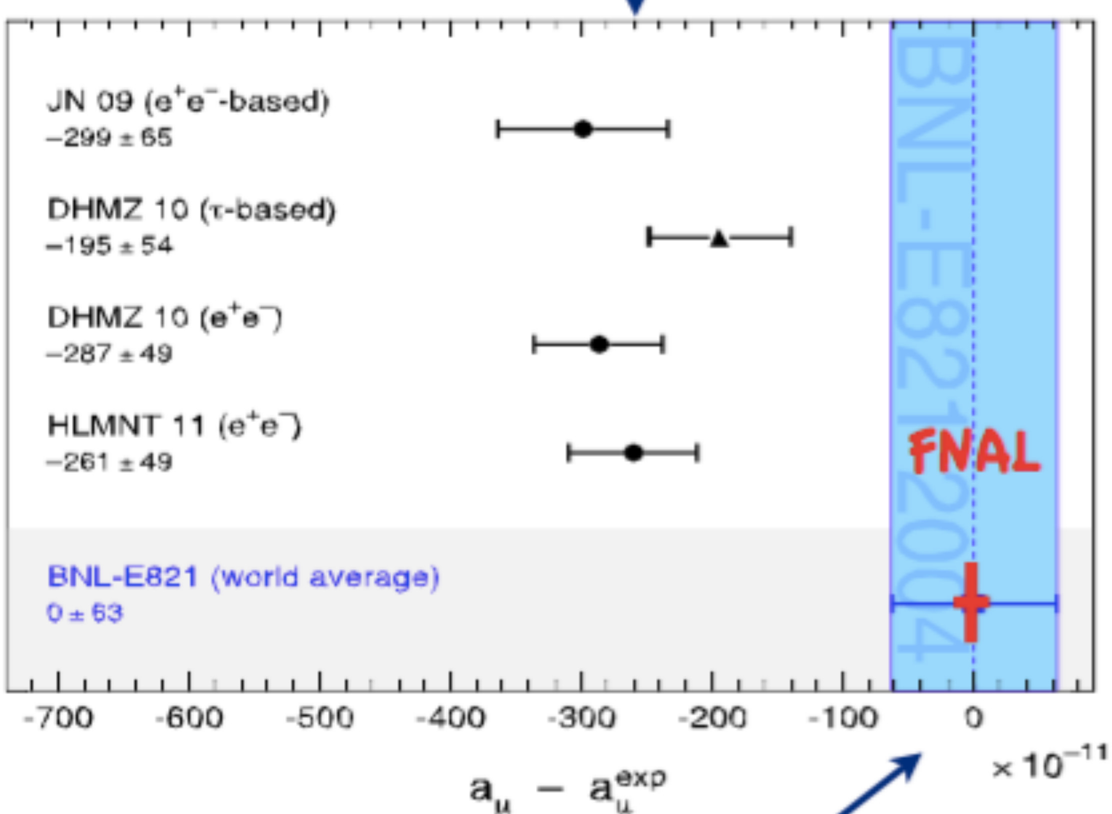
$$i\hbar \frac{\partial \varphi}{\partial t} = \left[\frac{(-i\hbar \boldsymbol{\nabla} - (e_\ell/c)\boldsymbol{\mathcal{A}})^2}{2m_\ell} - \underbrace{\frac{e_\ell \hbar}{2m_\ell c} \boldsymbol{\sigma} \cdot \mathsf{B}}_{\boldsymbol{\mu}_\ell \cdot \mathsf{B}} + e_\ell \mathcal{A}_0 \right] \varphi$$

with

$$\boldsymbol{\mu}_\ell = g_\ell \left(\frac{e_\ell}{2m_\ell c} \right) \mathsf{S} \,,\ \mathsf{S} = \hbar \frac{\boldsymbol{\sigma}}{2} \,,\ g_\ell^{\text{Dirac}} = 2$$

$(g-2)_\mu$: theory vs experiment

SM predictions for a_μ



BNL-E821 measurement of a_μ

$a_\mu^{exp} = (11\ 659\ 208.9 \pm 6.3) \times 10^{-10}$

$a_\mu^{exp} - a_\mu^{SM} =$
$(26.1 \pm 5.0_{\ th} \pm 6.3_{\ exp}) \times 10^{-10}$

Hagiwara et al. (2011)

3 - 4 σ deviation from SM value !

Errors or new physics ?

New FNAL, J-PARC experiments

$\delta a_\mu^{FNAL} = 1.6 \times 10^{-10}$ M. Lancaster

factor 4 improvement in exp. error

-> Improve theory !

Magic vs "New Magic"

- **Complimentary!**

$$\vec{\omega} = -\frac{e}{m}\left[a_\mu \vec{B} - \left(a_\mu - \frac{1}{\gamma^2 - 1}\right)\frac{\vec{\beta}\times\vec{E}}{c} + \frac{\eta}{2}\left(\vec{\beta}\times\vec{B} + \frac{\vec{E}}{c}\right)\right]$$

BNL/Fermilab Approach

$$a_\mu - \frac{1}{\gamma^2 - 1} = 0$$

$$\eta \approx 0$$

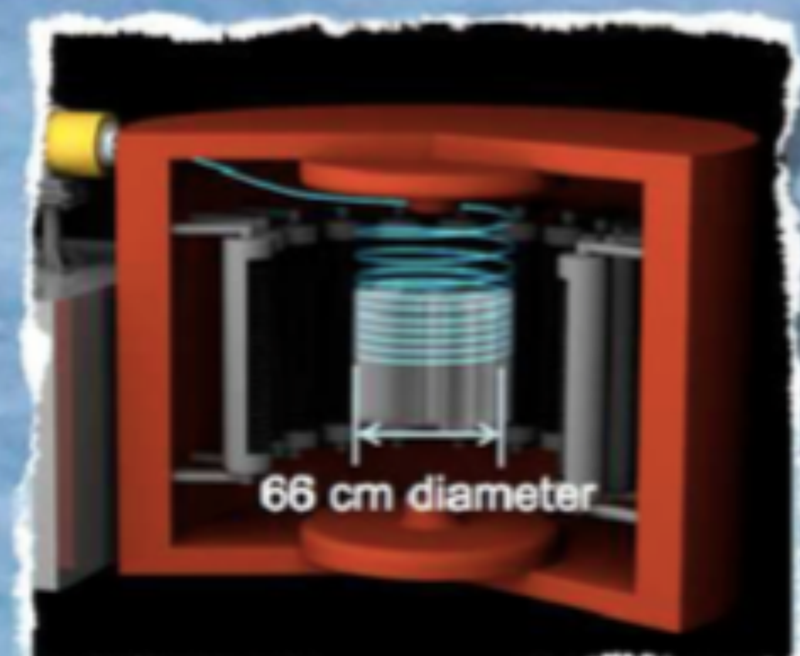
$$\gamma_{\text{magic}} = 29.3$$

$$p_{\text{magic}} = 3.09 \text{ GeV}/c$$

J-PARC Approach

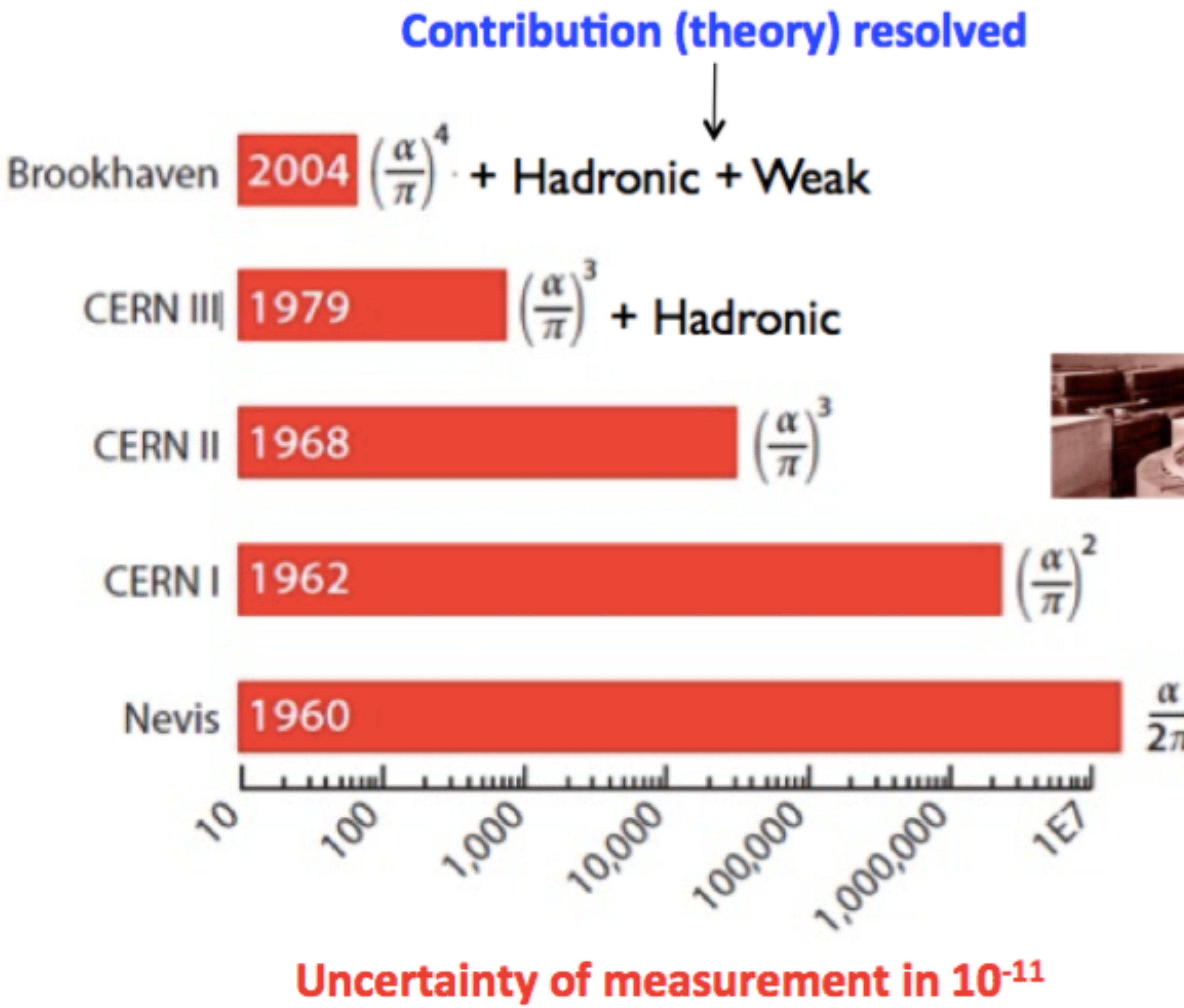
$$\vec{E} = 0$$

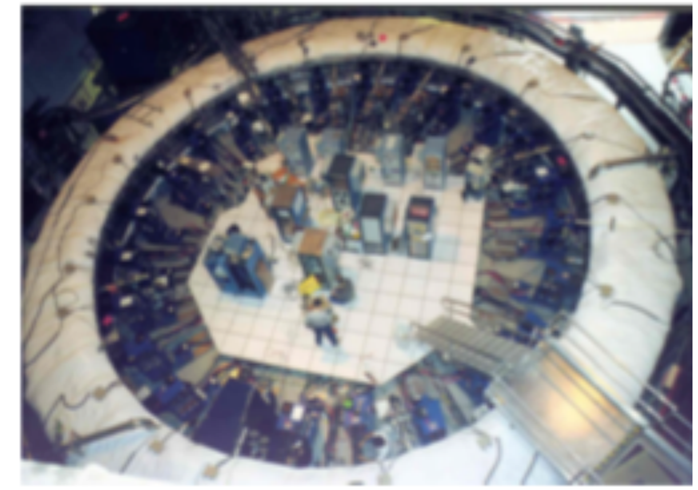
$$\vec{\omega} = \vec{\omega}_a + \vec{\omega}_\eta$$



$$\vec{\omega}_a = -\frac{e}{m} a_\mu \vec{B}$$

$(g\text{-}2)_\mu$: history of relevant corrections

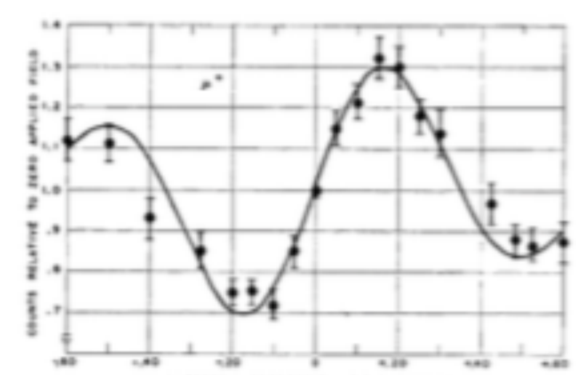




Brookhaven

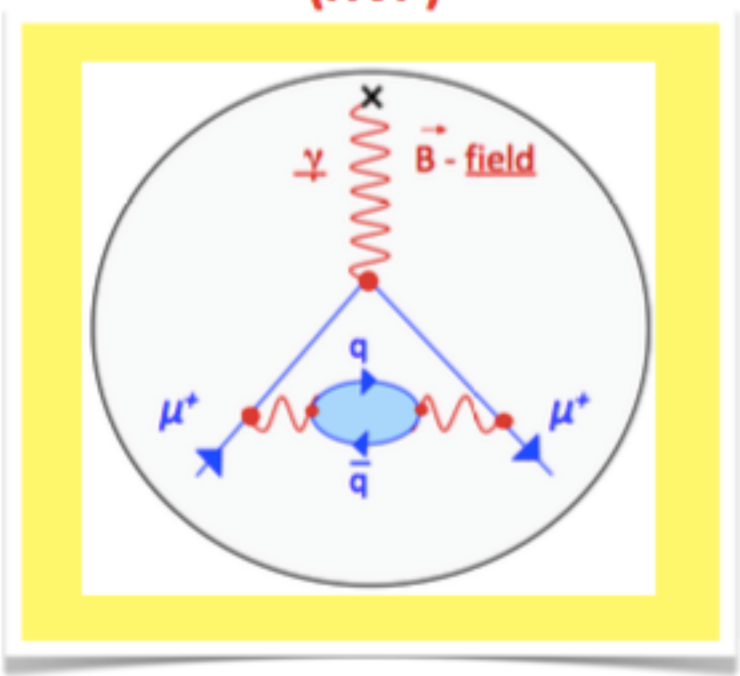
CERN I

Nevis

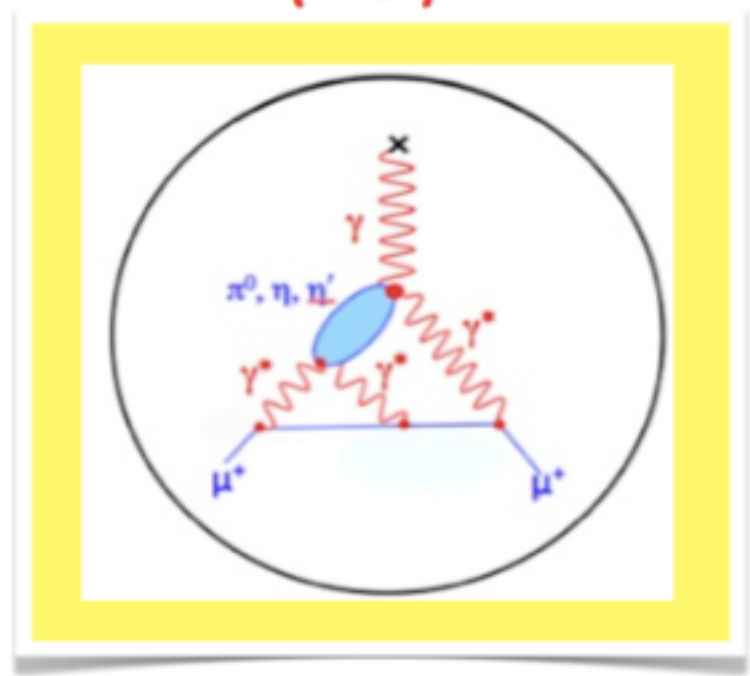


strong contributions to $(g-2)_\mu$

hadronic vacuum polarization (HVP)



hadronic light-by-light scattering (HLbL)



