Cuts and Feynman Integrals beyond multiple polylogarithms

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LHC and the Standard Model: Physics and Tools
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Based on collaboration with *A. von Manteuffel*, *A. Primo*, *E. Remiddi*

[arXiv:1602.01481], [arXiv:1610.08397], [arXiv:1701.05905], [arXiv:1704.05465]

Computation of multiloop Feynman Integrals is an essential step towards precise theoretical predictions in the **Standard Model**

1. The **Higgs discovery** at the LHC opened a window on the *electroweak symmetry breaking mechanism*

2. Thorough investigation of the consequences of *LHC measurements* requires **theoretical control** of different $2 \to 2$ and $2 \to 3$ processes, with very high precision (typically requires 2-loop calculations)

$$VV, \quad Hjet, \quad HH, \quad 2jet, \quad 3jet, \quad t\bar{t}, \ldots$$

3. In presence of **massive loops** and/or **high multiplicities** ($2 \to 3$!), many of these calculations are practically impossible

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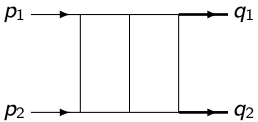
Any Feynman Diagrams is (*after some tedious but elementary algebra!*) nothing but a collection of scalar **Feynman Integrals**

$\mathcal{I}(p_1, p_2, q_1) =$



with $\quad q_2 = p_1 + p_2 - q_1$

$$\mathcal{I}(p_1, p_2, q_1) = \int \frac{\mathfrak{D}^d k \, \mathfrak{D}^d l}{k^2 \, l^2 \, (k-l)^2 \, (k-p_1)^2 \, (k-p_{12})^2 \, (l-p_{12})^2 \, (l-q_1)^2}$$

Typical **2-loop** Feynman Integral required for the computation of a $2 \to 2$ scattering process.

Dimensionally regularised Feynman Integrals fulfil differential equations!
[Kotikov '90, Remiddi '97, Gehrmann-Remiddi '00,..., C. Papadopoulos '14]

$\Downarrow$

Direct consequence of **Integration-by-parts (IBPs)** identities in d-dimensions!

$$\int \prod_{j=1}^{l} \frac{d^d\, k_j}{(2\pi)^d} \left(\frac{\partial}{\partial k_j^\mu}\, v_\mu\, \frac{S_1^{\sigma_1} \ldots S_s^{\sigma_s}}{D_1^{\alpha_1} \ldots D_n^{\alpha_n}} \right) = 0\,, \qquad v^\mu = k_j^\mu, p_k^\mu$$

Reduced to N master integrals, $I_i(d; x_k)$ with $i = 1, \ldots, N$.

$\Downarrow$

Differentiating the masters and using the **IBPs** we get a system of
N coupled differential equations

$$\frac{\partial}{\partial\, x_k} I_i(d; x_k) = \sum_{j=1}^{N} c_{ij}(d; x_k)\, I_j(d; x_k)\,.$$

A revolution in multi-loop calculations has started when physicists have re-discovered the so-called **multiple polylogarithms**
[E.Remiddi, J.Vermaseren '99; T. Gehrmann, E.Remiddi '00; ....]

$$G(0;x) = \ln(x)\,, \qquad G(a;x) = \ln\left(1 - \frac{x}{a}\right) \quad \text{for} \quad a \neq 0$$

$$G(\underbrace{0,\dots,0}_{n};x) = \frac{1}{n!}\ln^n(x)\,, \qquad G(a,\vec{w};x) = \int_0^x \frac{dy}{y-a}\,G(\vec{w};y)\,.$$

$$\Downarrow$$

Multiple polylogarithms are *special* because they satisfy
first order differential equations with rational coefficients

$$\frac{\partial}{\partial x}G(a,\vec{w};x) = \frac{1}{x-a}\,G(\vec{w};x) \quad \rightarrow \quad \text{purely non-homogeneous equation!}$$

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Interestingly enough, many multiloop Feynman integrals (mostly with massless propagators) satisfy **first order differential equations** in the mandelstam invariants, at least in the limit $d \to 4$!

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As a result, the coefficients of their **Laurent series** for $d \approx 4$ can be written as linear combinations of *rational functions* and *multiple polylogarithms*!
We understand almost everything about these functions now
[Goncharov '01, Duhr, Gangle, Rodes '12, Duhr '13, Panzer '14...]

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Explosion of **phenomenologically important** NNLO calculations for the LHC, all required solving effectively only **first order differential equations**.

