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On One Nonlocal Contact Problem for Elliptic Equation and its Numerical Solution

Tinatin Davitashvili
The Faculty of Exact and Natural Sciences,
Ivane javakhishvili Tbilisi State University

Hamlet Meladze
The School of Informatics, Mathematics and Natural Sciences,
St. Andrew the First Called Georgian University

Introduction

[1] J.R. Cannon, "The solution of the heat equation subject to the specification of energy", Quart.Aappl. Math.21,1963, pp.155-160.

In Cannon's paper the nonlocal problem was stated as

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},\ 0 < x < 1,\ 0 < t < T,$$

$$u(0,t) = \phi_1(t), \quad \int_0^1 u(x,t)dx = \phi_2(t),$$

$$u(x,0) = u_0(x),$$

where $\phi_1(.)$, $\phi_2(.)$, $u_0(.)$ are known smooth functions, which satisfy the coordination conditions.This was a nonlocal problem, which laid the foundation for the new direction in research of nonlocal boundary problems and methods of their numerical solution.

Introduction

[2] А.В. Бицадзе и А.А. Самарский, «Об одном простом обобщении линейных эллиптических краевых задач», ДАН АНСССР, 185, 1969, ст. 739-774.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad -l < x < l, \quad 0 < y < 1, \quad l = const > 0,$$

with boundary conditions

$$u(x,0) \quad \phi_1(x), \quad u(x,1) \quad \phi_2(x), \quad -l < x < l,$$

$$u(-l,y) \quad \phi_3(y), \quad 0 < y < 1,$$

$$u(0,y) = u(l,y), \quad 0 \le y \le 1.$$

Here $\phi_i(\,)$, $(i = 1,3)$ are known continuous functions.

In the last two decades, extensive studies of nonlocal initial-boundary and boundary value problems have been carried out, general theoretical fundamental principles of analysis were formulated, methods have been developed for the numerical solution of problems and for the construction of mathematical models of concrete problems in physics, ecology, biology, economics and other areas (see [3-20] the results of D.Gordeziani, H.Meladze, M.Sapagovas, V.Makarov, G.Berikelashvili, G.Avalishvili, D.Kapanadze, etc.).

Designations

Let us consider the rectangle domain D in two-dimensional space R^2

$D = \{(x_1, x_2) \mid 0 < x_1 < a, \quad 0 < x_2 < b\}$ with piecewise boundary $\gamma = \bigcup_{i=1}^{4} \gamma_i$, where

$$\gamma_1 = \{(x_1, x_2) \mid 0 \le x_1 \le a, \quad x_2 = 0\}, \quad \gamma_2 = \{(x_1, x_2) \mid 0 \le x_1 \le a, \quad x_2 = b\},$$

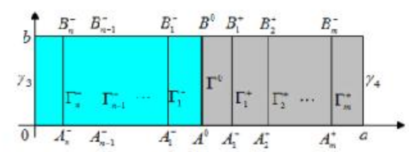
$$\gamma_3 = \{(x_1, x_2) \mid x_1 = 0, \quad 0 \le x_2 \le b\}, \quad \gamma_4 = \{(x_1, x_2) \mid x_1 = a, \quad 0 \le x_2 \le b\}.$$

Let us consider also the segments: $\Gamma^0 = \{(x_1, x_2) \mid x_1 = \xi^0, \quad 0 < \xi^0 < a, \quad 0 \le x_2 \le b\}$,

$$\gamma_1^- = \{(x_1, x_2) \mid 0 \le x_1 \le \xi^0, \quad x_2 = 0\}, \quad \gamma_2^- = \{(x_1, x_2) \mid 0 \le x_1 \le \xi^0, \quad x_2 = b\},$$

$$\gamma_1^+ = \{(x_1, x_2) \mid \xi^0 \le x_1 \le a, \quad x_2 = 0\}, \quad \gamma_2^+ = \{(x_1, x_2) \mid \xi^0 \le x_1 \le a, \quad x_2 = b\}$$

It's evident, that $\gamma_1^- \cup \gamma_1^+ = \gamma_1$ and $\gamma_2^- \cup \gamma_2^+ = \gamma_2$.