$a_\mu^{\text{l.o. had, VP}} = (692.3 \pm 4.2) \times 10^{-10}$

Teubner et al. (2011)

$a_\mu^{\text{had, LbL}} = (10.5 \pm 2.6) \times 10^{-10}$

$= (10.2 \pm 3.9) \times 10^{-10}$

Prades, de Rafael, Vainshtein (2009)

Jegerlehner, Nyffeler (2009)

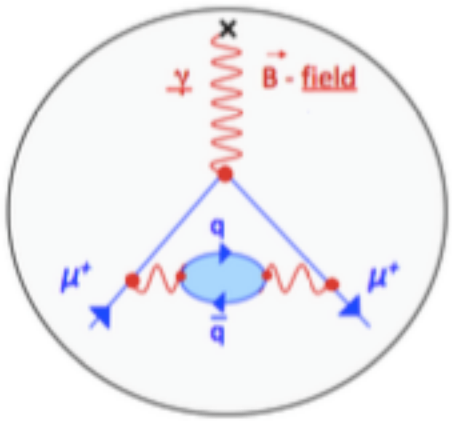
Jegerlehner (2015)

New FNAL and J-Parc $(g-2)_\mu$ expt. : $\delta a_\mu^{exp} = 1.6 \times 10^{-10}$

HVP determined by cross section measurements of e^+e^- -> hadrons

measurements of meson transition form factors required as input to reduce uncertainty

HVP corrections to $(g-2)_\mu$

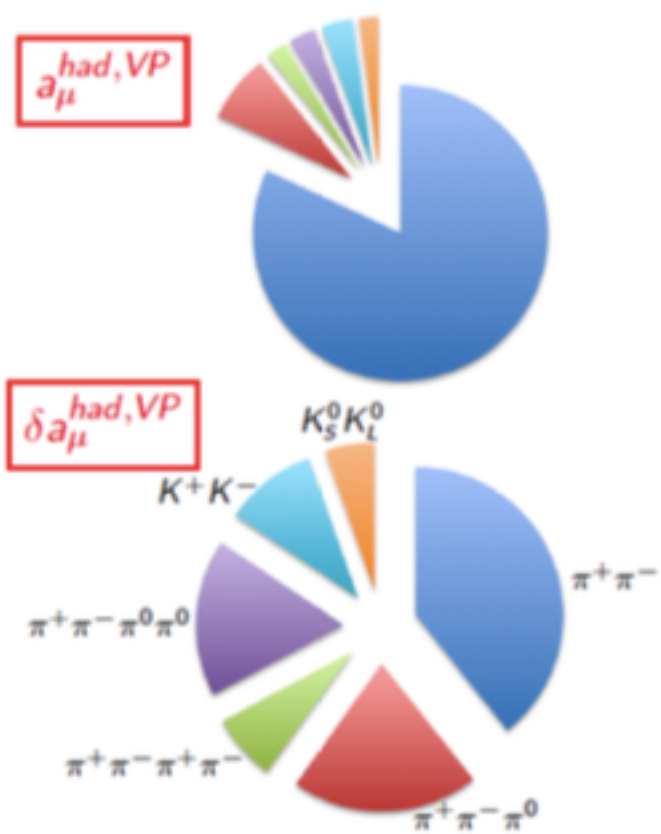


Optical theorem and analyticity allow to relate HVP contribution to $(g-2)_\mu$ with $\sigma_{had} = \sigma(e^+e^- \to \text{hadrons})$

$$a_\mu^{had,VP} = \frac{1}{4\pi^3} \int_{4m_\pi^2}^{\infty} ds\, K(s)\, \sigma_{had}$$

known Kernel function

Hadronic cross section



Future improvement of a_μ^{had} ?

1st priority:
Clarify situation regarding $\pi+\pi-$ (KLOE vs. BABAR puzzle)

2nd priority:
Measure 3π, 4π channels

Ongoing ISR analyses BESIII, BEPC-II collider

3rd priority:
KK and higher multiplicities

σ_{had}: Energy range up to 3 GeV essential !

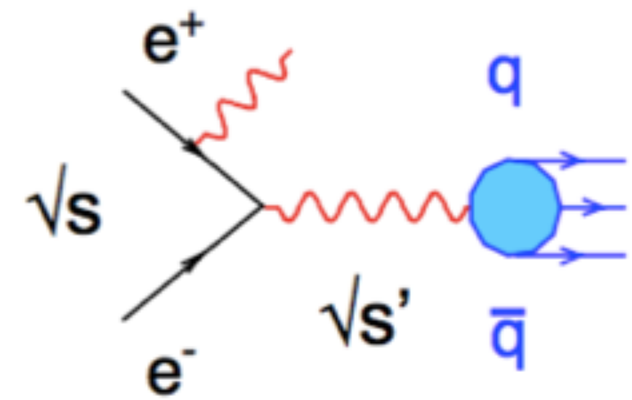
Y. Guo

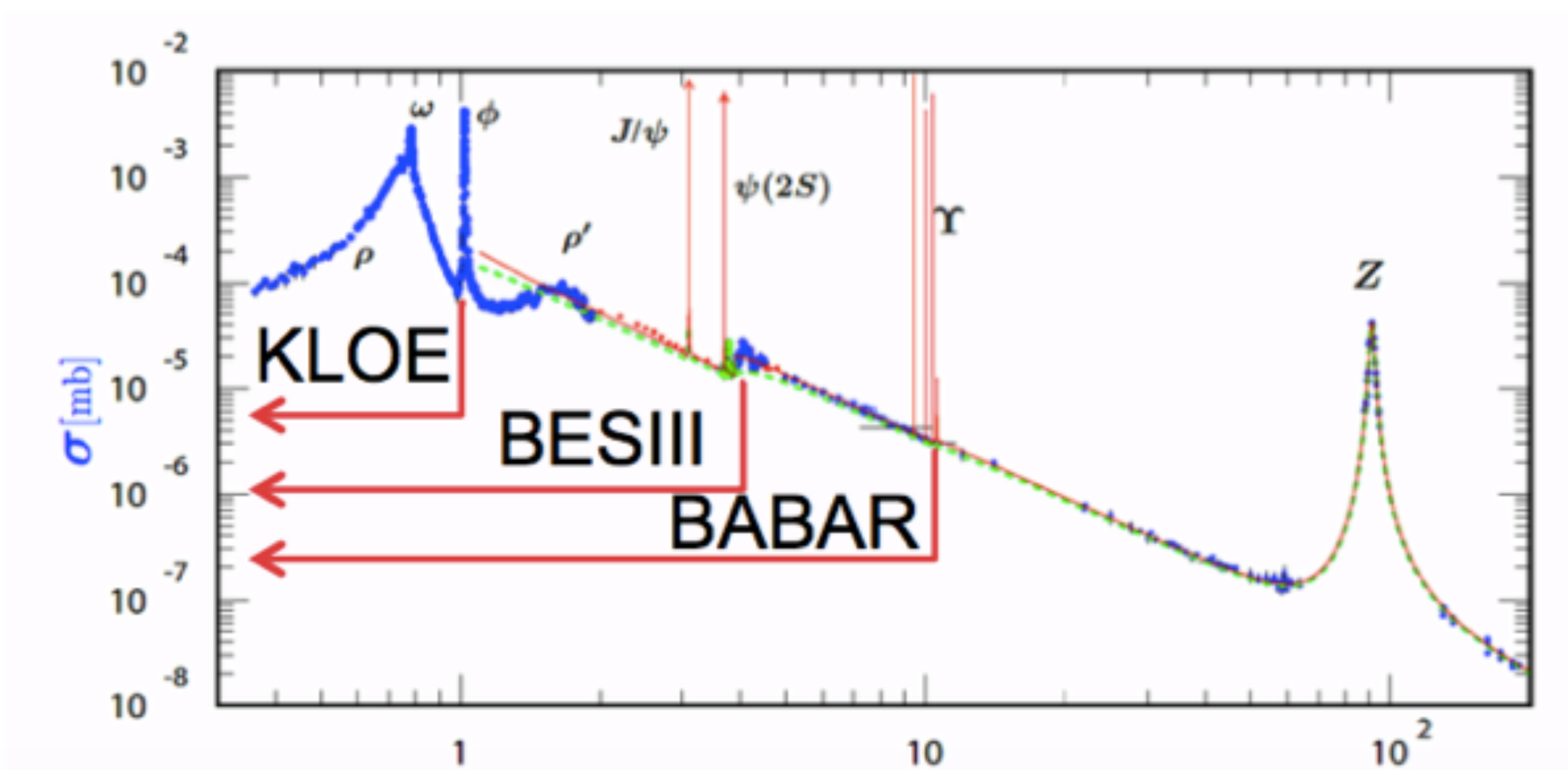
aim: reduction of current error by factor of 2

theory developments: - update $\pi\pi$ **I. Caprini**

- lattice QCD **A. Juettner, Ch. Davies**

- **Initial State Radiation:**
  - KLOE at DAφNE in Frascati
  - BABAR at PEP-II in Stanford
  - BESIII at BEPCII in Beijing