Exceptions (numerically!):
- $t\bar{t}$ at NNLO [Czakon et al, '13]
- HH at NLO [Borowka et al, '16]

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Important development was the introduction of a **Canonical Basis** [Henn '13]
→ Rotate to a new basis of master integrals of unit leading singularities
The equations become

$$d\vec{I}(d;x) = \epsilon\, A(x)\vec{I}(d;x)$$

with $A(x)$ in d-log form. In this case, master integrals are obviously (almost!) MPLs at every order in ϵ.

$$\Downarrow$$

In the present formulation, this is therefore limited to cases that integrate to **multiple-polylogarithms** (and fulfil therefore first order differential equations).

Let's look more in detail - *we should recall* that equations are in <u>block form</u>

$$I_j(d;x_k) = (m_j(d;x_k)\,, \quad sub_j(d;x_k))$$

$$\Downarrow$$

$$\frac{\partial}{\partial x_k} m_i(d;x_k) = \sum_{j=1}^{N} h_{ij}(d;x_k)\, m_j(d;x_k) + \sum_{j=1}^{M} nh_{ij}(d;x_k)\, sub_j(d;x_k)\,.$$

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homogeneous piece is MAIN source of complexity - whether differential equations are coupled

$$\Downarrow$$

No way to solve this in general...
Use some "physical" insight...

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non-homogeneous piece is the second source of complexity – we must integrate over it!

$\Downarrow$

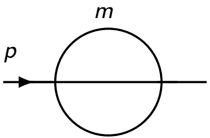
Can be simplified using dispersion relations [E.Remiddi, LT '16]

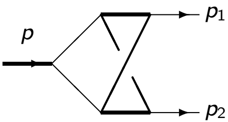
A canonical basis, when it exists, makes our life much easier
$\rightarrow$ integration in terms of multiple polylogarithms!

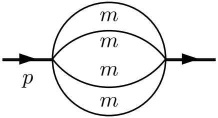
As a matter of fact, in many cases the equations remain **coupled** in $d = 4$ and a canonical basis in the sense above **does not exist**.

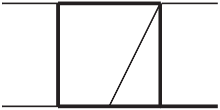
It's enough to start putting some masses in the loops or to increase the final state multiplicity

We know a more and more examples now







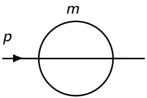


What all these examples have **in common** is a bulk 2×2 (or 3×3) irreducible system of differential equations

How do we solve them?

I want to go back to an older idea, in the context of the Sunrise graph
[S.Laporta, E.Remiddi '04]

$$\left(\frac{d^2}{d\, s^2} + A(d;s)\, \frac{d}{d\, s} + B(d;s) \right)$$



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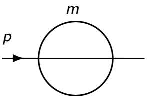
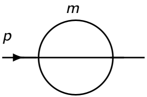


$$+ \; G(d;s)\, \mathrm{Tad}(d;m^2) = 0\,.$$

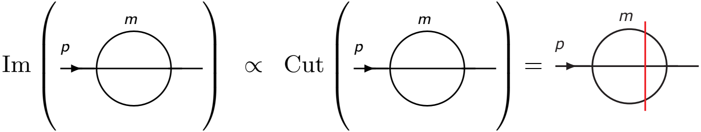
They realized that the imaginary part of the sunrise is precisely **ONE** solution of homogeneous equation, since the Tadpole doesn't have a branch cut in s!

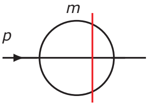
$$\left(\frac{d^2}{d\, s^2} + A(d;s)\, \frac{d}{d\, s} + B(d;s) \right) \mathrm{Im} \left(\right.$$



$$\left. \right) = 0\,.$$

How do we generalize this?
Computing the imaginary part means cutting *maximally* the graph!





$$= \oint_C \mathfrak{D}^d k\, \mathfrak{D}^d l\, \delta(k^2 - m^2)\, \delta(l^2 - m^2)\, \delta((k - l - p)^2 - m^2)$$

Clearly, any cut of a graph will fulfil a similar *(simpler)* differential equation

But, since all subtopologies have fewer propagators, if we **cut all propagators** (maximal cut) we are necessarily left only with the homogeneous equation!

The Maximal Cut provides us with **ONE solution** of the homogeneous system!
[A. Primo, L. T. '16]

$\Downarrow$

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The Maximal Cut provides us with **ONE solution** of the homogeneous system!