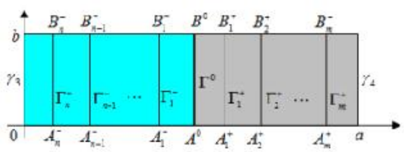


Designations

$\Gamma_i^- = \{(x_1,x_2) \mid x_1 = \xi_i^-,\ \ 0<\xi_i^-<\xi^0,\ \ 0\le x_2\le b\},\ \ i=\overline{1,n},\ 0<\xi_n^-<\xi_{n-1}^-<\dots<\xi_1^-<\xi^0,$
$\Gamma_j^+ = \{(x_1,x_2) \mid x_1 = \xi_j^+,\ \ \xi^0<\xi_j^+<a,\ \ 0\le x_2\le b\}\ \ j=\overline{1,m},\ \ \xi^0<\xi_1^+<\xi_2^+<\dots<\xi_m^+<b.$

Γ^0, Γ_i^-, $i=\overline{1,n}$, and Γ_j^+, $j=\overline{1,m}$, intersect γ_1 and γ_2, respectively, in the points:
$A^0(\xi^0,0),\ B^0(\xi^0,b),\ A_i^-(\xi_i^-,0),\ B_i^-(\xi_i^-,b),\ A_i^+(\xi_i^+,0)$ and $B_i^+(\xi_i^+,b)$.

It is obvious that Γ^0 divides the domain D into two parts (domains) D^- and D^+, where
$D^- = \{(x_1,x_2) \mid 0<x_1<\xi^0,\ \ 0<x_2<b\},\ \ D^+ = \{(x_1,x_2) \mid \xi^0<x_1<a,\ \ 0<x_2<b\}.$



Statement of the Problem

We consider the following problem: find in $\overline{D}$ a continuous function $u(x_1,x_2)$,

$$u(x_1,x_2)=\begin{cases} u^-(x_1,x_2), & if\ (x_1,x_2)\in D^-, \\ u^0(x_1,x_2), & if\ (x_1,x_2)\in \Gamma^0, \\ u^+(x_1,x_2), & if\ (x_1,x_2)\in D^+, \end{cases} \tag{1}$$

where $u^-(x_1,x_2)\in C^2(D^-)$, $u^+(x_1,x_2)\in C^2(D^+)$, $u^0(x_1,x_2)\in C(\Gamma^0)$,

which satisfies the equations

$$L^- u^-(x_1,x_2) \equiv \sum_{\sim,\ =1}^{2} \frac{\partial}{\partial x_\sim}\left(K_\sim^-\ (x_1,x_2)\frac{\partial u^-}{\partial x}\right) - k^-(x_1,x_2)u^- = -f^-(x_1,x_2),\quad (x_1,x_2)\in D^-, \tag{2}$$

$$L^+ u^+(x_1,x_2) \equiv \sum_{\sim,\ =1}^{2} \frac{\partial}{\partial x_\sim}\left(K_\sim^+\ (x_1,x_2)\frac{\partial u^+}{\partial x}\right) - k^+(x_1,x_2)u^+ = -f^+(x_1,x_2),\quad (x_1,x_2)\in D^+, \tag{3}$$

where $f^-(x_1,x_2)$ and $f^+(x_1,x_2)$ are known, sufficiently smooth functions. Suppose that coefficients $K_\sim^\pm\ (\cdot)$, $\sim,\ \ =1,2$, and $k^\pm(\cdot)$ satisfy all the conditions, sufficient for Dirichlet problems in D^- and D^+ to have the unique solutions for any continuous initial functions.

Statement of the Problem

The function $u(x_1,x_2)$ also satisfies the boundary conditions

$$u^-(x_1,x_2)=\varphi^-(x_1,x_2),\quad (x_1,x_2)\in\gamma_1^-\cup\gamma_2^-\cup\gamma_3, \tag{4}$$

$$u^+(x_1,x_2)=\varphi^+(x_1,x_2),\quad (x_1,x_2)\in\gamma_1^+\cup\gamma_2^+\cup\gamma_4, \tag{5}$$

the nonlocal contact conditions

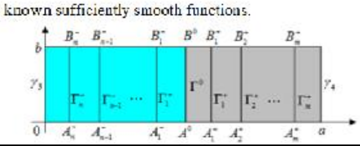
$$u^-(\Gamma^0)=u^+(\Gamma^0)=u(\Gamma^0)=\sum_{i=1}^{n}\beta_i^- u^-(\Gamma_i^-)+\sum_{j=1}^{m}\beta_j^+ u^+(\Gamma_j^+)+\varphi^0(\Gamma^0), \tag{6}$$

and the compatibility conditions

$$u(A^0)=\sum_{i=1}^{n}\beta_i^- u^-(A_i^-)+\sum_{j=1}^{m}\beta_j^+ u^+(A_j^+)+\varphi^0(A^0), \tag{7}$$

$$u(B^0)=\sum_{i=1}^{n}\beta_i^- u^-(B_i^-)+\sum_{j=1}^{m}\beta_j^+ u^+(B_j^+)+\varphi^0(B^0), \tag{8}$$

where $\beta_i^- - const>0$, $\beta_j^+ - const>0$, $0<\sum_{i=1}^{n}\beta_i^- +\sum_{j=1}^{m}\beta_j^+ <1$, $\varphi^0(\cdot)$, $\varphi^-(\cdot)$ and $\varphi^+(\cdot)$ are known sufficiently smooth functions.