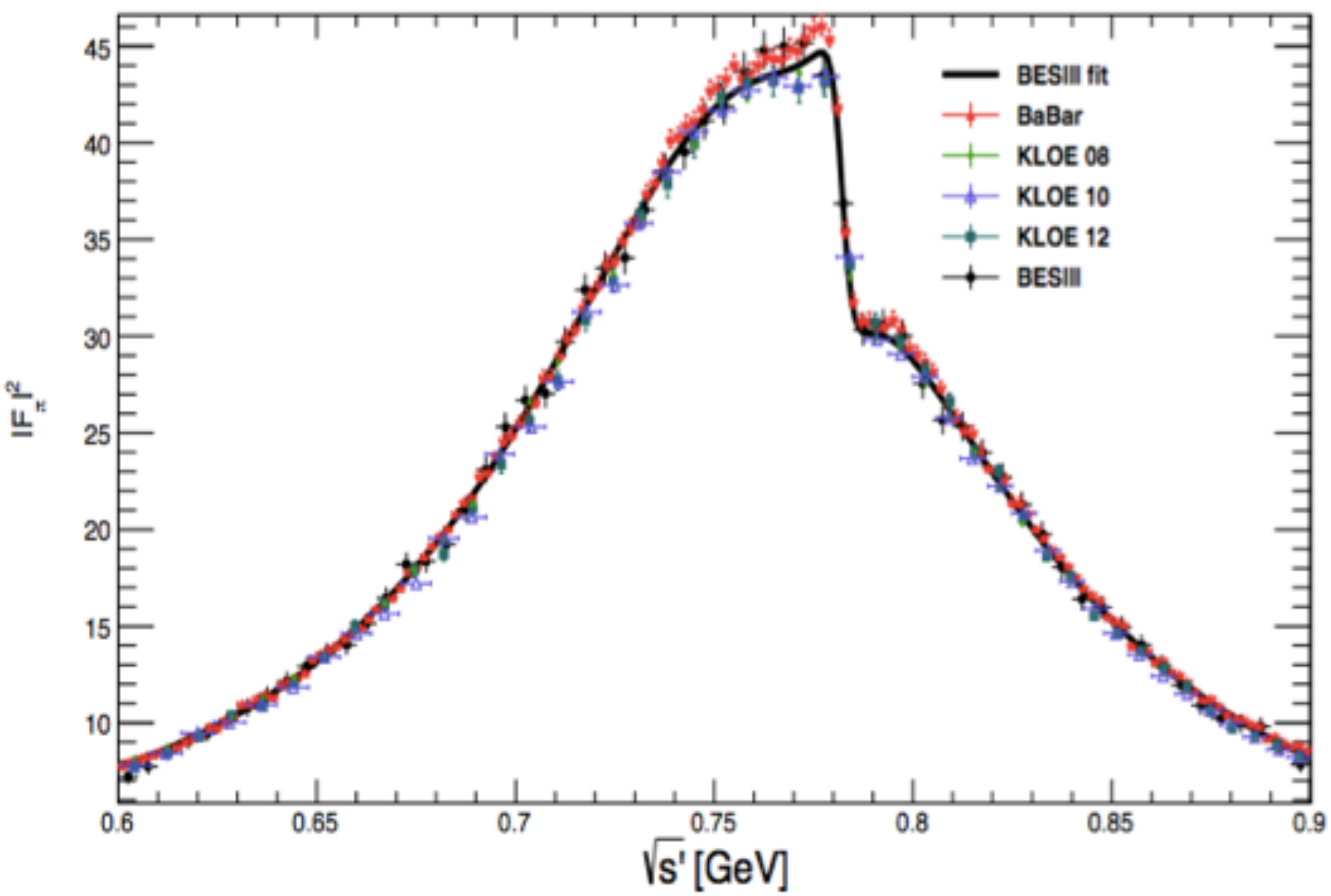


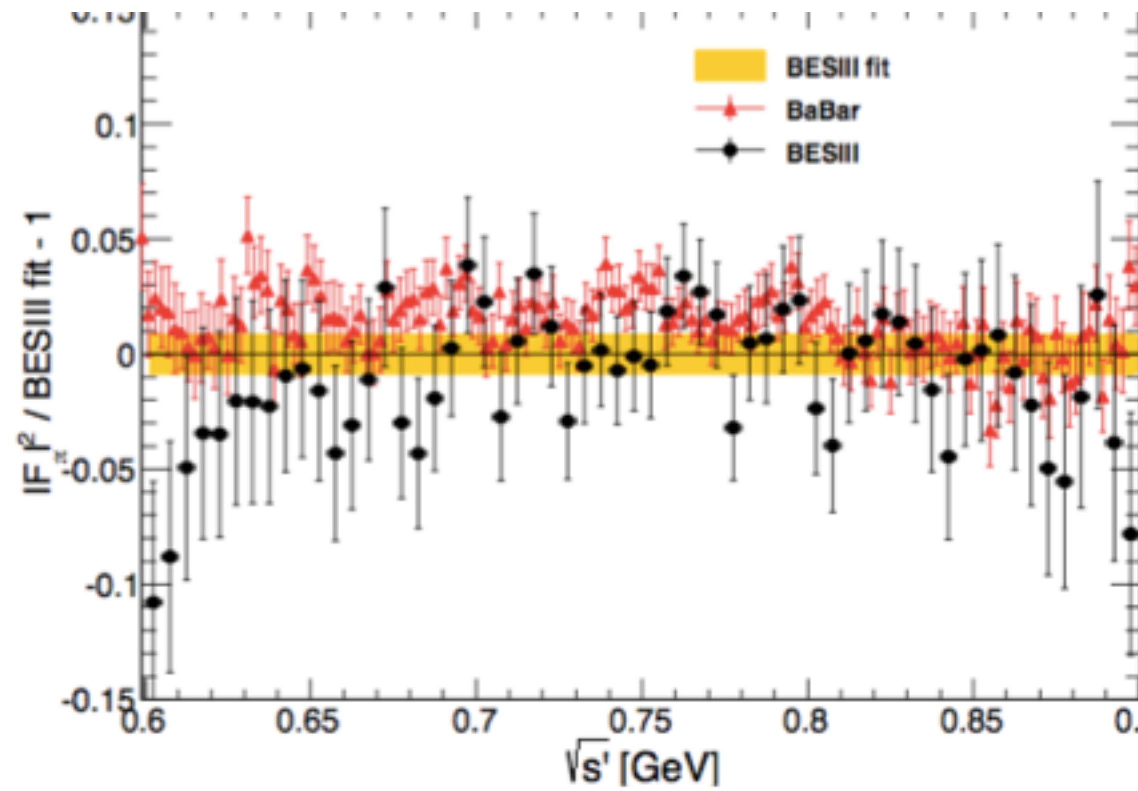


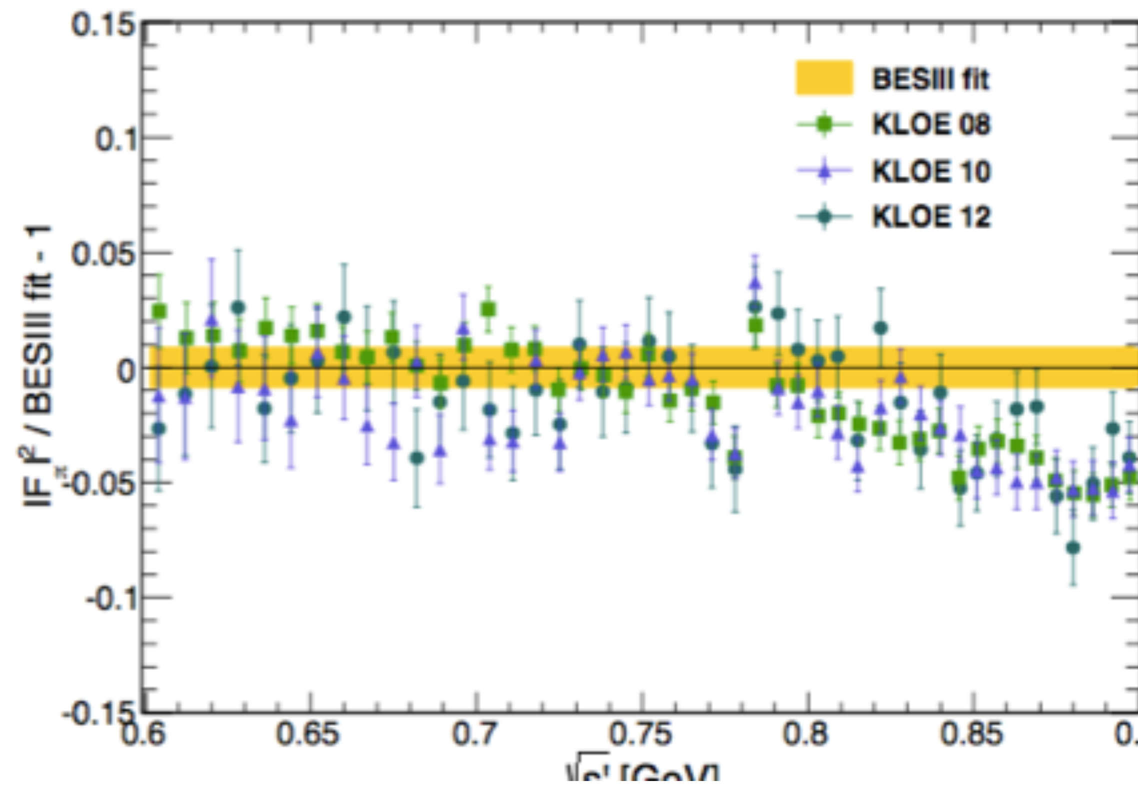
- Needs no systematic variation of beam energy
- High statistics thanks to hig integrated luminosities

Bess III data

Pion Form Factor F_π

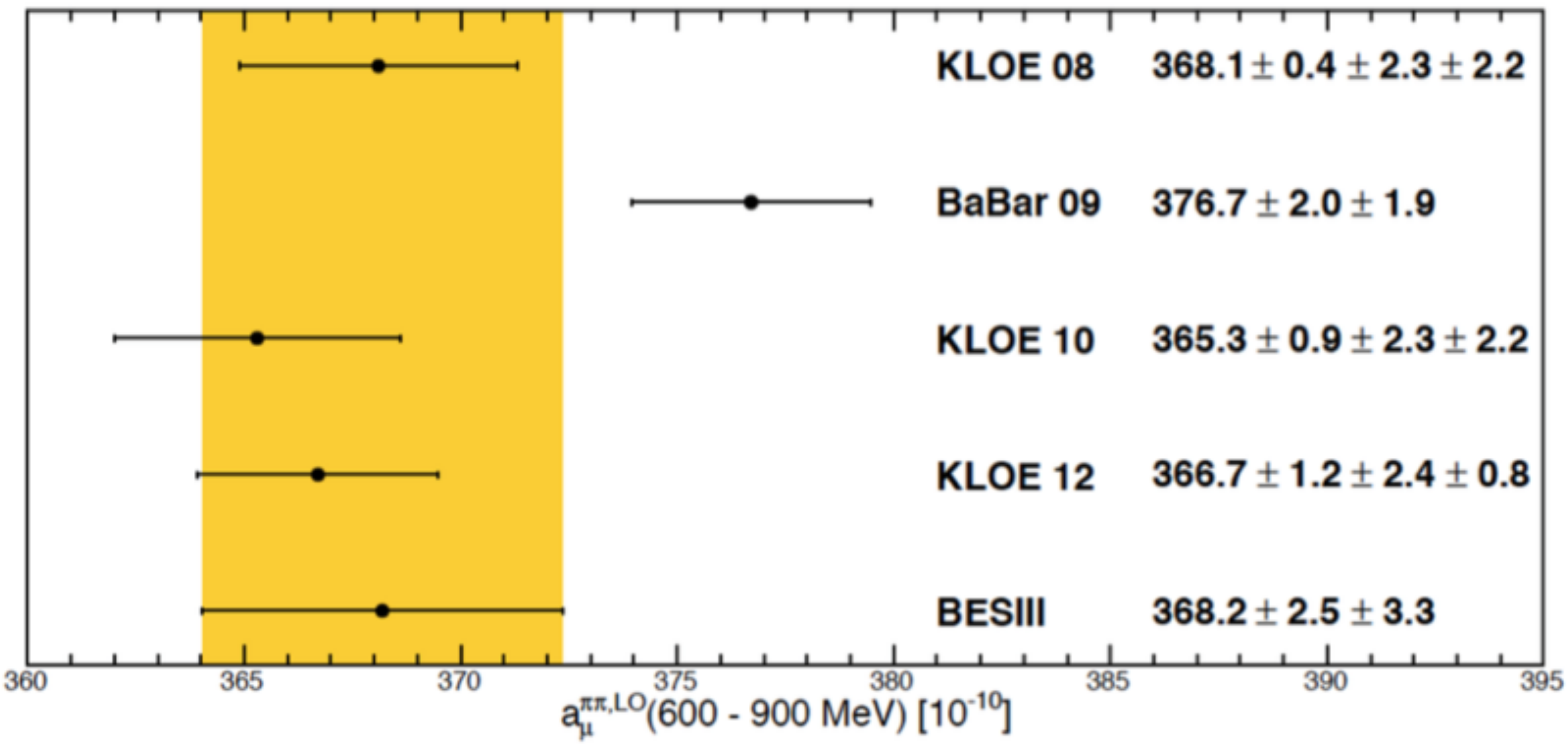






- Gounaris and Sakurai parameterization
- 0.9 % accuracy (dominated by theory)
- Normalization to luminosity × radiator function

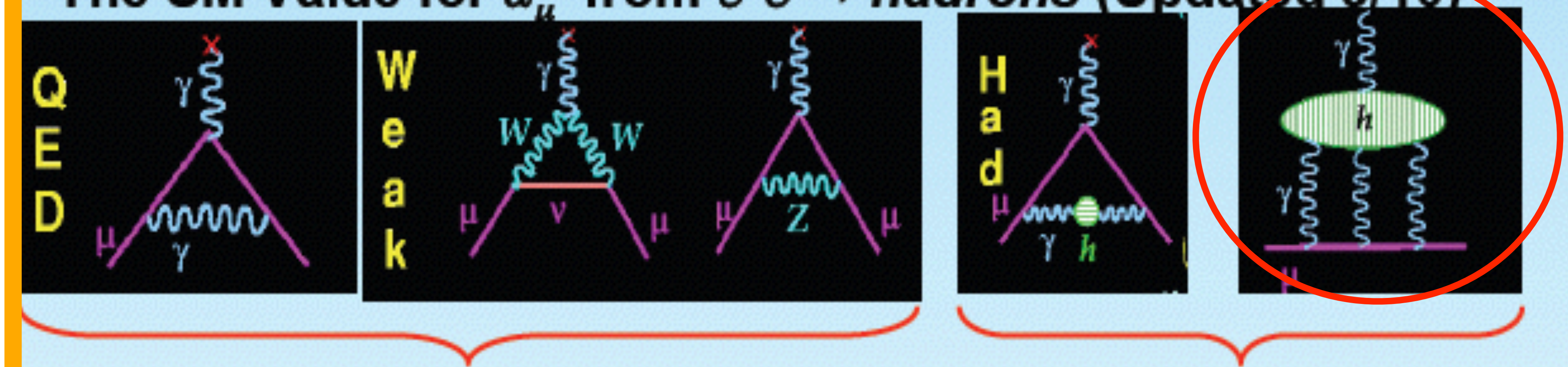
Impact on a_μ^{HVP}



Deviation on $(g-2)_\mu$ between experimental and SM: 3-4 sigma

Holographic QCD and Hadronic Light-by-Light Scattering Contribution to Muon g-2

The SM Value for a_μ from $e^+e^- \rightarrow$ *hadrons* (Updated 9/10)



well known

significant work ongoing

Contribution	Result ($\times 10^{-11}$) units
QED (leptons)	$116\ 584\ 718.09 \pm 0.14 \pm 0.04_\alpha$
HVP(lo)	$6\ 914 \pm 42_{\text{exp}} \pm 14_{\text{rad}} \pm 7_{\text{pQCD}}$
HVP(ho)	$-98 \pm 1_{\text{exp}} \pm 0.3_{\text{rad}}$
HLxL	105 ± 26
EW	$152 \pm 2 \pm 1$
Total SM	$116\ 591\ 793 \pm 51$

A. Höcker Tau 2010, U. Manchester September 2010 **116 592 089 +- 63**

EXP

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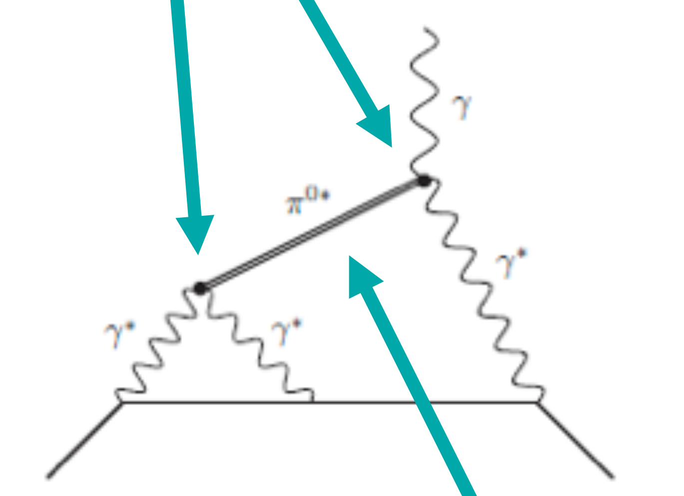
Pion exchange diagram dominates HLbL