[A. Primo, L. T. '16]

$$\Downarrow$$

$$\frac{\partial}{\partial x_k} \mathrm{Cut}\left(m_i(d; x_k)\right) = \sum_{j=1}^{N} h_{ij}(d; x_k)\, \mathrm{Cut}\left(m_j(d; x_k)\right)$$

We can use **Baikov representation** [Baikov '96] in order to compute the cuts very efficiently [Frellesvig, Papadopoulos '17] *(See Costas' Talk tomorrow!)*

...But we need a way to find all independent solutions...

Main Idea:
cutting along different (independent) contours that don't cross any branch cuts of the integrand, we can get all different homogeneous solutions.
[Bosma, Sogaard, Zhang, '17; A. Primo, L.T. '17; Harley, Moriello, Schabinger '17]

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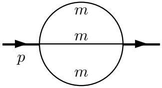
Let's see how this works for the sunrise graph (with $u = p^2/m^2$)



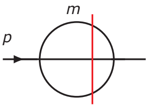
$$= S(d;u) = \int \frac{\mathfrak{D}^d k_1 \, \mathfrak{D}^d k_2}{[k_1^2 - m^2][k_2^2 - m^2][(k_1 - k_2 - p)^2 - m^2]} \, ,$$

For $\epsilon = (2-d)/2$ the sunrise fulfils a 2 **system of diff. equations**

$$\frac{d}{du}\begin{pmatrix} S_1(u) \\ S_2(u) \end{pmatrix} = B(u)\begin{pmatrix} S_1(u) \\ S_2(u) \end{pmatrix} + \epsilon \, D(u)\begin{pmatrix} S_1(u) \\ S_2(u) \end{pmatrix} + \begin{pmatrix} N_1(u) \\ N_2(u) \end{pmatrix}$$

We need to find a matrix of 2×2 independent homogeneous solutions!

But cutting the graph maximally we find

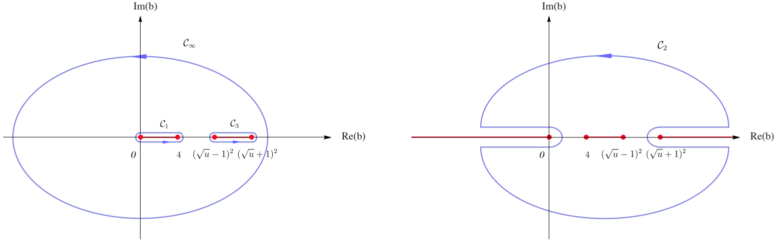


$$= \oint_{\mathcal{C}} \frac{db}{\sqrt{\pm b\,(b-4)\left(b-(\sqrt{u}-1)^2\right)\left(b-(\sqrt{u}+1)^2\right)}}$$

$$= \oint_{\mathcal{C}} \frac{db}{\sqrt{\pm R_4(b,u)}}$$

We want to get *independent homogeneous solutions*
by integrating along different contours

The problem of how many independent contours exist is a
cohomology problem, recent renewed interest from physics community
[... ; Abreu, Britto, Duhr, Gardi '17]



(a) (b)

(a)

$$\oint_{\mathcal{C}_\infty} \frac{db}{\sqrt{R_4(b,u)}} = 0 \longrightarrow \oint_{\mathcal{C}_1} \frac{db}{\sqrt{R_4(b,u)}} = -\oint_{\mathcal{C}_3} \frac{db}{\sqrt{R_4(b,u)}}$$

(b)

$$\oint_{\mathcal{C}_2} \frac{db}{\sqrt{-R_4(b,u)}}$$

Other two solutions by *differentiation* [or cutting the **second master integral**]

$$J_1(u) \propto \oint_{\mathcal{C}_1} \frac{db}{\sqrt{R_4(b,u)}} \qquad I_1(u) \propto \oint_{\mathcal{C}_2} \frac{db}{\sqrt{-R_4(b,u)}}$$

$$J_2(u) \propto \oint_{\mathcal{C}_1} \frac{db\; b^2}{\sqrt{R_4(b,u)}} \qquad I_2(u) \propto \oint_{\mathcal{C}_2} \frac{db\; b^2}{\sqrt{-R_4(b,u)}}$$

And by construction we find

$$\frac{d}{du}\begin{pmatrix} I_1(u) & J_1(u) \\ I_2(u) & J_2(u) \end{pmatrix} = B(u)\begin{pmatrix} I_1(u) & J_1(u) \\ I_2(u) & J_2(u) \end{pmatrix}$$