Uniqueness of a Solution of Problem (1)-(8)

The following theorem is true.

Theorem 1. If the regular solution of problem (1)-(8) exists and condition $0<\sum_{i=1}^{n}\ _i^- +\sum_{j=1}^{m}\ _j^+ <1$ is fulfilled, then the solution is unique.

Proof. Suppose that problem (1)-(8) has two solutions: $v(x_1,x_2)$ and $w(x_1,x_2)$. Then for the function $z(x_1,x_2)=v(x_1,x_2)-w(x_1,x_2)$ we have the following problem

$$L^- z^-(x_1,x_2)=0,\quad if\ (x_1,x_2)\in D^-, \tag{9}$$

$$L^+ z^+(x_1,x_2)=0,\quad if\ (x_1,x_2)\in D^+, \tag{10}$$

$$\begin{aligned} z^-(x_1,x_2)=0,\quad & if\ (x_1,x_2)\in \text{¥}_1^-\cup\text{¥}_2^-\cup\text{¥}_3, \\ z^+(x_1,x_2)=0,\quad & if\ (x_1,x_2)\in \text{¥}_1^+\cup\text{¥}_2^+\cup\text{¥}_4, \end{aligned} \tag{11}$$

$$z(\Gamma_0)=z^-(\Gamma_0)=z^+(\Gamma_0)=\sum_{i=1}^{n}\ _i^- z^-(\Gamma_i^-)+\sum_{j=1}^{m}\ _i^+ z^+(\Gamma_j^+). \tag{12}$$

Uniqueness of a Solution of Problem (1)-(8)

From the equality (12) it follows that

$$\max\left|z(\Gamma_0)\right| \le \max \sum_{i=1}^{r} \beta_i^- \left|z^-\left(\Gamma_i^-\right)\right| + \max \sum_{j=1}^{m} \beta_j^+ \left|z^+\left(\Gamma_j^+\right)\right| \le$$

$$\le \max_{0\le i\le n}\left|z^-\left(\Gamma_i^-\right)\right| \cdot \sum_{i=1}^{n} \beta_i^- + \max_{0\le j\le m}\left|z^+\left(\Gamma_j^+\right)\right| \cdot \sum_{j=1}^{m} \beta_j^+ .$$

Taking into account the condition $0 < \sum_{i=1}^{r} \beta_i^- + \sum_{j=1}^{m} \beta_j^+ < 1$, we obtain

$$\max\left|z(\Gamma_0)\right| < \max_{0\le i\le n}\left|z^-\left(\Gamma_i^-\right)\right| \quad or \quad \max\left|z(\Gamma_0)\right| < \max_{0\le j\le m}\left|z^+\left(\Gamma_j^+\right)\right|$$

This means that the function z does not attain a maximum on Γ_0, but attains its maximum on D^- or D^+, that contradicts the maximum principle. So, $z = const$ and taking into account condition (11), we obtain $z = 0$, i.e. the solution of the problem (1)-(8) is unique. ◆

Iteration Process for Problem (1)-(8)

Let us consider the following iteration process for the problem (1)-(8):

$$L^-\left(u^-(x_1,x_2)\right)^{(k)} = -f^-(x_1,x_2), \quad if\ (x_1,x_2)\in D^-, \qquad (13)$$

$$L^+\left(u^+(x_1,x_2)\right)^{(k)} = -f^+(x_1,x_2), \quad if\ (x_1,x_2)\in D^+, \qquad (14)$$

$$\left(u^-(x_1,x_2)\right)^{(k)} = \cdot\cdot^{-}(x_1,x_2), \quad if\ (x_1,x_2)\in \yen_1^- \cup \yen_2^- \cup \yen_3, \qquad (15)$$

$$\left(u^+(x_1,x_2)\right)^{(k)} = \cdot\cdot^{+}(x_1,x_2), \quad if\ (x_1,x_2)\in \yen_1^+ \cup \yen_2^+ \cup \yen_4, \qquad (16)$$

$$u^{(k)}(\Gamma_0) = \left[u^-(\Gamma_0)\right]^{(k)} = \left[u^+(\Gamma_0)\right]^{(k)} = \sum_{i=1}^{n} {}_i^-\left[u^-\left(\Gamma_i^-\right)\right]^{(k-1)} + \sum_{j=1}^{m} {}_i^+\left[u^+\left(\Gamma_j^+\right)\right]^{(k-1)} + \cdot\cdot^{0}(\Gamma_0), \qquad (17)$$

where $k = 0,1,2,\dots$ and $\left(u^-\right)^{(-1)}\left(\Gamma_i^-\right) = 0$, $\left(u^+\right)^{(-1)}\left(\Gamma_j^+\right) = 0$, $i = \overline{1,n}$, $j = \overline{1,m}$.