Pion Form Factor

$F_{\pi^0\gamma^*\gamma^*}(Q_1^2, Q_2^2)$



Off-Shell Pion

HLbL scattering: Summary of selected results

Some results for the various contributions to $a_\mu^{\mathrm{HLbL}} \times 10^{11}$:

Contribution	BPP	HKS, HK	KN	MV	BP, MdRR	PdRV	N, JN
π^0, η, η'	85±13	82.7±6.4	83±12	114±10	–	114±13	99 ± 16
axial vectors	2.5±1.0	1.7±1.7	–	22±5	–	15±10	22±5
scalars	−6.8±2.0	–	–	–	–	−7±7	−7±2
π, K loops	−19±13	−4.5±8.1	–	–	–	−19±19	−19±13
π,K loops +subl. N_C	–	–	–	0±10	–	–	–
quark loops	21±3	9.7±11.1	–	–	–	2.3 (c-quark)	21±3
Total	83±32	89.6±15.4	80±40	136±25	110±40	105 ± 26	116 ± 39

BPP = Bijnens, Pallante, Prades '95, '96, '02; HKS = Hayakawa, Kinoshita, Sanda '95, '96; HK = Hayakawa, Kinoshita '98, '02; KN = Knecht, AN '02; MV = Melnikov, Vainshtein '04; BP = Bijnens, Prades '07; MdRR = Miller, de Rafael, Roberts '07; PdRV = Prades, de Rafael, Vainshtein '09; N = AN '09, JN = Jegerlehner, AN '09

Hadronic light-by-light: the really complicated thing

- Need some organizing principle: ChPT, large-N_c (turns out to be most relevant in practice) [E. de Rafael, Phys. Lett. B 322, 239 (1994)]

$$a_\mu^{\mathrm{HLxL}} = N_c \left(\frac{\alpha}{\pi}\right)^3 \frac{N_c}{F_\pi^2} \frac{m_\mu^2}{48\pi^2} \left[\ln^2 \frac{M_\rho}{M_\pi} + c_\chi \ln \frac{M_\rho}{M_\pi} + \kappa\right] + \mathcal{O}(N_c^0)$$

[M. Knecht, A. Nyffeler, Phys. Rev. D 65, 073034 (2002)]
[M. Knecht, A. Nyffeler, M. Perrottet, E. de Rafael, Phys. Rev. Lett. 88, 071802 (2002)]
M. J. Ramsey-Musolf, M. B. Wise, Phys. Rev. Lett. 89, 041601 (2002)]
[J. Prades, E. de Rafael, A. Vainshtein, Glasgow White Paper (2008)]

- Impose QCD short-distance properties [K. Melnikov, A. Vainshtein, Phys. Rev. D, 113006 (2004)]

- Only two (so far) attemps at a “complete”, but model-dependent calculation...

$$a_\mu^{\mathrm{HLxL}} = +(8.3 \pm 3.2) \cdot 10^{-10}$$

[J. Bijnens, E. Pallante, J. Prades, Phys. Rev. Lett. 75, 1447 (1995) [Err.-ibid. 75, 3781 (1995)]; Nucl. Phys. B 474, 379 (1995); Nucl. Phys. B 626, 410 (2002)]

$$a_\mu^{\mathrm{HLxL}} = +(89.6 \pm 15.4) \cdot 10^{-11}$$

[M. Hayakawa, T. Kinoshita, A. I. Sanda, Phys. Rev. Lett. 75, 790 (1995); Phys. Rev. D **54**, 3137 (1996)]
[M. Hayakawa, T. Kinoshita, Phys. Rev. D 57, 365 (1998) [Err.-ibid. 66, 019902(E) (2002)]

...after the sign change [M.K. and A. Nyffeler, Phys. Rev. D 65, 073034 (2002)]

Hadronic light-by-light: the really complicated thing

Recent (partial) reevaluations

$a_\mu^{\mathrm{HLxL}} = (10.5 \pm 2.6) \cdot 10^{-10}$ [J. Prades, E. de Rafael, A. Vainshtein, arXiv:0901.0306]

"best estimate"

$a_\mu^{\mathrm{HLxL}} = (11.5 \pm 4.0) \cdot 10^{-10}$ [A. Nyffeler, Phys. Rev. D 79, 073012 (2009)]

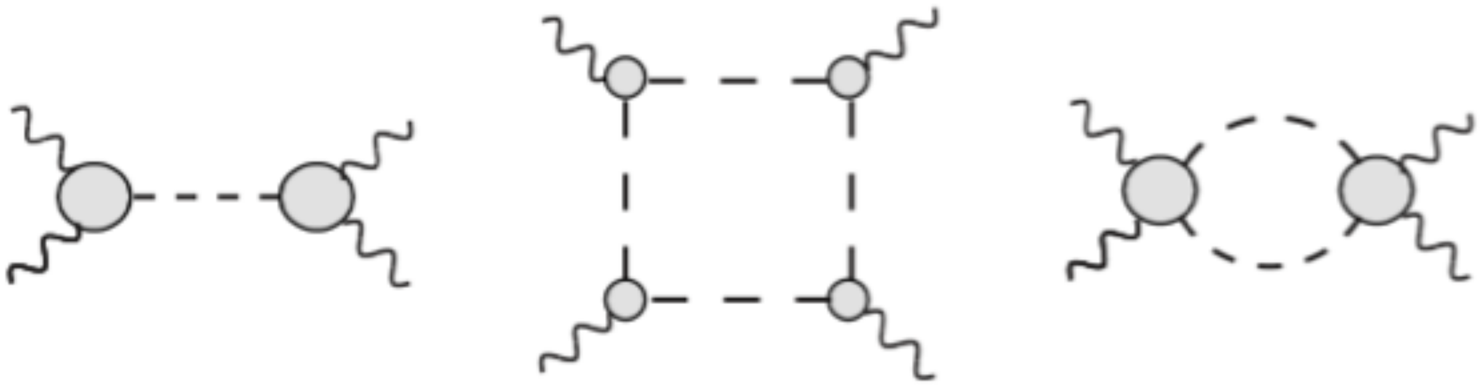
more conservative estimate

$a_e^{\mathrm{HLxL}} = (0.035 \pm 0.010) \cdot 10^{-12}$ [J. Prades, E. de Rafael, A. Vainshtein, in *Lepton Dipole Moments*]

Hadronic light-by-light: the really complicated thing

- More recently: dispersive approaches

- for $\Pi_{\mu\nu\rho\sigma}$



$$\Pi = \Pi^{\pi^0,\eta,\eta' \text{ poles}} + \Pi^{\pi^\pm,K^\pm \text{ loops}} + \Pi^{\pi\pi} + \Pi^{\text{residual}}$$

[G. Colangelo, M. Hoferichter, M. Procura, P. Stoffer, JHEP09, 091 (2014); arXiv:1506.01386 [hep-ph]]

Needs input from data (transition form factors,...)

[G. Colangelo, M. Hoferichter, B. Kubis, M. Procura, P. Stoffer, Phys. Lett. B 738, 6 (2014)]

[A. Nyffeler, arXiv:1602.03398 [hep-ph]]

Main unanswered issues:

- how will short-distance constraints be imposed?
- how will Π^{residual} be estimated? Cf. axial vectors (leading in large-N_c) $\to 3\pi$ channel

- for $F_2^{\text{HLxL}}(k^2)$

only pion pole with VMD form factor (two-loop graph) reconstructed this way so far

[V. Pauk and M. Vanderhaeghen, Phys. Rev. D 90, 113012 (2014) [arXiv:1409.0819 [hep-ph]]]

Leading Log and large Nc

M. J. Ramsey-Musolf and Mark B. Wise

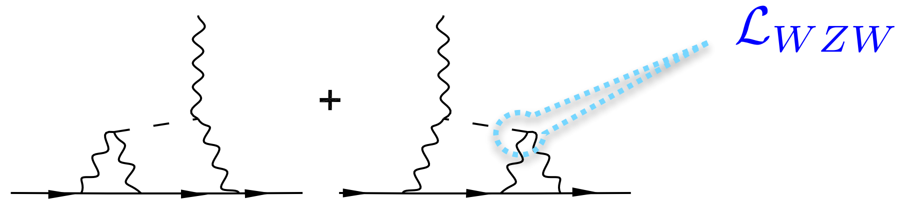
$$\mathcal{O}(\ N_c\ \alpha^3\ \frac{p^2}{\Lambda^2}\ )$$

$$a_\mu^{\rm LbyL} = \left(\frac{\alpha}{\pi}\right)^3 \left(\frac{N_c}{4\pi}\right)^2 \frac{1}{3}\left(\frac{m}{F_\pi}\right)^2 \left[\ln^2\left(\frac{M}{m}\right) + c_\chi \ln\left(\frac{M}{m}\right) + K\right] + \mathcal{O}(N_c^0)$$

Subleading terms from CHPT CT's: $\pi^0 \to e^+e^-$

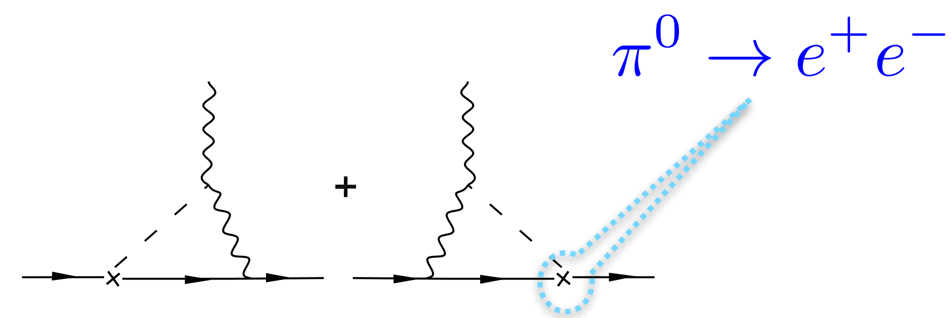
M. J. Ramsey-Musolf and Mark B. Wise PRL 2002

$$a_\mu^{\rm LbyL} = \left(\frac{\alpha}{\pi}\right)^3 \left(\frac{N_c}{4\pi}\right)^2 \frac{1}{3} \left(\frac{m}{F_\pi}\right)^2 \left[\ln^2\left(\frac{M}{m}\right) + c_\chi \ln\left(\frac{M}{m}\right) + K\right] + \mathcal{O}(N_c^0)$$



(a)

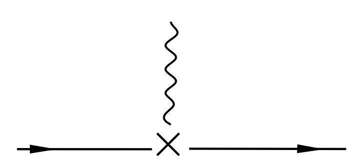
$$a_\mu = (57^{+50}{}_{-60} + 31\tilde{C}) \times 10^{-11}$$



(b)

$\mathcal{L}_{WZW}$

lattice?



$$\frac{3}{16}\left(\frac{\alpha}{\pi}\right)^3\left(\frac{m_\mu}{F_\pi}\right)^2\left(\frac{N_C}{3\pi}\right)^2\left\{\ln^2\left(\frac{\Lambda}{\mu}\right) + \left[-f(r) + \frac{1}{2} + \frac{1}{6}\chi(\Lambda)\right]\ln\left(\frac{\Lambda}{\mu}\right) + \tilde{C}\right\},$$

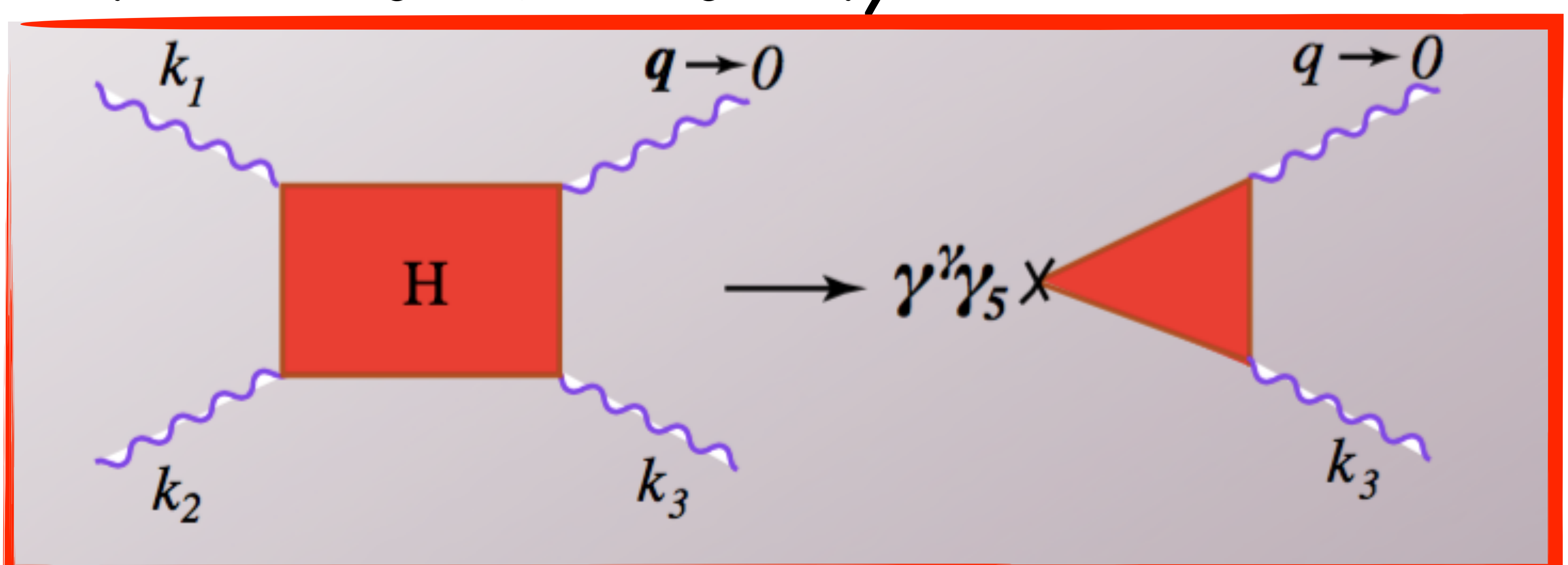
$\pi^0 \to e^+e^-$

Melnikov-Vainshtein Limit $q_1^2 \approx q_2^2 \gg q_3^2$

$$a_\mu^{\rm LbyL} = \left(\frac{\alpha}{\pi}\right)^3 \left(\frac{N_c}{4\pi}\right)^2 \frac{1}{3} \left(\frac{m}{F_\pi}\right)^2 \left[\ln^2\left(\frac{M}{m}\right) + \ldots\right]$$

