Investigation and numerical solution of some 3D internal dirichlet generalized harmonic problems for finite domains

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Abstract

A Dirichlet generalized harmonic problem for finite right circular cylindrical domains is considered. The term "generalized" indicates that a boundary function has a finite number of first kind discontinuity curves. It is shown that if a finite domain is bounded by several surfaces and the mentioned curves are disposed in arbitrary form, then the generalized problem has a unique solution depending continuously on the data. The problem is considered for the case when the curves of discontinuity are circles with centers situated on the axis of the cylinder. An algorithm of numerical solution by the probabilistic method is given, which in its turn is based on a computer simulation of the Wiener process. A numerical example is considered to illustrate the effectiveness and simplicity of the proposed method.

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1. Introduction

It is known (see e.g., [1, 2, 3, 4, 5]) that in practical stationary problems (for example, for determination of temperature of the thermal field or the potential of the electric field, and so on) there are cases when the corresponding boundary function has a finite number of first kind discontinuity points (in the case of 2D) or curves (in the case of 3D). Problems of such type are known as Dirichlet generalized problems [1], and their solutions represent generalized solutions, respectively. In general, it is known (see [3, 6]) that methods used to obtain an approximate solution to ordinary boundary problems are less suitable (or not suitable at all) for solving boundary problems with singularities. In particular, the convergence is very slow in the neighborhood of boundary singularities and, consequently, the accuracy of the approximate solution of the generalized problem is very low.

Choice and construction of computational schemes (algorithms) mainly depend on problem class, its dimension, geometry and location of singularities on the boundary. E.g., plane Dirichlet generalized problems for harmonic functions with concrete location of discontinuity points in the cases of simply connected domains are considered in [3, 7], and general cases for finite and infinite domains are studied in [8, 9, 10, 11, 12].

In the case of spatial (3D) harmonic generalized problems, due to higher dimension, the difficulties become more significant. On the other hand, study of such problems from the viewpoint of correctness and approximate solution is of certain interest, since in the nature and practice, some processes occur whose investigation is reduced to solution of problems of the indicated type (see e.g., [3, 4]). In the case of 3D, there does not exist a standard scheme which can be applied to a wide class of domains. In the classical literature, simplified, or so called "solvable" generalized problems (problems whose "exact" solutions can be constructed by series, whose terms are represented by special functions) are considered, and for their solution classical method of separation of variables is mainly applied and therefore the accuracy of the solution is sufficiently low. In particular, in the mentioned problems, boundary functions

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(conditions) are mainly constants, and in general case, the analytic form of the "exact" solution is so difficult in the sense of numerical implementation, that it has only theoretical significance (see e.g., [5]).

As a consequence of the above mentioned, from our viewpoint, construction of high accuracy and effectively realizable computational schemes for approximate solution of 3D generalized harmonic problems (whose application is possible to a wide class of domains) has both theoretical and practical importance and belongs to a number of actual problems.

2. Statement of the Problem and Properties of its Solution

Let D be a finite right circular cylindrical domain in the Euclidian space E_3, bounded by surface S. Without loss of generality we assume that coordinate axis ox_3 of Cartesian coordinates $ox_1x_2x_3$ is taken in the role of the axis of the cylinder D. We consider the Dirichlet generalized problem for the Laplace equation.

Problem A. Function $g(y)$ is given on the boundary S of the domain D and is continuous everywhere, except a finite number of circles $l_1, l_2, \dots, l_n$ which represent discontinuity curves of the first kind for the function $g(y)$. Besides, it is assumed that the centers of the mentioned circles are situated on the axis of the cylinder D. It is required to find a function $u(x) \equiv u(x_1, x_2, x_3) \in C^2(D) \bigcap C(\overline{D} \setminus \bigcup_{k=1}^{n} l_k)$ satisfying the conditions

$$\Delta u(x) = 0, \qquad x \in D, \tag{2.1}$$

$$u(y) = g(y), \qquad y \in S, \quad y \,\overline{\in}\, l_k \ \ (k = 1, 2, \dots, n), \tag{2.2}$$

$$|u(y)| < c, \qquad\qquad y \in \overline{D}, \tag{2.3}$$

where $\Delta = \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator and c is a real constant.