Denote

$$\left[z^-(x_1,x_2)\right]^{(k)} = \left[u^-(x_1,x_2)\right]^{(k)} - u^-(x_1,x_2), \quad if \quad u^-(x_1,x_2)\in \overline{D}^-,$$

$$\left[z^+(x_1,x_2)\right]^{(k)} = \left[u^+(x_1,x_2)\right]^{(k)} - u^+(x_1,x_2), \quad if \quad u^+(x_1,x_2)\in \overline{D}^+.$$

Iteration Process for Problem (1)-(8)

Then for the function $z(x_1,x_2)$ we obtain the problem

$$L^-\left(z^-(x_1,x_2)\right)^{(k)} = 0, \quad if\ (x_1,x_2)\in D^-, \qquad (18)$$

$$L^+\left(z^+(x_1,x_2)\right)^{(k)} = 0, \quad if\ (x_1,x_2)\in D^+, \qquad (19)$$

$$\left(z^-(x_1,x_2)\right)^{(k)} = 0, \quad if\ (x_1,x_2)\in \yen_1^- \cup \yen_2^- \cup \yen_3, \qquad (20)$$

$$\left(z^+(x_1,x_2)\right)^{(k)} = 0, \quad if\ (x_1,x_2)\in \yen_1^+ \cup \yen_2^+ \cup \yen_4, \qquad (21)$$

$$\left[z(\Gamma_0)\right]^{(k)} = \left[z^-(\Gamma_0)\right]^{(k)} = \left[z^+(\Gamma_0)\right]^{(k)} = \sum_{i=1}^{n} {}_i^-\left[z^-\left(\Gamma_i^-\right)\right]^{(k-1)} + \sum_{j=1}^{m} {}_i^+\left[z^+\left(\Gamma_j^+\right)\right]^{(k-1)}, \qquad (22)$$

where $k = 0,1,2,\dots$ and $\left(z^-\right)^{(-1)}\left(\Gamma_i^-\right) = 0$, $\left(z^+\right)^{(-1)}\left(\Gamma_j^+\right) = 0$, $i = \overline{1,n}$, $j = \overline{1,m}$.

From equality (22) we have

$$\max_{\Gamma_0}\left|\left[z(\Gamma_0)\right]^{(k)}\right| \le \max_{1\le j\le m}\left|\left[z^+\left(\Gamma_j^+\right)\right]^{(k-1)}\right| \cdot \sum_{j=1}^{m} {}_i^+ + \max_{1\le i\le n}\left|\left[z^-\left(\Gamma_i^-\right)\right]^{(k-1)}\right| \cdot \sum_{i=1}^{n} {}_i^- .$$

Iteration Process for Problem (1)-(8)

If we use Schwarz' lemma, we will get inequalities:

$$\max_{1\le j\le m}\left|\left[z^+\left(\Gamma_j^-\right)\right]^{(k-1)}\right| \le q^+ \max_{\Gamma_0}\left|\left[z(\Gamma_0)\right]^{(k-1)}\right|, \qquad (23)$$

$$\max_{1\le i\le n}\left|\left[z^-\left(\Gamma_i^-\right)\right]^{(k-1)}\right| \le q^- \max_{\Gamma_0}\left|\left[z(\Gamma_0)\right]^{(k-1)}\right|, \qquad (24)$$

where $q^+ = const$, $0 < q^+ < 1$, $q^- = const$, $0 < q^- < 1$. Note, that these constants depend only on geometric properties of domains D^- and D^+.

If we use inequalities (23), (24), then we have

$$\max_{\Gamma_0}\left|\left[z(\Gamma_0)\right]^{(k)}\right| \le q^+ \cdot \sum_{j=1}^{m} \beta_i^+ \cdot \max_{\Gamma_0}\left|\left[z(\Gamma_0)\right]^{(k-1)}\right| + q^- \cdot \sum_{i=1}^{n} \beta_i^- \cdot \max_{\Gamma_0}\left|\left[z(\Gamma_0)\right]^{(k-1)}\right|$$

or
$$\max\left|\left[z(\Gamma_0)\right]^{(k)}\right| < Q \max_{\Gamma_0}\left|\left[z(\Gamma_0)\right]^{(k-1)}\right|, \qquad (25)$$

where $Q = q^+ \cdot \sum_{j=1}^{m} \beta_i^+ + q^- \cdot \sum_{i=1}^{n} \beta_i^-$.