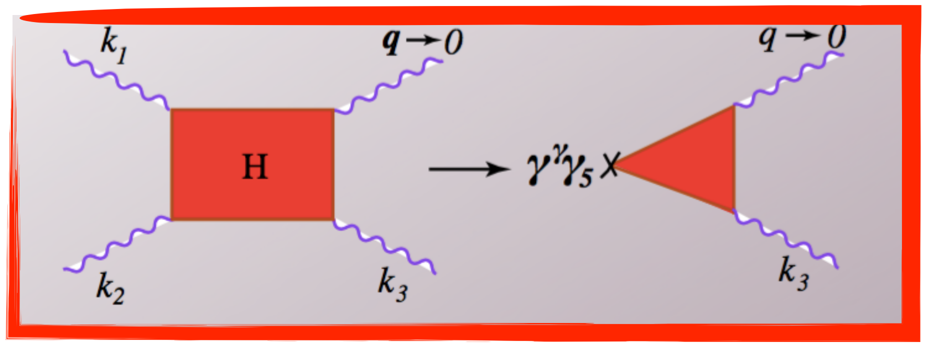
- Other Large N contributions
- Large N Short distance limit directly in the 4-point function

- In this limit it is possible to write an OPE relation linked to the anomaly term



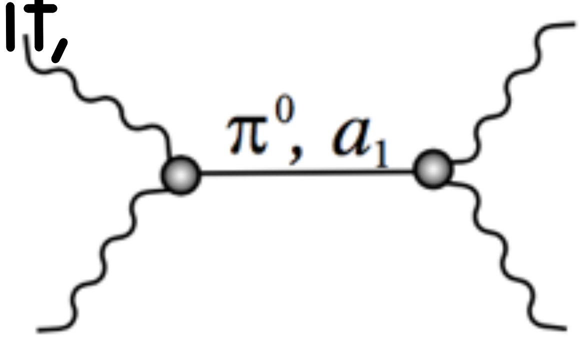
$q_1^2 \approx q_2^2 \gg q_3^2$

Melnikov-Vainshtein Limit



- Contribution to **two** helicity amplitudes: π^0 and a_1 exchange

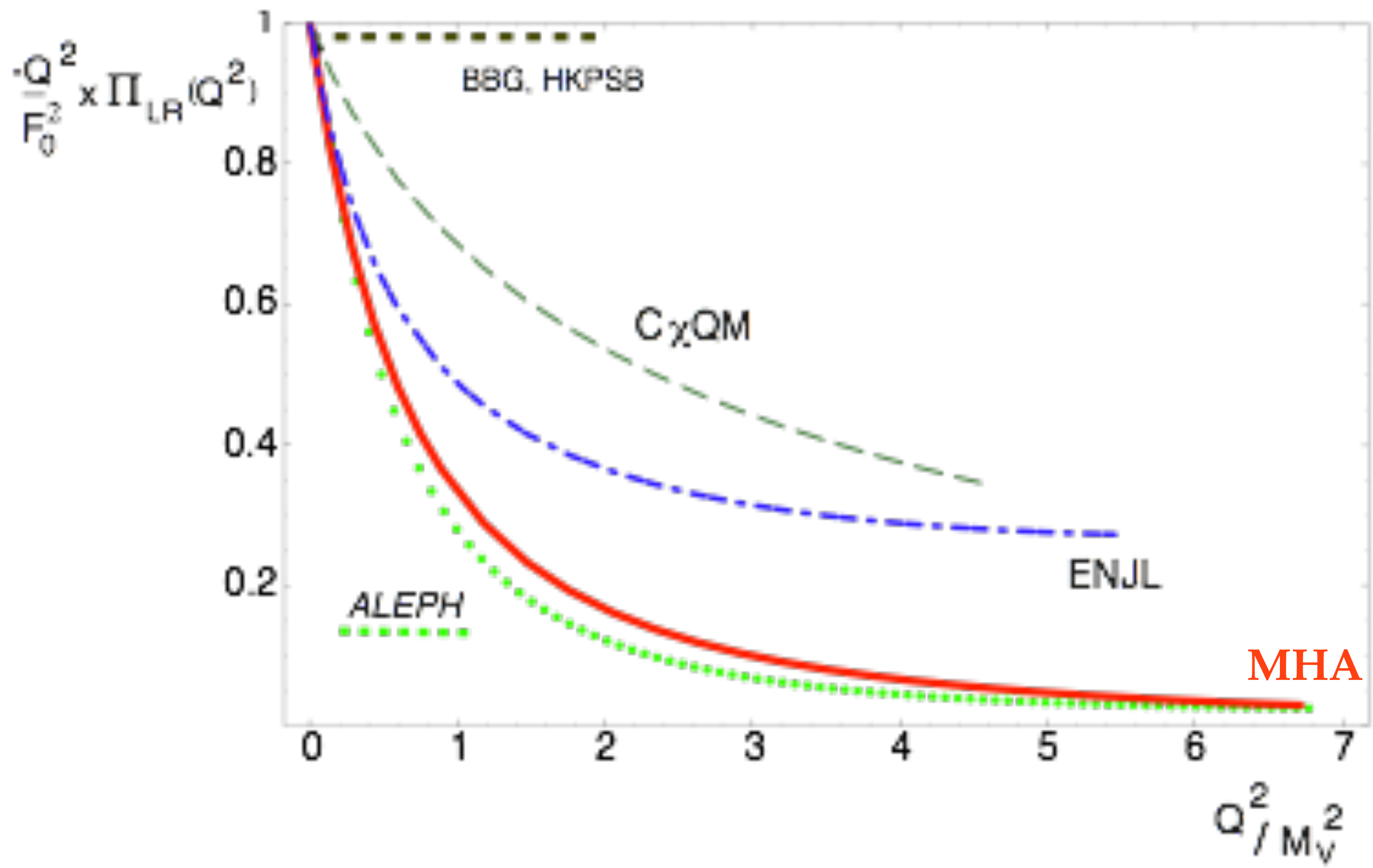
- Model to correctly reproduce this s.d. limit, numerically important



Contribution	BPP	HKS	KN	MV	BP	PdRV	N/JN
π^0, η, η'	85 ± 13	82.7 ± 6.4	83 ± 12	114 ± 10	–	114 ± 13	99 ± 16
π, K loops	-19 ± 13	-4.5 ± 8.1	–	–	–	-19 ± 19	-19 ± 13
π, K l. + subl. in Nc	–	–	–	0 ± 10	–	–	–
axial vectors	2.5 ± 1.0	1.7 ± 1.7	–	22 ± 5	–	15 ± 10	22 ± 5

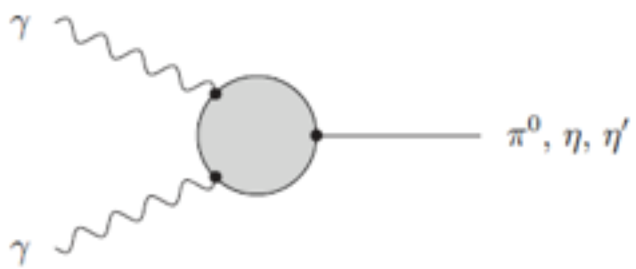
Minimal Hadronic Ansatz vs holographic models

- VMD vs holography: complementary



De Rafael

Anomalous AdS/CFT three point function Cappiello Cata G.D.



Grigoryan and A.V. Radyushkin

- From CS

$$K(Q_1^2, Q_2^2) = -\int_0^{z_0} \mathcal{J}(Q_1, z)\mathcal{J}(Q_2, z)\, \partial_z \Psi(z)\, dz$$

$$\mathcal{J}(Q, z) = Qz\left[K_1(Qz) + I_1(Qz)\frac{K_0(Qz_0)}{I_0(Qz_0)}\right] .$$

- short distance naturally implemented

- low energy, various models discriminated:

acceptable phenom. linear slope measured

$$F_{\pi^0\gamma^*\gamma^*}(Q_1^2, Q_2^2) \simeq -\frac{N_C}{12\pi^2 f_\pi}\left[1 + \hat{\alpha}\,(Q_1^2 + Q_2^2) + \hat{\beta}\; Q_1^2 Q_2^2 + \hat{\gamma}\;(Q_1^4 + Q_2^4)\right]$$

Good models=>phenon. slopes fixed !

Pseudoscalar exchanges

Our result

Model for $\mathcal{F}_{P^{(*)}\gamma^*\gamma^*}$	$a_\mu(\pi^0) \times 10^{11}$	$a_\mu(\pi^0, \eta, \eta') \times 10^{11}$
modified ENJL (off-shell) [BPP]	59(9)	85(13)
VMD / HLS (off-shell) [HKS,HK]	57(4)	83(6)
LMD+V (on-shell, $h_2 = 0$) [KN]	58(10)	83(12)
LMD+V (on-shell, $h_2 = -10$ GeV^2) [KN]	63(10)	88(12)
LMD+V (on-shell, constant FF at ext. vertex) [MV]	77(7)	114(10)
nonlocal χQM (off-shell) [DB]	65(2)	—
LMD+V (off-shell) [N]	72(12)	99(16)
AdS/QCD (off-shell ?) [HoK]	69	107
AdS/QCD/DIP (off-shell) [CCD]	65.4(2.5)	—
DSE (off-shell) [FGW]	58(7)	84(13)
[PdRV]	—	114(13)
[JN]	72(12)	99(16)

BPP = Bijnens, Pallante, Prades '95, '96, '02 (ENJL = Extended Nambu-Jona-Lasinio model); HK(S) = Hayakawa, Kinoshita, Sanda '95, '96; Hayakawa, Kinoshita '98, '02 (HLS = Hidden Local Symmetry model); KN = Knecht, Nyffeler '02; MV = Melnikov, Vainshtein '04; DB = Dorokhov, Broniowski '08 (χQM = Chiral Quark Model); N = Nyffeler '09; HoK = Hong, Kim '09; CCD = Cappiello, Catà, D'Ambrosio '10 (used AdS/QCD to fix parameters in DIP (D'Ambrosio, Isidori, Portolés) ansatz); FGW = Fischer, Goecke, Williams '10, '11 (Dyson-Schwinger equation)

Reviews on LbyL: PdRV = Prades, de Rafael, Vainshtein '09; JN = Jegerlehner, Nyffeler '09

A. Nyffeler Seattle 2011

There are many competing models:
ENJL
(Chiral quark model)
Lowest Meson Dominance
Hidden Symmetry
Non-Local ChQM
Bethe-Salpeter
Holographic QCD
Lattice QCD

A theoretical effort should be done to make them talk to each other

Uncertainty can increase of 10-15 % due to poor knowledge of the parameter χ_0 which we used to encode the pion off-shellness by the high-Q^2 constraint

$$\lim_{Q^2\to\infty} F_{\pi^{0*}\gamma^*\gamma^*}(Q^2, Q^2, 0) = -\frac{f_\pi}{3}\chi_0 + \cdots$$

Notice that the low-Q^2 predictions for PFF of the holographic models could be tested at KLOE-2

$$\lim_{Q_1^2, Q_2^2\to 0} F_{\pi^0\gamma^*\gamma^*}(Q_1^2, Q_2^2) \simeq -\frac{N_C}{12\pi^2 f_\pi}\times \left[1 + \hat\alpha\,(Q_1^2+Q_2^2) + \hat\beta\, Q_1^2 Q_2^2 + \hat\gamma\,(Q_1^4+Q_2^4)\right]$$

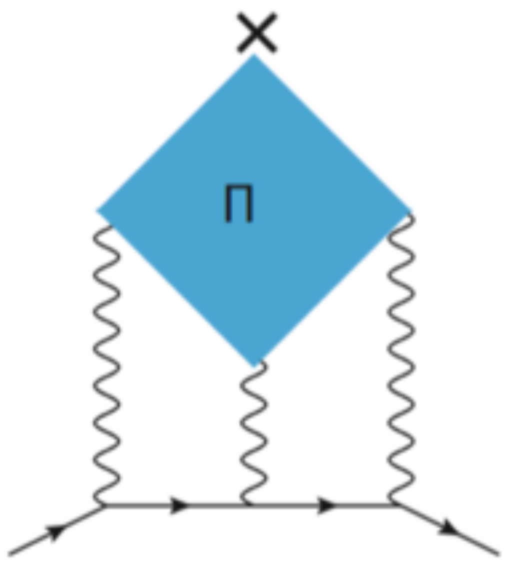
Exp.

$$\hat\alpha = -1.76(22)\ \mathrm{GeV}^{-2}$$

$$\hat\beta = 3.33(32)\ \mathrm{GeV}^{-4},\quad \hat\gamma = 2.84(21)\ \mathrm{GeV}^{-4}.$$

The Hadronic Light-by-Light contribution

Cappiello GD Greynat



$$= -ie^6 \int \frac{d^4 q_1}{(2\pi)^4} \int \frac{d^4 q_2}{(2\pi)^4} \frac{1}{q_1^2\, q_2^2\, (q_1 + q_2 - k)^2}$$

$$\times \frac{\bar{u}(p')\gamma^\mu(\not{p}' - \not{q}_1 + m)\gamma^\nu(\not{p}' - \not{q}_1 - \not{q}_2 + m)\gamma^\lambda u(p)}{[(p' - q_1)^2 - m^2]\,[(p' - q_1 - q_2)^2 - m^2]}$$

$$\times \frac{\partial}{\partial k^\rho} \Pi_{\mu\nu\lambda\sigma}(q_1, q_2, k - q_1 - q_2)$$

where

$$\Pi_{\mu\nu\lambda\rho}(q_1, q_2, q_3) = \int d^4x_1 \int d^4x_2 \int d^4x_3\, e^{i|\mathbf{q}\cdot\mathbf{x}|} \langle\, \Omega \mid \mathrm{T}\,\{j_\mu(x_1) j_\nu(x_2) j_\lambda(x_3) j_\rho(0)\} \mid \Omega\, \rangle$$

and

$$j_\rho(x) = \frac{2}{3} :(\bar{u}\gamma_\rho u)(x): \; -\frac{1}{3} :(\bar{d}\gamma_\rho d)(x): \; -\frac{1}{3} :(\bar{s}\gamma_\rho s)(x): \;=\; :(\bar{q} Q_{\bar{q}q} \gamma_\rho q)(x):$$