Matrix of solutions can be therefore written as the matrix of the maximal cuts

$$G(u) = \begin{pmatrix} I_1(u) & J_1(u) \\ I_2(u) & J_2(u) \end{pmatrix} = \begin{pmatrix} \mathrm{Cut}_{C_1}\left(S_1(u)\right) & \mathrm{Cut}_{C_2}\left(S_1(u)\right) \\ \mathrm{Cut}_{C_1}\left(S_2(u)\right) & \mathrm{Cut}_{C_2}\left(S_2(u)\right) \end{pmatrix}$$

and recall that

$$G^{-1}(u) = \frac{1}{W(u)} \begin{pmatrix} J_2(u) & -J_1(u) \\ -I_2(u) & I_1(u) \end{pmatrix} \quad \to \quad W(u) = \det\left(G(u)\right) = I_1(u)J_2(u) - I_2(u)J_1(u)$$

where $W(u)$ is the Wronskian of the solutions!

Let's rotate the system to a more convenient form

$$G(u) = \begin{pmatrix} I_1(u) & J_1(u) \\ I_2(u) & J_2(u) \end{pmatrix} \quad \rightarrow \quad \begin{pmatrix} S_1(u) \\ S_2(u) \end{pmatrix} = G(u) \begin{pmatrix} m_1(u) \\ m_2(u) \end{pmatrix}$$

Such that

$$\frac{d}{du} \begin{pmatrix} m_1(u) \\ m_2(u) \end{pmatrix} = \epsilon \underbrace{G^{-1}(u) D(u) G(u)} \begin{pmatrix} m_1(u) \\ m_2(u) \end{pmatrix} + G^{-1}(u) \begin{pmatrix} N_1(u) \\ N_2(u) \end{pmatrix}$$

$$\Downarrow$$

Iterated integrals over products of **two** elliptic integrals and rational functions!

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Iterated integrals over products of **two** elliptic integrals and rational functions!

Interestingly, the iteration matrix can be written as a total differential!

$$M(u) = G^{-1}(u)D(u)G(u)\,,$$

$$M_{11}(u) = -\frac{d}{d\,u}\left[\left(\frac{(u+3)^2}{6} I_1(u)\,J_1(u)\right) + \frac{\pi}{4}\left(2\ln\left(u-9\right) + 2\ln\left(u-1\right) - \ln\left(u\right)\right)\right]\,,$$

$$M_{12}(u) = -\frac{d}{d\,u}\left(\frac{(u+3)^2}{6} I_1(u)\,I_1(u)\right)\,,$$

$$M_{21}(u) = +\frac{d}{d\,u}\left(\frac{(u+3)^2}{6} J_1(u)\,J_1(u)\right)\,,$$

$$M_{22}(u) = +\frac{d}{d\,u}\left[\left(\frac{(u+3)^2}{6} I_1(u)\,J_1(u)\right) + \frac{\pi}{4}\left(2\ln\left(u-9\right) + 2\ln\left(u-1\right) - \ln\left(u\right)\right)\right]\,,$$

Physical intepretation of this rotation

$$\begin{pmatrix} m_1(u) \\ m_2(u) \end{pmatrix} = G^{-1}(u) \begin{pmatrix} S_1(u) \\ S_2(u) \end{pmatrix} = \frac{1}{W(u)} \begin{pmatrix} J_2(u)S_1(u) - J_1(u)S_2(u) \\ -I_2(u)S_1(u) + I_1(u)S_2(u) \end{pmatrix}$$

Let **maximal-cut** it along the two independent contours that we found earlier

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Physical intepretation of this rotation

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You see what is happening here...

Remember, given a system of differential equations, the matrix of the maximal cuts is the matrix of the homogeneous solutions!

What about our new basis? For $\epsilon = 0$ it's homogeneous equations is

$$\frac{d}{du}\begin{pmatrix} m_1(u) \\ m_2(u) \end{pmatrix} = 0$$

And indeed

$$\begin{pmatrix} \mathrm{Cut}_{C_1}(m_1(u)) & \mathrm{Cut}_{C_2}(m_1(u)) \\ \mathrm{Cut}_{C_1}(m_2(u)) & \mathrm{Cut}_{C_2}(m_2(u)) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \rightarrow \quad \text{The } \textbf{identity !!!!}$$

Unit leading singularity!!! Generalization (?) of [Henn '13]