Iteration Process for Problem (1)-(8)

Taking into account the conditions $\beta_i^- = const \geq 0$, $\beta_j^+ = const \geq 0$, $0 < \sum_{i=1}^{n} \beta_i^- + \sum_{j=1}^{m} \beta_j^+ < 1$, we obtain $0 < Q < 1$. This implies that

$$\lim_{k\to\infty} \left[z(\Gamma_0)\right]^{(k)} = 0 .$$

If the solution of problem (1)-(8) exists, then by the maximum principle we obtain

$$\max_{D^-} \left| \left[u^-(x_1,x_2)\right]^{(k)} - u^-(x_1,x_2) \right| = O\left(Q^k\right),$$

$$\max_{D^+} \left| \left[u^+(x_1,x_2)\right]^{(k)} - u^+(x_1,x_2) \right| = O\left(Q^k\right),$$

and, accordingly,

$$\max_{\overline{D}} \left| \left[u(x_1,x_2)\right]^{(k)} - u(x_1,x_2) \right| = O\left(Q^k\right).$$

Iteration Process for Problem (1)-(8)

Thereby we proved the following theorem.

Theorem 2. If the solution of problem (1)-(8) exists, then the iteration process (13)-(17) converges to this solution at the rate of an infinitely decreasing geometric progression.

Remark. By using the iteration algorithm (13)-(17) the solution of a nonclassical contact problem (1)-(8) is reduced to the solution of a sequence of classical Dirichlet problems.

For solving classical problems are developed powerful, well-tested numerical algorithms and programs, among them parallel methods.

Existence of a Solution of Problem (1)-(8)

Let us now prove the existence of a regular solution of the problem (1)-(8) in case $f^-(x_1,x_2) \equiv 0$ and $f^+(x_1,x_2) \equiv 0$. We introduce the notation $£^{(k)}(x_1,x_2) = u^{(k)}(x_1,x_2) - u^{(k-1)}(x_1,x_2)$. Then for the function $£^{(k)}$ we obtain the problem

$$L^-\left[£^-(x_1,x_2)\right]^{(k)} = 0, \quad if \quad (x_1,x_2) \in D^-, \tag{26}$$

$$L^+\left[£^+(x_1,x_2)\right]^{(k)} = 0, \quad if \quad (x_1,x_2) \in D^+, \tag{27}$$

$$\left[£^-(x_1,x_2)\right]^{(k)} = 0, \quad if \ (x_1,x_2) \in ¥_1^- \cup ¥_2^- \cup ¥_3, \tag{28}$$

$$\left[£^+(x_1,x_2)\right]^{(k)} = 0, \quad if \ (x_1,x_2) \in ¥_1^+ \cup ¥_2^+ \cup ¥_4, \tag{29}$$

$$\left[£(\Gamma_0)\right]^{(k)} = \left[£^-(\Gamma_0)\right]^{(k)} = \left[£^+(\Gamma_0)\right]^{(k)} = \sum_{i=1}^{n} {}_i^- \left[£^-\left(\Gamma_i^-\right)\right]^{(k-1)} + \sum_{j=1}^{m} {}_i^+ \left[£^+\left(\Gamma_j^+\right)\right]^{(k-1)}, \tag{30}$$

where $k = 0,1,2,\ldots$ and $\left[£^-\left(\Gamma_i^-\right)\right]^{(-1)} = 0$, $\left[£^+\left(\Gamma_j^+\right)\right]^{(-1)} = 0$, $i = \overline{1,n}$, $j = \overline{1,m}$.

Existence of a Solution of Problem (1)-(8)

Then, similarly to (25), we obtain the estimate

$$\max_{\Gamma_0} \left| \left[£(\Gamma_0)\right]^{(k)} \right| \leq Q \max_{\Gamma_0} \left| \left[£(\Gamma_0)\right]^{(k-1)} \right|, \quad 0 < Q < 1,$$

or

$$\max_{\Gamma_0} \left| u^{(k)}(\Gamma_0) - u^{(k-1)}(\Gamma_0) \right| \leq Q \max_{\Gamma_0} \left| u^{(k-1)}(\Gamma_0) - u^{(k-2)}(\Gamma_0) \right|, \quad 0 < Q < 1.$$

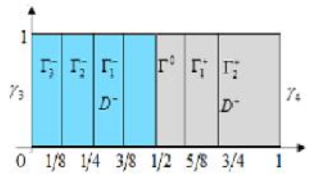
This means that the sequence $\left\{u^{(k)}(x_1,x_2)\right\}$ converges uniformly on Γ_0. Then the functions $\left[u^+(x_1,x_2)\right]^{(k)}$ and $\left[u^-(x_1,x_2)\right]^{(k)}$ converge to the functions $u^+(x_1,x_2)$ and $u^-(x_1,x_2)$, respectively, for the domains D^- and D^+ on the base of Harnak's first theorem [22].