Mellin transform

Cappiello GD Greynat

$$a_\mu^{\mathrm{LbyL}} = \left(\frac{\alpha}{\pi}\right)^3 \left(\frac{N_c}{4\pi}\right)^2 \frac{1}{3} \left(\frac{m}{F_\pi}\right)^2 \left[\ln^2\left(\frac{M}{m}\right) + c_\chi \ln\left(\frac{M}{m}\right) + K\right] + \mathcal{O}(N_c^0)$$

$$a_\mu^{\mathrm{LbyL}} = \left(\frac{\alpha}{\pi}\right)^3 \left(\frac{N_c}{4\pi}\right)^2 \frac{1}{3} \left(\frac{m}{F_\pi}\right)^2 \int \frac{\underset{\cdot}{\mathrm{s}}}{2i\pi} \left(\frac{M}{m}\right)^{-s} \widehat{\mathcal{A}}(s)$$

$$\widehat{\mathcal{A}}(s) \asymp -\frac{1}{2}\,\frac{1}{s^3} + c_\chi\,\frac{1}{s^2} - K\frac{1}{s} + \mathcal{O}(N_c^0)$$

DIP form factor to warm up

D. Isidori
Portoles

Cappiello GD Greynat

$$F_{\pi\gamma^*\gamma^*}(Q_1^2,Q_2^2) = 1 + \lambda\left[\frac{Q_1^2}{Q_1^2+M^2} + \frac{Q_2^2}{Q_2^2+M^2}\right] + \eta\,\frac{Q_1^2 Q_2^2}{(Q_1^2+M^2)(Q_2^2+M^2)}$$

$$F_{\pi\gamma^*\gamma^*}(\lambda Q_1^2,\lambda Q_2^2) \underset{\lambda\to 0}{\sim} = 1 + \hat{\alpha}(Q_1^2+Q_2^2) + \hat{\beta} Q_1^2\, Q_2^2$$

$$a_\mu^{\rm LbyL} = \left(\frac{\alpha}{\pi}\right)^3 \left(\frac{N_c}{4\pi}\right)^2 \frac{1}{3}\left(\frac{m}{F_\pi}\right)^2 \left[\ln^2\left(\frac{M}{m}\right) + c_\chi \ln\left(\frac{M}{m}\right) + K\right] + \mathcal{O}(N_c^0)$$

$c_\chi = 9.5$ Cappiello GD Greynat

$c_\chi = 19$

M. J. Ramsey-Musolf and Mark B. Wise PRL 200

Work in progress

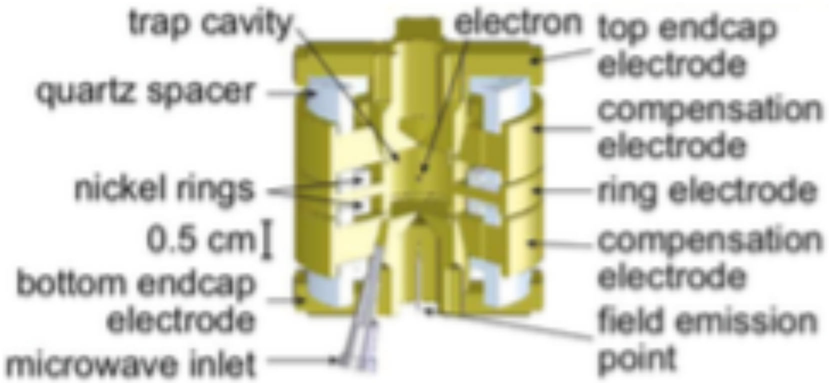
Cappiello GD Greynat

- MV limit in holographic model (form factor)
- duality low energy slopes and MV limit (the subheading coefficient)
- Compute the constant term

Outlook

- Beautiful and precise experiment require theoretical work

a_e and a_μ are experimentally measured to very high precision:



$$a_e^{\text{exp}} = 1\,159\,652\,180.73(0.28) \cdot 10^{-12}$$

$$\Delta a_e^{\text{exp}} = 2.8 \cdot 10^{-13} \text{ [0.24ppb]}$$

D. Hanneke et al, Phys. Rev. Lett. 100, 120801 (2008)

$$\tau_\mu = (2.19703 \pm 0.00004) \times 10^{-6} \text{ s}$$

$$\gamma \sim 29.3,\ p \sim 3.094 \text{ GeV/c}$$

$$a_\mu^{\text{exp}} = 116\,592\,089(63) \cdot 10^{-11}$$

$$\Delta a_\mu^{\text{exp}} = 6.3 \cdot 10^{-10} \text{ [0.54ppm]}$$

G. W. Bennett et al, Phys Rev D 73, 072003 (2006)

Note: $\tau_\tau = (290.6 \pm 1.1) \times 10^{-15}\,\text{s}$

$-0.052 < a_\tau^{exp} < +0.013$ (95% CL) $[e^+e^- \to e^+e^-\tau^+\tau^-]$ DELPHI, Eur. Phys. J. C 35, 159 (2004)

theory: $a_\tau = 117721(5) \cdot 10^{-8}$

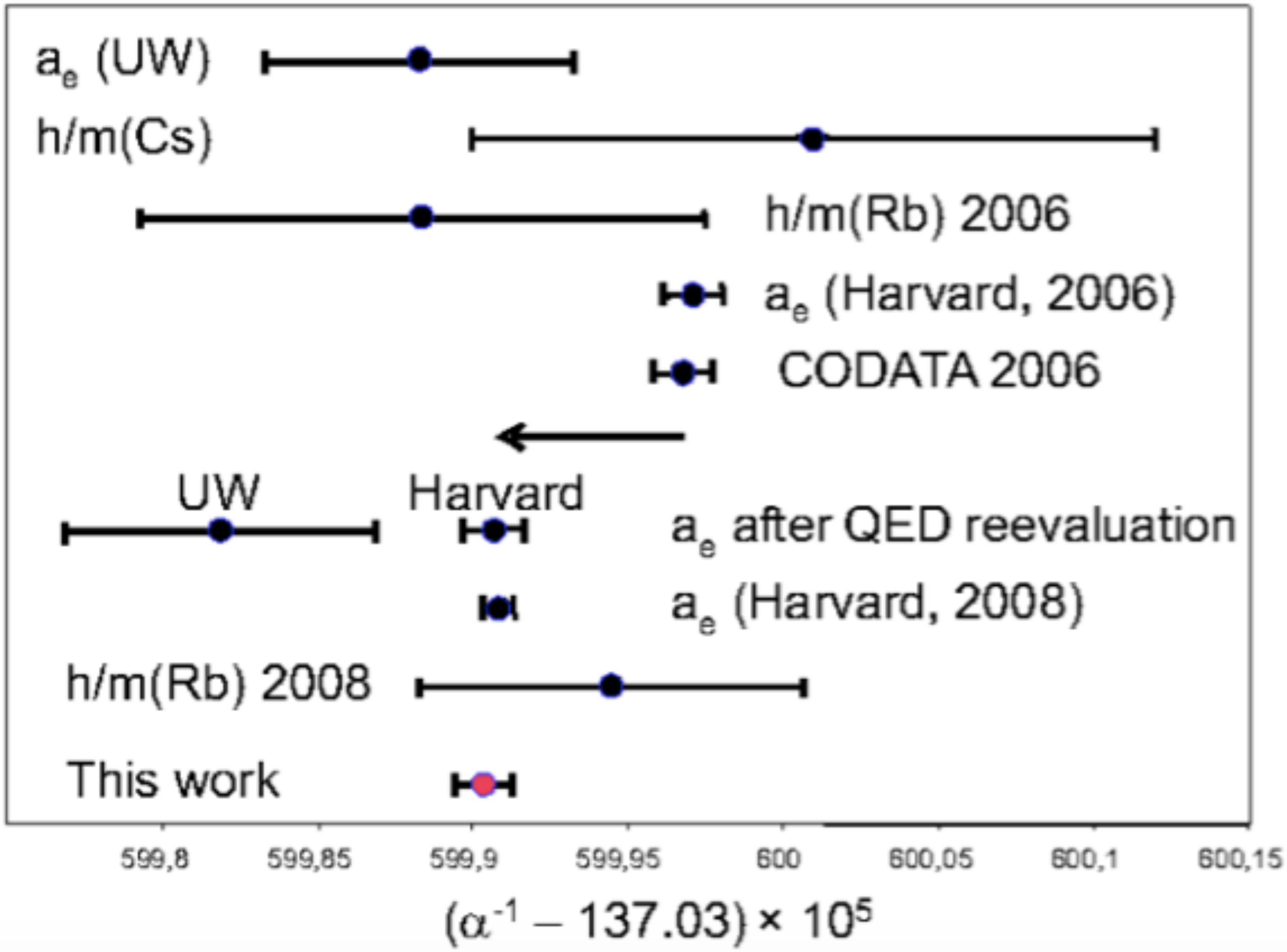
S. Eidelman, M. Passera, Mod. Phys. Lett. A 22, 159 (2007)

S. Narison, Phys Lett B 513 (2001); err. B 526, 414 (2002)

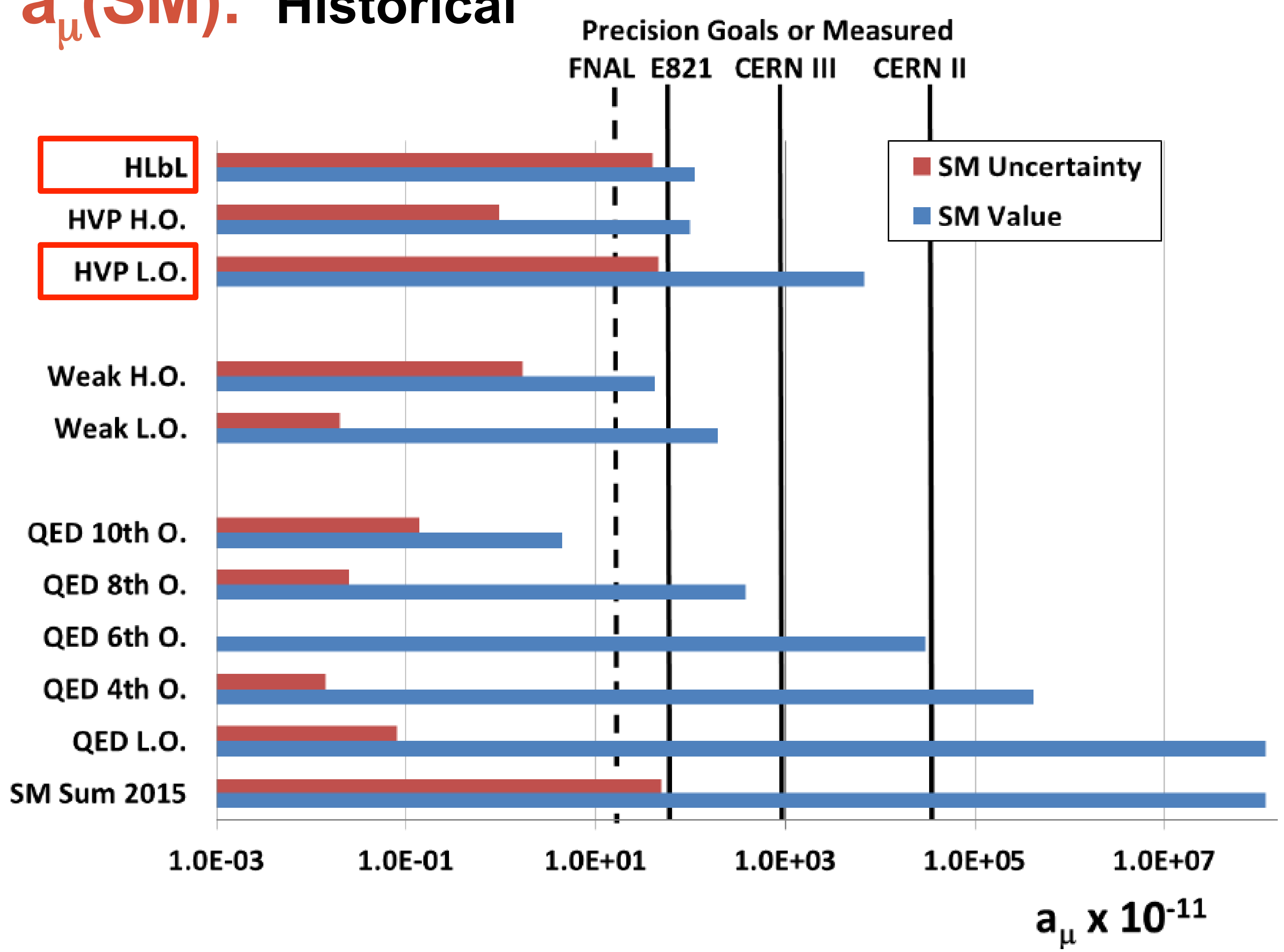
$$a_e^{\mathrm{QED}} = 1\,159\,652\,180.07(6)_{\alpha^4}(4)_{\alpha^5}(77)_{\alpha(Rb11)} \cdot 10^{-12} \qquad a_e^{\mathrm{exp}} - a_e^{\mathrm{QED}} = 0.67(82) \cdot 10^{-12}$$