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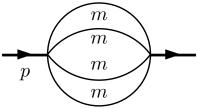
Unit leading singularity!!! Generalization (?) of [Henn '13]

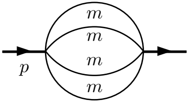
It turns out to be a formidable tool to simplify the solution of differential equations, even **beyond** the 2×2 cases which give rise to **elliptic integrals**!

$$\Downarrow$$

Let's see how it works for a 3×3 example



We consider the three-loop two-point integral family defined by



$$= I_{a_1,a_2,a_3,a_4,a_5,a_6,a_7,a_8,a_9}\Big|_{a_5,\cdots,a_9<0}$$

$$= \int \frac{\mathfrak{D}^d k_1 \, \mathfrak{D}^d k_2 \mathfrak{D}^d k_3 \;\; (k_3^2)^{-a_5}(k_1\cdot p)^{-a_6}(k_2\cdot p)^{-a_7}(k_3\cdot p)^{-a_8}(k_1\cdot k_2)^{-a_9}}{[k_1^2-m^2]^{a_1}[k_2^2-m^2]^{a_2}[(k_1-k_3)^2-m^2]^{a_3}[(k_2-k_3-p)^2-m^2]^{a_4}} \,,$$

Has three master integrals (for simplicity $\epsilon = (2-d)/2$)

$$\begin{aligned}
\mathcal{I}_1(\epsilon; s) =& (1+2\epsilon)(1+3\epsilon)(m^2)^{-2} I_{1,1,1,1,0,0,0,0,0} \,, \\
\mathcal{I}_2(\epsilon; s) =& (1+2\epsilon)(m^2)^{-1} I_{2,1,1,1,0,0,0,0,0} \,, \\
\mathcal{I}_3(\epsilon; s) =& I_{2,2,1,1,0,0,0,0,0} \,,
\end{aligned}$$

$$\frac{d}{dx}\begin{pmatrix}\mathcal{I}_1(\epsilon;x)\\ \mathcal{I}_2(\epsilon;x)\\ \mathcal{I}_3(\epsilon;x)\end{pmatrix} = B(x)\begin{pmatrix}\mathcal{I}_1(\epsilon;x)\\ \mathcal{I}_2(\epsilon;x)\\ \mathcal{I}_3(\epsilon;x)\end{pmatrix} + \epsilon\, D(x)\begin{pmatrix}\mathcal{I}_1(\epsilon;x)\\ \mathcal{I}_2(\epsilon;x)\\ \mathcal{I}_3(\epsilon;x)\end{pmatrix} + \begin{pmatrix}0\\ 0\\ -\frac{1}{2(4x-1)}\end{pmatrix}$$

where $B(x)$ and $D(x)$ are 3×3 matrices, with $x = 4m^2/p^2$

$$B(x) = \begin{pmatrix} \frac{1}{x} & \frac{4}{x} & 0 \\ -\frac{1}{4(x-1)} & \frac{1}{x}-\frac{2}{x-1} & \frac{3}{x}-\frac{3}{x-1} \\ \frac{1}{8(x-1)}-\frac{1}{8(4x-1)} & \frac{1}{x-1}-\frac{3}{2(4x-1)} & \frac{1}{x}-\frac{6}{4x-1}+\frac{3}{2(x-1)} \end{pmatrix}$$

$$D(x) = \begin{pmatrix} \frac{3}{x} & \frac{12}{x} & 0 \\ -\frac{1}{x-1} & \frac{2}{x}-\frac{6}{x-1} & \frac{6}{x}-\frac{6}{x-1} \\ \frac{1}{2(x-1)}-\frac{1}{2(4x-1)} & \frac{3}{x-1}-\frac{9}{2(4x-1)} & \frac{1}{x}-\frac{12}{4x-1}+\frac{3}{x-1} \end{pmatrix}$$

We need to find now three independent solutions, i.e. a matrix

$$G(x) = \begin{pmatrix} H_1(x) & J_1(x) & I_1(x) \\ H_2(x) & J_2(x) & I_2(x) \\ H_3(x) & J_3(x) & I_3(x) \end{pmatrix} \qquad \rightarrow \qquad \frac{d}{dx} G(x) = B(x)\, G(x)\,.$$