From this we conclude that the limit function is the regular solution of the problem (1)-(8):

$$\lim_{k\to\infty} u^{(k)}(x_1,x_2) = u(x_1,x_2).$$

Example

Let us consider the area $D=\{(x_1,x_2)\,|\,0<x_1<1,\quad 0<x_2<1\}$



find in $\overline{D}$ a continuous function $u(x_1,x_2)$,

$$u(x_1,x_2)=\begin{cases} u^-(x_1,x_2), & \text{if } (x_1,x_2)\in D^-, \\ u^0(x_1,x_2), & \text{if } (x_1,x_2)\in \Gamma_0, \\ u^+(x_1,x_2), & \text{if } (x_1,x_2)\in D^+, \end{cases} \qquad (31)$$

Example

which satisfies the equations

$$\Delta u^-(x_1,x_2)=2x_1^2(x_1-1)(3x_2-1)+2x_2^2(3x_1-1)(x_2-1),\quad \text{if } (x_1,x_2)\in D^-,$$

$$\Delta u^+(x_1,x_2)=2x_1x_2^2(x_2-1)+\frac{2}{3}x_1(x_1^2-1)(3x_2-1),\quad \text{if } (x_1,x_2)\in D^+,$$

The function $u(x_1,x_2)$ also satisfies the boundary conditions

$$u^-(x_1,x_2)=0,\quad \text{if } (x_1,x_2)\in \gamma_1^-\cup\gamma_2^-\cup\gamma_3,$$

$$u^+(x_1,x_2)=0,\quad \text{if } (x_1,x_2)\in \gamma_1^+\cup\gamma_2^+\cup\gamma_4,$$

the nonlocal contact condition

$$u^-\left(\frac{1}{2},x_2\right)=u^+\left(\frac{1}{2},x_2\right)=u(\Gamma_0)=\frac{1}{8}u^-\left(\frac{1}{8},x_2\right)+\frac{1}{8}u^-\left(\frac{1}{4},x_2\right)+$$

$$+\frac{1}{8}u^-\left(\frac{3}{8},x_2\right)+\frac{1}{8}u^+\left(\frac{3}{4},x_2\right)+\frac{1}{8}u^+\left(\frac{5}{8},x_2\right)-\frac{315}{4096}x_2^2(x_2-1)$$

and the compatibility conditions are fulfilled.

Example

Let us consider the following iteration process:

$$\Delta\left(u^-(x_1,x_2)\right)^{(k)}=2x_1^2(x_1-1)(3x_2-1)+2x_2^2(3x_1-1)(x_2-1),\quad \text{if } (x_1,x_2)\in D^-,$$

$$\Delta\left(u^+(x_1,x_2)\right)^{(k)}=2x_1x_2^2(x_2-1)-\frac{2}{3}x_1(x_1^2-1)(3x_2-1),\quad \text{if } (x_1,x_2)\in D^+,$$

$$\left(u^-(x_1,x_2)\right)^{(k)}=0,\quad \text{if } (x_1,x_2)\in\gamma_1^-\cup\gamma_2^-\cup\gamma_3,\qquad \left(u^+(x_1,x_2)\right)^{(k)}=0,\quad \text{if } (x_1,x_2)\in\gamma_1^+\cup\gamma_2^+\cup\gamma_4,$$

$$u^{(k)}(\Gamma_0)=\left[u^-(\Gamma_0)\right]^{(k)}=\left[u^+(\Gamma_0)\right]^{(k)}=\frac{1}{8}\left[u^-\left(\frac{1}{8},x_2\right)\right]^{(k-1)}+\frac{1}{8}\left[u^-\left(\frac{1}{4},x_2\right)\right]^{(k-1)}+\frac{1}{8}\left[u^-\left(\frac{3}{8},x_2\right)\right]^{(k-1)}+$$

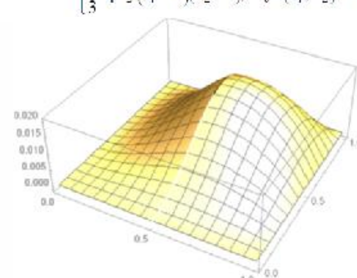
$$+\frac{1}{8}\left[u^+\left(\frac{3}{4},x_2\right)\right]^{(k-1)}+\frac{1}{8}\left[u^+\left(\frac{5}{8},x_2\right)\right]^{(k-1)}-\frac{315}{4096}x_2^2(x_2-1),$$