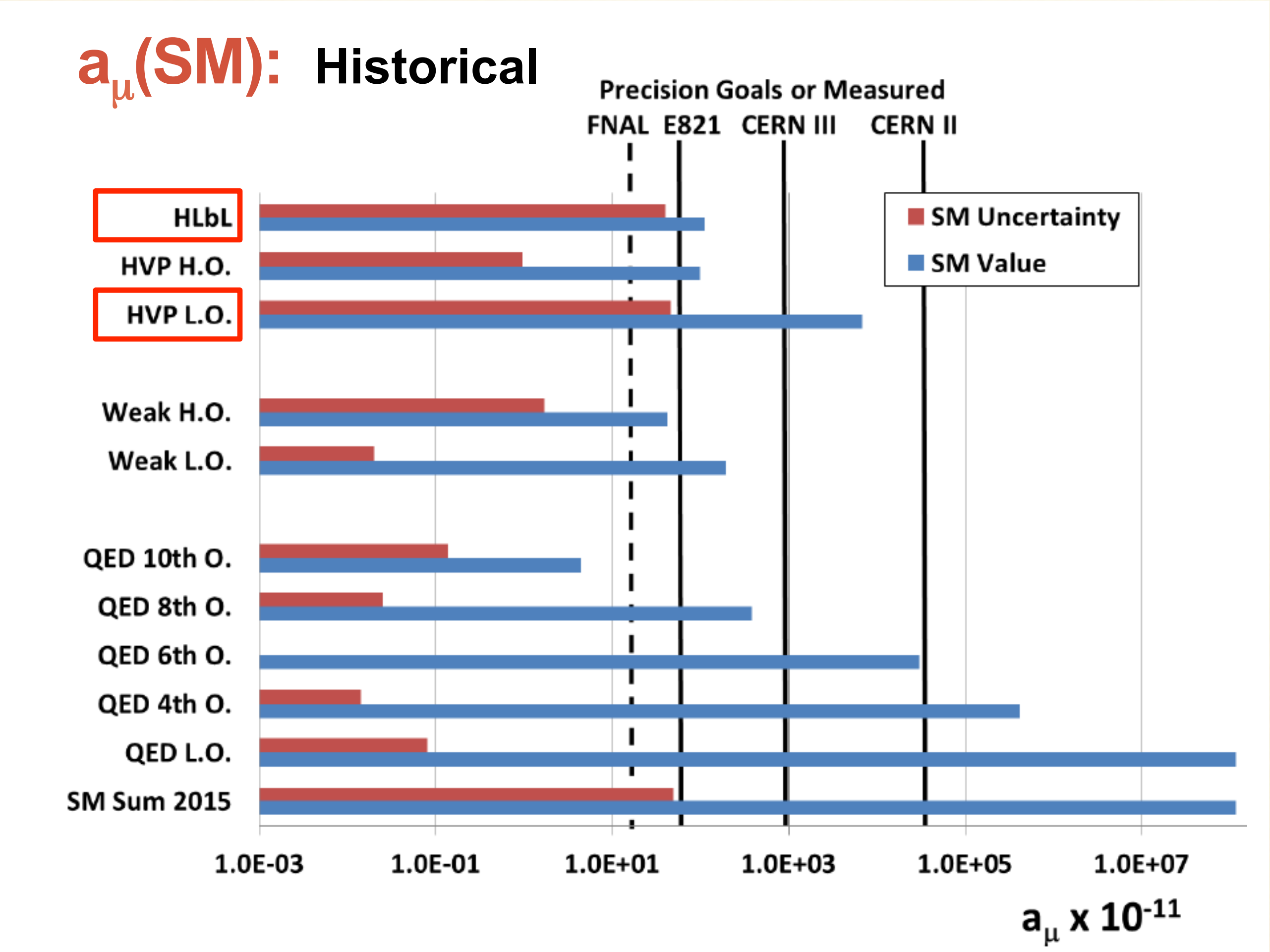
$$\alpha[a_e(HV\,08)] = 137.035\,999\,172\,2(68)_{\alpha^4}(46)_{\alpha^5}(19)_{\mathrm{had}}(331)_{\mathrm{exp}} \qquad [0.25\mathrm{ppb}]$$

Aoyama et al., Phys. Rev. Lett. 109, 111807 (2012)



a_μ(SM): Historical





aμ(SM): Historical
Precision Goals or Measured
FNAL E821 CERN III CERN II
SM Uncertainty
SM Value
HLbL
HVP H.O.
HVP L.O.
Weak H.O.
Weak L.O.
QED 10th O.
QED 8th O.
QED 6th O.
QED 4th O.
QED L.O.
SM Sum 2015
1.0E-03
1.0E-01
1.0E+01
1.0E+03
1.0E+05
1.0E+07
aμ x 10-11

Precision physics → Solid Theory (QED)

$$\mathcal{L}_{QED} = \bar{\psi}(i\gamma_\mu D^\mu - m)\psi - \frac{F^2}{4}$$

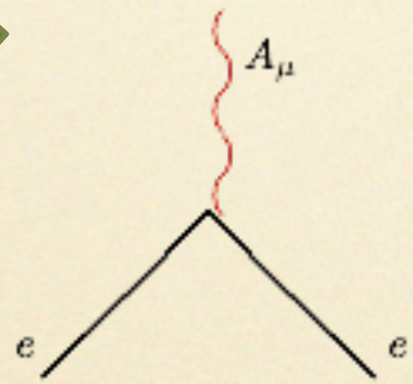
Lorentz + gauge inv.

$$D^\mu\psi \equiv \partial_\mu\psi + i\, e\, A_\mu\psi$$

gauge inv

$$\psi \to e^{i\alpha(x)}\psi$$
$$A_\mu \to A_\mu + \partial_\mu\alpha(x)$$
$$D^\mu\psi \to e^{i\alpha(x)}D^\mu\psi$$

Two crucial precise measurements

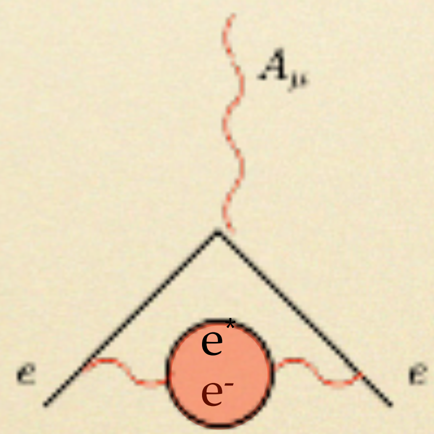


Lamb shifts (hydrogen level splittings)

$g_e\, \mu_B\, H$ 1.00118(3)

e.m. corrections needed ! $\alpha = \frac{e^2}{4\pi}$

Quantum field theory calculation⇒ virtual particles ($e^+ e^-$ pairs), inf.degrees of freedom, div. theory?



$\mathcal{L}_{QED}$

sufficient to describe all QED processes,

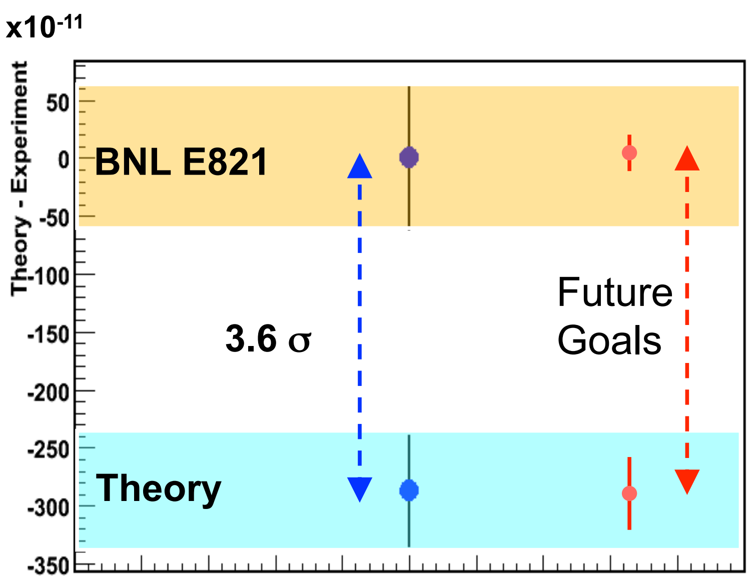
no need to add higher, gauge inv., dimensional terms ($\frac{(\bar{\psi}\psi)^2}{\Lambda^2}$)

renormalizable ⇒ finite # terms, they must have dimensions ≤ 4

Gauge inv. → VERY Predictive theory

Observables! Parameters (m, e) in $\mathcal{L}_{QED}$ can be divergent. Phys. obs. NO!

Graphically: Present situation and Goals



Goal: 140 ppb

>7.5 σ if same central values

Expected Improvement

$$a_\mu^{exp} - a_\mu^{SM} = (261 - 287 \pm 80)^{*} \times 10^{-11}$$

➡ 3.3 to 3.6 σ

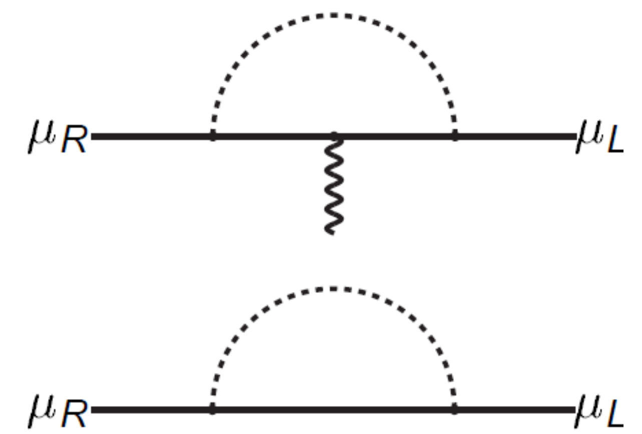
What is nature trying to tell us?

*range of typical SM evaluations

g-2 feature: **Chirality flipping interactions for mass and charge (moment) terms**

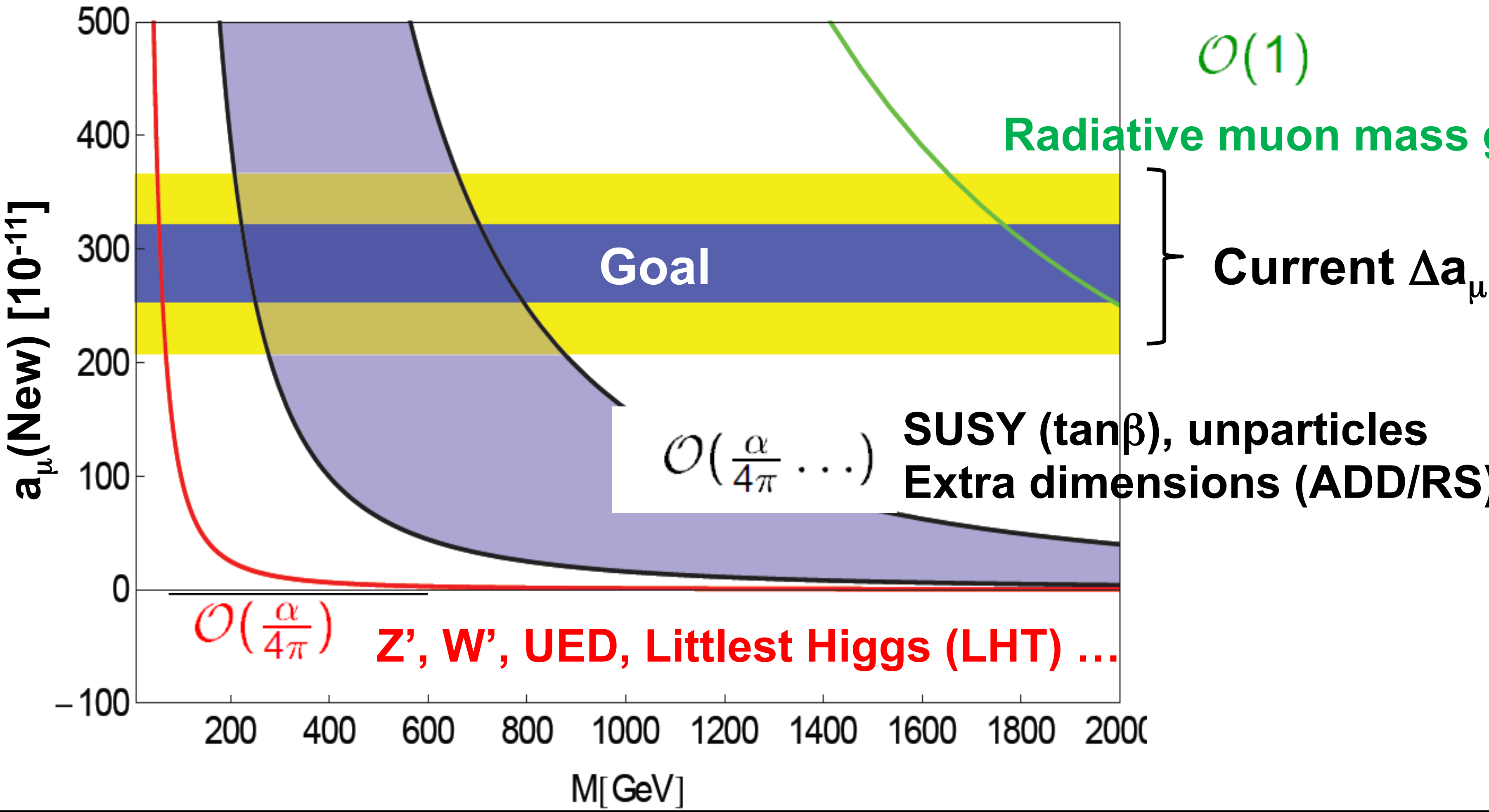
g – 2

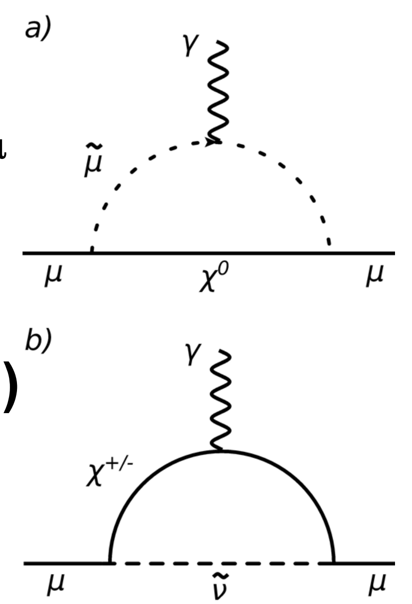
m_μ



$$C = \frac{\delta m_\mu(\mathrm{N.P.})}{m_\mu}, \quad \delta a_\mu(\mathrm{N.P.}) = \mathcal{O}(C)\left(\frac{m_\mu}{M}\right)^2$$

The coupling *C* is VERY model dependent





From D. Stockinger (See many of this g-2 presentations about new physics impact)