Or, if our idea is correct, there should exists **three independent** integration contours $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ such that (for $\epsilon = 0$)

$$G(x) = \begin{pmatrix} \mathrm{Cut}_{\mathcal{C}_1}(\mathcal{I}_1(x)) & \mathrm{Cut}_{\mathcal{C}_2}(\mathcal{I}_1(x)) & \mathrm{Cut}_{\mathcal{C}_3}(\mathcal{I}_1(x)) \\ \mathrm{Cut}_{\mathcal{C}_1}(\mathcal{I}_2(x)) & \mathrm{Cut}_{\mathcal{C}_2}(\mathcal{I}_2(x)) & \mathrm{Cut}_{\mathcal{C}_3}(\mathcal{I}_2(x)) \\ \mathrm{Cut}_{\mathcal{C}_1}(\mathcal{I}_3(x)) & \mathrm{Cut}_{\mathcal{C}_2}(\mathcal{I}_3(x)) & \mathrm{Cut}_{\mathcal{C}_3}(\mathcal{I}_3(x)) \end{pmatrix}$$

It is very easy to compute the maximal cut of $\mathcal{I}_1(x)$ (at least it's integrand!)

$$\mathrm{Cut}(\mathcal{I}_1(x)) = x \oint_{\mathcal{C}'} \frac{da\, db}{\sqrt{a\, b\, (a\, b - x)}\sqrt{(a+1)(b+1)\,((a+1)(b+1)-x)}}$$
$$= x \oint_{\mathcal{C}'} \frac{da\, db}{\sqrt{R(a,b,x)}}\,,$$

$$R(a,b,x) = a\, b\, (a\, b - x)(a+1)(b+1)\,((a+1)(b+1)-x)\;.$$

It is relatively easy to verify that there are only 3 possible independent integration contours, that can in turn be shrunk to three **two-dimensional real integrals**

$$f_1^{\mathrm{V}}(x) = x \oint_{\mathcal{C}_1} \frac{da\,db}{\sqrt{R(a,b,x)}} = x \int_0^\infty db \int_{-1}^{x/(b+1)-1} \frac{da}{\sqrt{-R(a,b,x)}}$$

$$f_2^{\mathrm{V}}(x) = x \oint_{\mathcal{C}_2} \frac{da\,db}{\sqrt{R(a,b,x)}} = x \int_0^\infty db \int_{x/(b+1)-1}^{0} \frac{da}{\sqrt{R(a,b,x)}}\,,$$

$$f_1^{\mathrm{IV}}(x) = x \oint_{\mathcal{C}_3} \frac{da\,db}{\sqrt{R(a,b,x)}} = x \int_{x-1}^{0} db \int_{x/b}^{-1} \frac{da}{\sqrt{-R(a,b,x)}}$$

Interestingly enough, with some effort, and following:
[Bailey, Borwein, Broadhurst '08]

$$f_1^{\mathrm{V}}(x) = 2x\,\mathrm{K}(k_-^2)\,\mathrm{K}(k_+^2)$$
$$f_2^{\mathrm{V}}(x) = 4x\left(\mathrm{K}(k_-^2)\,\mathrm{K}(1-k_+^2) + \mathrm{K}(k_+^2)\,\mathrm{K}(1-k_-^2)\right)\,,$$

$$k_\pm = \frac{\sqrt{(\gamma+\alpha)^2-\beta^2} \pm \sqrt{(\gamma-\alpha)^2-\beta^2}}{2\gamma} \qquad \text{with} \qquad k_- = \left(\frac{\alpha}{\gamma}\right)\frac{1}{k_+} = \frac{2\alpha}{k_+}$$

$$\alpha = \frac{\sqrt{x}+\sqrt{x(1-x)}}{2}\,, \qquad \beta = \frac{\sqrt{x}-\sqrt{x(1-x)}}{2}\,, \qquad \gamma = \frac{1}{2}$$

Result expected from studies of Joyce '73 on cubic lattice Green functions!
Elliptic Tri-Log by [Bloch, Kerr, Vanhove '14]