where $k=0,1,2,\ldots$ The initial value for $u^{(k)}(\Gamma_0)$ is equal to 0

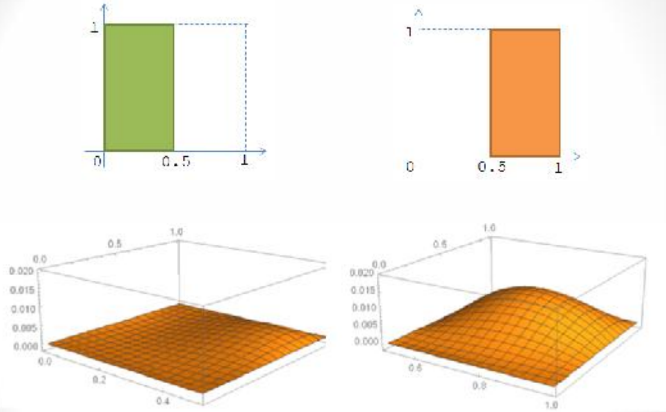
The Calculation Results

The exact solution of this problem is

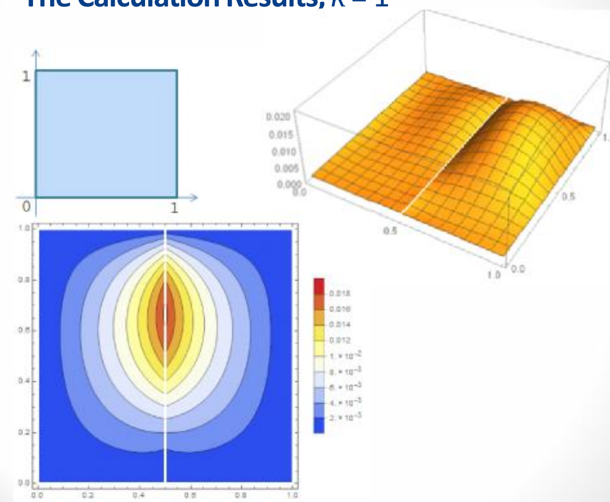
$$u(x_1,x_2)=\begin{cases} x_1^2x_2^2(x_1-1)(x_2-1), & \text{if } (x_1,x_2)\in D^-, \\ -\frac{1}{8}x_2^2(x_2-1), & \text{if } (x_1,x_2)\in \Gamma_0, \\ \frac{1}{3}x_1x_2^2(x_1^2-1)(x_2-1), & \text{if } (x_1,x_2)\in D^+, \end{cases}$$