The dominant contribution

Decomposing the fields as

$$\int d^4x\, e^{-iqx}\, \mathbb{V}^a_\mu(x,z) = f_V(q,z)\, \mathcal{L}_{\mu\nu}\, v^{a\,\nu}(q)$$

$$\int d^4x\, e^{-iqx}\, \mathbb{A}^a_\mu(x,z) = f_A(q,z)\, \mathcal{L}_{\mu\nu}\, a^{a\,\nu}(q) + \phi(q,z)\, \mathcal{T}_{\mu\nu}\, p^{a\,\nu}(q)\,,$$

for the longitudinal $\mathcal{L}_{\mu\nu} = \eta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}$ and transverse $\mathcal{T}_{\mu\nu} = \frac{q_\mu q_\nu}{q^2}$ parts.
The first contribution is given by

$$\Pi^{\mathbf{a}}_{\mu\nu\lambda\rho}(q_1,q_2,q_3,z) = \frac{\delta^4 S^2_{CS}}{\delta v^{a_1\,\mu}(q_1)\, \delta v^{a_2\,\nu}(q_2) \delta v^{a_3\,\lambda}(q_3) \delta v^{a_4\,\rho}(q_4)}$$

$$= W^{\mathbf{a}}\left((q_1+q_2)^2\right) \varepsilon_{\mu\nu\alpha\beta}\varepsilon_{\lambda\rho\sigma\tau} q_1^\alpha q_2^\beta q_3^\sigma (q_1+q_2)^\tau$$
$$+ W^{\mathbf{a}}\left((q_2+q_3)^2\right) \varepsilon_{\mu\rho\alpha\beta}\varepsilon_{\nu\lambda\sigma\tau}\, q_2^\sigma q_3^\tau q_1^\alpha (q_2+q_3)^\beta$$
$$+ W^{\mathbf{a}}\left((q_1+q_3)^2\right) \varepsilon_{\mu\lambda\alpha\beta}\varepsilon_{\nu\rho\sigma\tau} q_1^\alpha q_3^\beta q_2^\sigma (q_1+q_3)^\tau\,,$$

One obtains (a general formula for Holographic QCD models)

$$W^{\mathbf{a}}(k^2) = i\,\frac{4}{k^2}\left(\frac{N_c}{4\pi^2}\right)^2 \mathrm{Tr}\left[T^{a_1}T^{a_2}T^{a_3}T^{a_4}\right] \times \int \mathrm{d}z\,\mathrm{d}z' \left[f_V(q_1,z)f_V'(q_2,z) - f_V'(q_1,z)f_V(q_2,z)\right] \times \left[f_V(q_3,z')f_V'(q_4,z') - f_V'(q_3,z')f_V(q_4,z')\right] \times \left[G_{A,L}(k^2;z,z') + G_{A,T}(k^2;z,z')\right]$$

where we have in our peculiar model,

$$f_V(Q^2,z) = \frac{Q^2}{4\kappa^2}\int_0^1 \mathrm{d}u\, u^{\frac{Q^2}{4\kappa^2}-1}\exp\left[-\frac{u}{1-u}\kappa^2 z^2\right]$$

$$G_{A,T}(Q^2;x,y) = \frac{F_\pi^2}{2}xy\delta(x-y) + \Gamma\left(1+\frac{Q^2}{4\kappa^2}\right)\mathrm{U}\left(1,\frac{Q^2}{4\kappa^2};\kappa^2(x-y)^2\right)$$

$$G_{A,L}(Q^2;x,y) = -\frac{xy}{2}\int_0^1 \mathrm{d}t\,\frac{t^{\frac{Q^2}{4\kappa^2}+\frac{1}{2}}}{1-t}\exp\left[-\frac{t}{1-t}\kappa^2(x^2+y^2)\right]\mathrm{I}_1\left(2\kappa^2 xy\frac{\sqrt{t}}{1-t}\right).$$

Therefore the contribution to the anomaly is given by

$$a_\mu^{\mathrm{LbyL}} = -e^6 \int \frac{d^4 q_1}{(2\pi)^4} \int \frac{d^4 q_2}{(2\pi)^4} \frac{[W_1 \; T_1(q_1, q_2; p) + W_2 \; T_2(q_1, q_2; p)]}{q_1^2 q_2^2 (q_1 + q_2)^2 [(p + q_1)^2 - m^2][(p - q_2)^2 - m^2]}$$

where

$$\begin{aligned} T_1(q_1, q_2; p) = & \frac{16}{3} (p \cdot q_1)(p \cdot q_2)(q_1 \cdot q_2) - \frac{16}{3} (p \cdot q_2)^2 q_1^2 \\ & - \frac{8}{3} (p \cdot q_1)(q_1 \cdot q_2) q_2^2 + 8 (p \cdot q_2) q_1^2 q_2^2 - \frac{16}{3} (p \cdot q_2)(q_1 \cdot q_2)^2 \\ & + \frac{16}{3} m^2 q_1^2 q_2^2 - \frac{16}{3} m^2 (q_1 \cdot q_2)^2 , \\ T_2(q_1, q_2; p) = & \frac{16}{3} (p \cdot q_1)(p \cdot q_2)(q_1 \cdot q_2) - \frac{16}{3} (p \cdot q_1)^2 q_2^2 \\ & + \frac{8}{3} (p \cdot q_1)(q_1 \cdot q_2) q_2^2 + \frac{8}{3} (p \cdot q_1) q_1^2 q_2^2 \\ & + \frac{8}{3} m^2 q_1^2 q_2^2 - \frac{8}{3} m^2 (q_1 \cdot q_2)^2 . \end{aligned}$$

For $q_4 = 0$ and $q_3 = q_1 + q_2$

$$W_1 = \frac{1}{16} \sum_{\mathbf{a}=0}^{N_f^2 - 1} \mathrm{Tr}\left[Q T^{\mathbf{a}} \right] W^{\mathbf{a}}(q_2^2) \quad W_2 = \frac{1}{16} \sum_{\mathbf{a}=0}^{N_f^2 - 1} \mathrm{Tr}\left[Q T^{\mathbf{a}} \right] W^{\mathbf{a}}((q_1 + q_2)^2)$$

The contribution to the anomaly can be expended as

$$a_\mu^{\rm LbyL} = \left(\frac{\alpha}{\pi}\right)^3 \left(\frac{N_c}{4\pi}\right)^2 \frac{1}{3}\left(\frac{m}{F_\pi}\right)^2 \left[\ln^2\left(\frac{M}{m}\right) + c_\chi \ln\left(\frac{M}{m}\right) + K\right] + \mathcal{O}(N_c^0)$$

The constants c_χ and K are model and schema dependent.

Actually, the regularized WZW contribution to the form factor of $P \to \ell^* \ell^-$ to c_χ as

$$c_\chi = \frac{1}{2} - f(r) + \frac{1}{6}\chi(M^2)$$

where for $r = \frac{m_\pi}{m_\mu}$,

$$f(r) = \ln\left(\frac{m_\mu^2}{\mu^2}\right) + \frac{1}{6}r^2 \ln r - \frac{1}{6}(2r+13) + \frac{1}{3}(2+r)\sqrt{r(4-r)}\cos^{-1}\left(\frac{\sqrt{r}}{2}\right)$$

Using the approximation that the pion is massless and then the lower scale is the muon mass, one deduces that

$$c_\chi \simeq \frac{5}{3} + \frac{1}{6}\chi(M^2)\ .$$

Moreover, in our limits and conventions,

$$\frac{1}{6}\chi(M^2) \simeq \frac{\hat{\alpha}}{3} - \frac{5}{3}\,,$$

where $\hat{\alpha}$ is the *slope* at the origin of the normalized form factor

$$F_{\pi\gamma^*\gamma^*}(q_1^2, q_2^2, q_3^2) \underset{q_1^2, q_2^2 \to 0}{\sim} 1 + \hat{\alpha}(Q_1^2 + Q_2^2)$$

Therefore

$$c_\chi \simeq \frac{\hat{\alpha}}{3}$$

Using the parametrization "LMD+V"

$$F_{\pi\gamma^*\gamma^*}(q_1^2,q_2^2,q_3^2) = \frac{4\pi^2 F_\pi^2}{N_c}\,\frac{q_1^2 q_2^2(q_1^2+q_2^2) - h_2 q_1^2 q_2^2 + h_5(q_1^2+q_2^2) + (N_c M_1^4 M_2^4/4\pi^2 F_\pi^2)}{(q_1^2+M_1^2)(q_1^2+M_2^2)(q_2^2+M_1^2)(q_2^2+M_2^2)}\,,$$

then

$$F_{\pi\gamma^*\gamma^*}(\lambda^2 q_1^2, \lambda^2 q_2^2, q_3^2) \underset{\lambda\to 0}{\sim} 1 + \left[-\frac{1}{M_1^2} - \frac{1}{M_2^2} + \frac{4\pi^2}{N_c}\frac{F_\pi^2}{M_1^4 M_2^4} h_5\right](Q_1^2+Q_2^2)\lambda^2$$

In this context the relevant quantity leading to the anomaly is

$$\begin{aligned}
&\frac{\partial}{\partial k^\rho}\Pi_{\mu\nu\lambda\sigma}(q_1,q_2,k-q_1-q_2) = \\
&i\,\frac{\mathcal{F}_{\pi\gamma^*\gamma^*}(q_1^2,q_2^2)\,\mathcal{F}_{\pi\gamma^*\gamma^*}((q_1+q_2)^2,0)}{(q_1+q_2)^2 - M_\pi^2}\,\varepsilon_{\mu\nu\alpha\beta}\,q_1^\alpha q_2^\beta\,\varepsilon_{\lambda\sigma\rho\tau}\,(q_1+q_2)^\tau \\
&+\ i\,\frac{\mathcal{F}_{\pi\gamma^*\gamma^*}(q_1^2,0)\,\mathcal{F}_{\pi\gamma^*\gamma^*}(q_2^2,(q_1+q_2)^2)}{q_1^2 - M_\pi^2}\,\varepsilon_{\mu\sigma\tau\rho}\,q_1^\tau\,\varepsilon_{\nu\lambda\alpha\beta}\,q_1^\alpha q_2^\beta \\
&+\ i\,\frac{\mathcal{F}_{\pi\gamma^*\gamma^*}(q_1^2,(q_1+q_2)^2)\,\mathcal{F}_{\pi\gamma^*\gamma^*}(q_2^2,0)}{q_2^2 - M_\pi^2}\,\varepsilon_{\mu\lambda\alpha\beta}\,q_1^\alpha q_2^\beta\,\varepsilon_{\nu\sigma\rho\tau}\,q_2^\tau \\
&+\ \mathcal{O}(k)
\end{aligned}$$

If one considers the Melnikov-Vainshtein Limit, we notice that

$$\frac{\partial}{\partial k^\rho}\,\Pi_{\mu\nu\lambda\sigma}(\lambda q_1, q_2-\lambda q_1, k-q_2)$$
$$\underset{\lambda\to\infty}{\sim}\left(\frac{N_c}{4\pi^2F_\pi^2}\right)^2\frac{1}{q_1^2}\frac{1}{q_2^2}\frac{1}{\lambda}\left[1+\frac{2F_\pi^2h_5}{M_1^2M_2^2}\frac{1}{q_2^2}\frac{1}{\lambda}\right]\varepsilon_{\mu\nu\alpha\beta}\,q_1^\alpha q_2^\beta\,\varepsilon_{\lambda\sigma\rho\tau}\,q_2^\tau\,,$$

clearly one has shown explicitly the relation with the subleading term in the MVL and the slope of the form factor.

$$\left.\frac{\partial}{\partial k^\rho}\,\Pi_{\mu\nu\lambda\sigma}(\lambda q_1, q_2-\lambda q_1, k-q_2)\right|_{\lambda^{-2}}\sim\left(\frac{N_c}{4\pi^2}\right)^2\frac{1}{q_1^2}\frac{1}{q_2^4}$$
$$\times\;F_\pi^2\int_0^1 \mathrm{d}\tilde{u}\,\mathrm{d}\tilde{v}\;\tilde{u}\,\ln\tilde{u}\;\ln\tilde{v}\,\frac{\tilde{v}}{1-\tilde{v}}\,\frac{1}{\left(\frac{\tilde{u}}{1-\tilde{u}}+\frac{\tilde{v}}{1-\tilde{v}}\right)^2}$$

$$F_{\pi\gamma^*\gamma^*}(\lambda^2q_1^2,\lambda^2q_2^2)\underset{\lambda\to 0}{\sim}1$$
$$+\,\lambda^2(Q_1^2+Q_2^2)\frac{2}{\kappa^2}\int_0^1\mathrm{d}u\,\mathrm{d}v\;u\,\ln u\;\ln v\,\frac{v}{1-v}\,\frac{1}{\left(\frac{u}{1-u}+\frac{v}{1-v}\right)^2}\,.$$

How to extract the general behaviour?

The contribution to the anomaly expended

$$a_\mu^{\mathrm{LbyL}} = \left(\frac{\alpha}{\pi}\right)^3 \left(\frac{N_c}{4\pi}\right)^2 \frac{1}{3} \left(\frac{m}{F_\pi}\right)^2 \left[\ln^2\left(\frac{M}{m}\right) + c_\chi \ln\left(\frac{M}{m}\right) + K\right] + \mathcal{O}(N_c^0)$$

corresponds in the Mellin plane

$$a_\mu^{\mathrm{LbyL}} = \left(\frac{\alpha}{\pi}\right)^3 \left(\frac{N_c}{4\pi}\right)^2 \frac{1}{3} \left(\frac{m}{F_\pi}\right)^2 \int \frac{\mathrm{d}s}{2i\pi} \left(\frac{M}{m}\right)^{-s} \left[-\frac{1}{s^3} + c_\chi \frac{1}{s^2} - K\frac{1}{s}\right]$$

Once you perform the angular integrals one gets

$$a_\mu^{\mathrm{LbyL}} = \left(\frac{\alpha}{\pi}\right)^3 \int_0^\infty \mathrm{d}Q_1^2 \int_0^\infty \mathrm{d}Q_2^2 \; \mathcal{A}(Q_1^2, Q_2)$$

By the help of the change of variables $Q_j = 2\frac{M}{m}\, m\, x_j$,

$$a_\mu^{\mathrm{LbyL}} = \left(\frac{\alpha}{\pi}\right)^3 \int \frac{\mathrm{d}s}{2i\pi} \left(\frac{M}{m}\right)^{-s} \widehat{\mathcal{A}}(s) \;\; \text{and} \;\; \int_0^\infty \mathrm{d}x_1 \mathrm{d}x_2 \; x_1^{s-1} x_2^{s-1} \; \mathcal{A}(4m^2 x_1^2, 4m^2 x_2^2)$$

We can extract then singularities.