We choose as three independent functions

$$\begin{aligned}
\mathcal{H}_1(x) =& x \ \mathrm{K}\left(k_+^2\right) \mathrm{K}\left(k_-^2\right) , \\
\mathcal{J}_1(x) =& x \ \mathrm{K}\left(k_+^2\right) \mathrm{K}\left(1-k_-^2\right) , \\
\mathcal{I}_1(x) =& x \ \mathrm{K}\left(1-k_+^2\right) \mathrm{K}\left(k_-^2\right) ,
\end{aligned}$$

where the remaining rows of the matrix $G(x)$ can be obtained by differentiation. With this choice we have

$$W(x) = -\frac{\pi^3 x^3}{512\sqrt{(1-4x)^3(1-x)}}$$

We can perform the same rotation as for the sunrise, but now 3×3

$$\begin{pmatrix} \mathcal{I}_1(x) \\ \mathcal{I}_2(x) \\ \mathcal{I}_3(x) \end{pmatrix} = G(x) \begin{pmatrix} M_1(x) \\ M_2(x) \\ M_3(x) \end{pmatrix}$$

which by construction fulfil the system

$$\frac{d}{dx} \begin{pmatrix} M_1(x) \\ M_2(x) \\ M_3(x) \end{pmatrix} = \epsilon \, \underbrace{G^{-1}(x) D(x) G(x)} \begin{pmatrix} M_1(x) \\ M_2(x) \\ M_3(x) \end{pmatrix} + G^{-1}(x) \begin{pmatrix} 0 \\ 0 \\ -\frac{1}{2(4x-1)} \end{pmatrix}$$

$$\Downarrow$$

Iterated integrals over products of **6** elliptic integrals!

The method works!
It allows us to write integral representations for the solutions.
It is **NOT** limited to the elliptic case !

Still problem remains, what are these functions???
Different approaches to **Elliptic Polylogarithms** for the sunrise case
[Adams, Bogner, Weinzierl, Bloch, Vanhove, ...]

Topic of a totally different talk !

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CONCLUSIONS

1- Computation of multiloop Feynman integrals is essential ingredient towards **precision physics** (at LHC and beyond).

2- Last years developments allowed us to compute huge number of new processes in terms of **multiple polylogarithms**
$\rightarrow$ NNLO pheno revolution is starting...

3- Multiple polylogarithms are not the end of the story at two-loops.
All our techniques would break down when approaching this case:
Elliptic functions (and maybe beyond?) $\rightarrow$ *hic sunt leones*

4- I think we can say, today we understand something more:
maximal cut can be practically used to solve the homogeneous solution:
it is a fundamental step towards a complete solution!

5- What is missing?

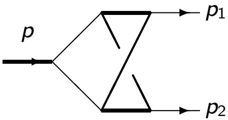
a) Understanding of these new functions –
   – some promising steps have been taken
   [S. Weinzierl et al. '14,'15,'16, '17]

b) Handling the algebraic complexity and swell of expressions at intermediate stages –
   – brute force (more RAM, faster CPUs,... ?)
   – more clever reduction codes?
   – finite fields? [A.Manteuffel, R.Schabinger '15; T.Peraro '16]
   – Two- (multi-)loop unitarity? [S.Badger et al '13,'14,'15; H. Ita '16; P.Mastrolia et al '11,'13,'16; ....]

THANKS!

Backup slides

An explicit example [A.von Manteuffel, L.T. '17]



$$, \qquad x = -\frac{p^2}{m^2}$$

There are 11 MIs m_j j=1,...,11.
All 9 subtopologies can be written in terms of polylogs (*not trivial!*)

$$\ln\left(f(l_i)\right), \quad \mathrm{Li}_n(f(l_i)), \quad \mathrm{Li}_{2,2}(f(l_i), g(l_j)),$$

with

$$l_i = \{\sqrt{x}, \tfrac{1}{2}(\sqrt{x}+\sqrt{x+4}), \sqrt{x+4}, \tfrac{1}{2}(\sqrt{x}+\sqrt{x-4}), \sqrt{x-4}\}$$

$$\frac{d}{dx}\begin{pmatrix} m_{10} \\ m_{11} \end{pmatrix} = B(x)\begin{pmatrix} m_{10} \\ m_{11} \end{pmatrix} + \epsilon\, D(x)\begin{pmatrix} m_{10} \\ m_{11} \end{pmatrix} + \begin{pmatrix} N_{10}(\epsilon;x) \\ N_{11}(\epsilon;x) \end{pmatrix}$$

where $B(x)$ and $D(x)$ are two 2×2 **matrices** that do not depend on ϵ,

$$B(x) = \begin{pmatrix} 0 & \frac{1}{2(x-16)} - \frac{1}{2\,x} \\ \frac{1}{2\,x} & \frac{1}{x} \end{pmatrix}, \qquad D(x) = \begin{pmatrix} -\frac{2}{x} & \frac{1}{x-16} - \frac{1}{x} \\ \frac{2}{x} & \frac{1}{x} - \frac{1}{x-16} \end{pmatrix},$$