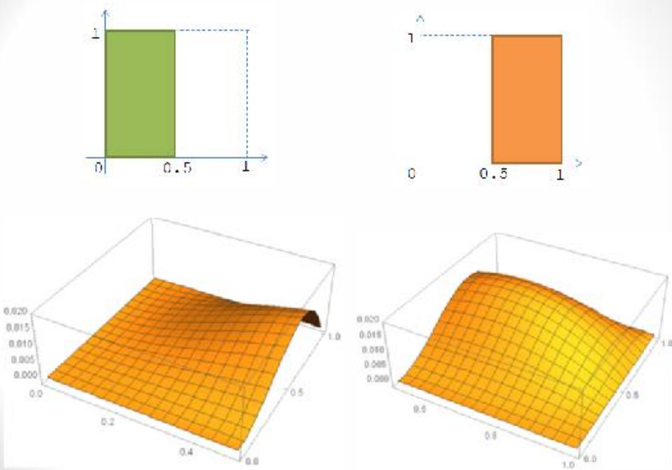
The Calculation Results, $k = 1$



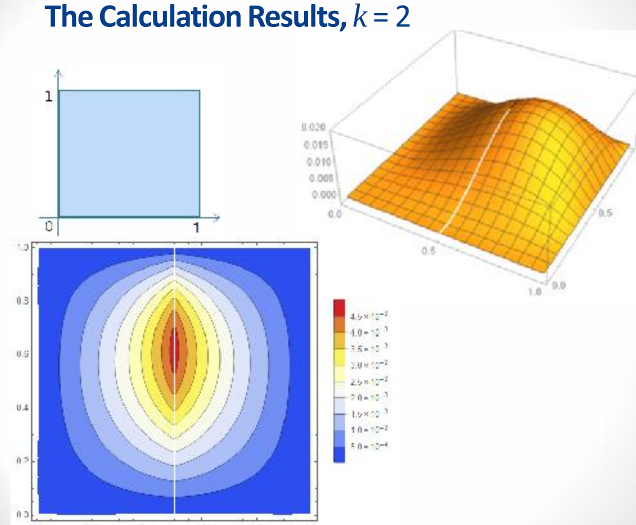
The Calculation Results, $k = 1$



The Calculation Results, $k = 2$

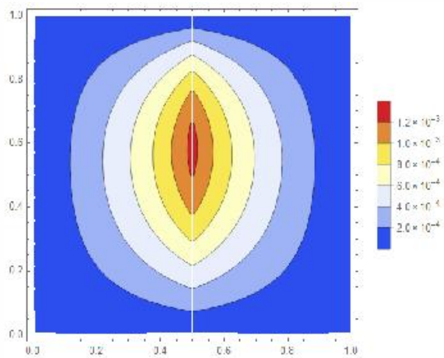


The Calculation Results, $k = 2$



The Calculation Results

Difference between exact and approximate solutions, $k = 3$



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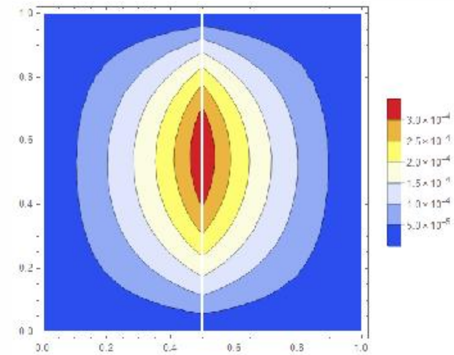
The Calculation Results

Difference between exact and approximate solutions, $k = 4$



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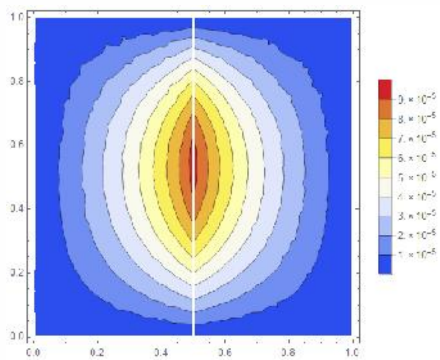
The Calculation Results

Difference between exact and approximate solutions, $k = 5$



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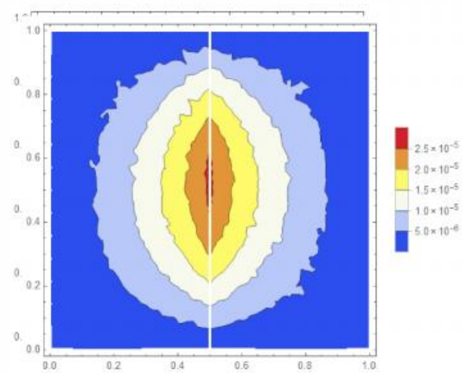
The Calculation Results

Difference between exact and approximate solutions, $k = 6$



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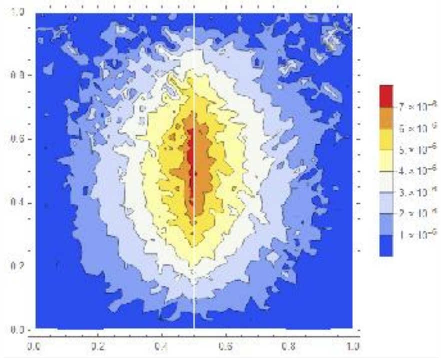
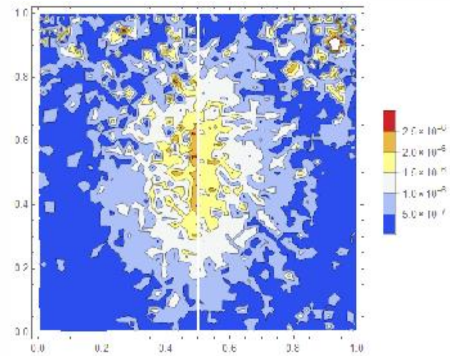
The Calculation Results

Difference between exact and approximate solutions, $k = 7$



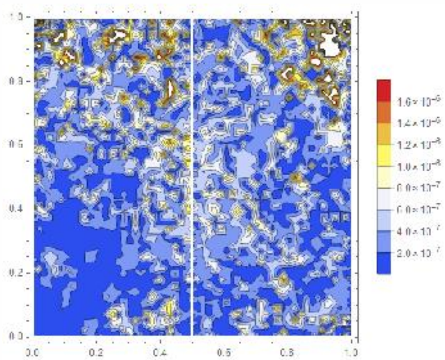
The Calculation Results

Difference between exact and approximate solutions, $k = 8$



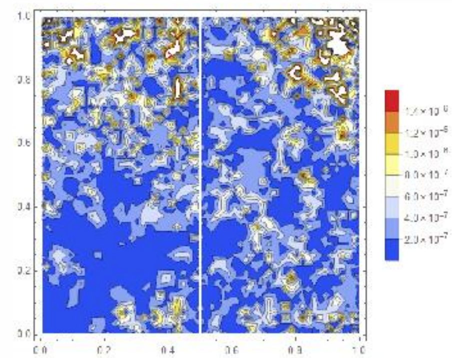
The Calculation Results

Difference between exact and approximate solutions, $k = 9$



The Calculation Results

Difference between exact and approximate solutions, $k = 10$



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Thank you for your attention