$N_{10}(\epsilon;x)$ and $N_{11}(\epsilon;x)$ are **subtopologies** (polylogs)

$$N_{10}^{(j)}(x) = 0\,, \quad N_{11}^{(j)}(x) = 0\,, \qquad j = 0,1,2,3$$
$$N_{10}^{(4)}(x) = 0$$
$$N_{11}^{(4)}(x) = 5\ln^2(l_2) - l_1\,\frac{3/2\,\zeta_2 + 3\ln^2(l_4) + 3\mathrm{Li}_2(-1/l_4^2)}{l_5}\,,$$

First step: Solve homogeneous system for $\epsilon = 0$

$$\frac{d}{dx}\begin{pmatrix} m_{10} \\ m_{11} \end{pmatrix} = B(x)\begin{pmatrix} m_{10} \\ m_{11} \end{pmatrix} \longrightarrow \quad \text{complete solution is a } 2 \times 2 \text{ matrix}$$

First two solutions from **second order differential equation** for m_{10}

$$\frac{d^2\, m_{10}(x)}{dx^2} + \left(\frac{1}{x-16}\right)\frac{d\, m_{10}(x)}{dx} + \frac{1}{64}\left(\frac{1}{x} + \frac{16}{x^2} - \frac{1}{x-16}\right) m_{10}(x) = 0$$

Use the **maximal cut** so find it's **two independent solutions** (as for sunrise!)

$$\mathrm{Cut}_1(m_{10}) = I_1(x) = \sqrt{x}\ \mathrm{K}\left(\frac{x}{16}\right), \qquad \mathrm{Cut}_2(m_{10}) = J_1(x) = \sqrt{x}\ \mathrm{K}\left(1 - \frac{x}{16}\right).$$

From the system of differential equations the other two solutions are obtained

$$I_2(x) = -\sqrt{x}\,\mathrm{E}\left(\frac{x}{16}\right), \qquad J_2(x) = \sqrt{x}\left[\mathrm{E}\left(1-\frac{x}{16}\right) - \mathrm{K}\left(1-\frac{x}{16}\right)\right],$$

where $\mathrm{E}(x)$ is the complete elliptic integral of the second kind.

$$\mathrm{E}(x) = \int_0^1 dt\,\sqrt{\frac{1-x\,t^2}{1-t^2}}$$

They correspond to the maximal cut of the second master integral m_{11}!

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They correspond to the maximal cut of the second master integral m_{11}!

Writing integral representation is now straightforward. The two master integrals can be written as

$$m_{10}^{(4)}(x) = \frac{2}{\pi} \left[J_1(x) \int_0^x dt\, I_1(t)\, Q(t) - I_1(x) \int_0^x dt\, J_1(t)\, Q(t) \right] ,$$
$$m_{11}^{(4)}(x) = \frac{2}{\pi} \left[J_2(x) \int_0^x dt\, I_1(t)\, Q(t) - I_2(x) \int_0^x dt\, J_1(t)\, Q(t) \right] .$$

All building blocks are **well understood functions**

$$Q(t) = 5 \ln^2(l_2) - l_1 \frac{3/2\, \zeta_2 + 3 \ln^2(l_4) + 3 \mathrm{Li}_2(-1/l_4^2)}{l_5}$$

$$I_1(t) = \sqrt{x}\, \mathrm{K}\left(\frac{x}{16}\right) , \qquad J_1(x) = \sqrt{x}\, \mathrm{K}\left(1 - \frac{x}{16}\right) .$$

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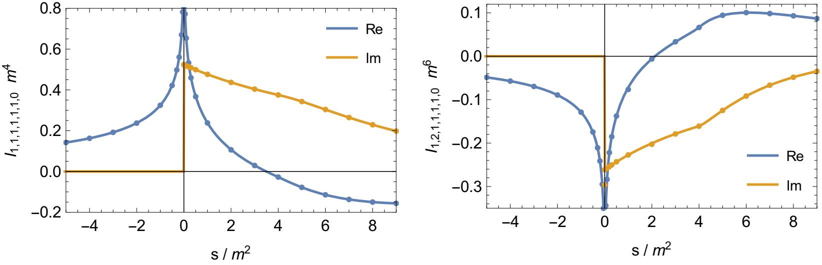
The integral representation of the solution is well suited for analytic continuation and, **very importantly**, fast and precise numerical evaluation

$\Downarrow$



Comparison against SecDec 3 [S.Borowka et al., '15] for the two masters