For the sake of simplicity, in the following we assume that the circles $l_k (k = \overline{1, n})$ are situated on S preserving the order of succession in the direction of axis ox_3. It is evident that the surface S is divided into parts $S_k (k = 1, \overline{n+1})$ by circles l_k or $S = \bigcup_{k=1}^{n+1} S_k$. On the basis of the mentioned above the boundary function $g(y)$ has the following form

$$g(y) = \begin{cases} g_1(y), & y \in S_1, \\ g_2(y), & y \in S_2, \\ \dots\dots\dots\dots\dots & \\ g_{n+1}(y), & y \in S_{n+1}, \end{cases} \tag{2.4}$$

where the functions $g_k(y) = g_k(y_1, y_2, y_3), y \in S_k$ are continuous on the parts S_k of S, respectively.

Note that the additional requirement (2.3) of boundedness concerns actually only the neighborhoods of discontinuity curves of the function $g(y)$ and it plays an important role in the extremum principle (see Theorem 1).

Remark 1. If inside the surface S there is a vacuum then we have the generalized problem with respect to a right circular cylindrical shell.

In order to study properties of solution of Problem ((2.1), (2.2), (2.3)), we will firstly prove generalized extremum principle in more general case. Let us consider in space E_3 finite domain D with surface S (D may be bounded by several surfaces).

Theorem 1. *If function $u(x)$ is harmonic in D, bounded in $\overline{D}$ and takes a value $g(y)$ on the boundary S, which is continuous on S everywhere, except a finite number of curves $l_1, l_2, \dots, l_n$ (with discontinuities of first kind), then*

$$\min_{x \in S} u(x) < \underset{x \in D}{u(x)} < \max_{x \in S} u(x), \tag{2.5}$$

where for $x \in S$ it is meant that $x \,\overline{\in}\, l_k$ ($k = \overline{1, n}$).

Proof. Let $M = \max u(x)$, $x \in S'$, $S' = S \setminus \bigcup_{k=1}^{n} l_k$ and consider function

$$v(x) = M + \varepsilon \sum_{k=1}^{n} \frac{1}{r_k}, \quad x \in D. \tag{2.6}$$

In (2.6): ε is an arbitrary positive number, r_k is minimal distance from the considered point x to k-th curve of discontinuity l_k or $r_k = \min \rho(x; y^k)$, where y^k- is a current point of curve l_k. Evidently, function $v(x)$ is harmonic and more than M in D, continuous in $\overline{D}$ everywhere, except curves l_k and $\lim v(x) = \infty$ for $x \to l_k$. Assume that $C(y^k, \delta)$ are current spheres with radius δ and with centers at current points y^k of the curves $l_k (k = \overline{1, n})$. At passing by point $y^k$the line l_k or sphere $C(y^k, \delta)$ we obtain certain domain T_k. It is evident that actually T_k is a closed pipe if l_k is a closed curve and T_k is an open pipe which is terminated with hemispheres, if l_k is an open curve. Also $T_k \to l_k$ when $\delta \to 0$.

Let us consider closed domain $\overline{D_\delta} = \overline{D} \setminus \bigcup_{k=1}^{n} T_k$. Function $v(x) - u(x)$ is continuous in $\overline{D}_\delta$, harmonic in D_δ and $v(x) - u(x) > 0$ on the common part of boundaries D and D_δ. For sufficiently small δ the above inequality is also valid on surfaces of domains T_k (since function $u(x)$ is bounded in $\overline{D}$ and for $\delta \to 0$ values $v(x)$ increase infinitely on the surfaces of domains T_k). Thus, on the basis of usual extremum principle we have $u(x) < v(x)$, $x \in D_\delta$, and consequently

$$u(x) < v(x), \quad x \in D. \tag{2.7}$$

Indeed, any point x from domain D belongs to some domain D_δ for arbitrarily small δ .

Since $u(x)$ does not depend on ε, from (2.7) we obtain $u(x) < M, x \in D$ or

$$\underset{x \in D}{u(x)} < \max_{x \in S'} u(x)$$

for any fixed point x of domain D when $\varepsilon \to 0$.

Now, if in the role of function $v(x)$ we take

$$v(x) = m - \varepsilon \sum_{k=1}^{n} \frac{1}{r_k}, \quad x \in D,$$

where $m = \min u(x)$, $x \in S'$, then the inequality

$$\underset{x \in D}{u(x)} > \min_{x \in S'} u(x),$$

can be proved similarly.

Thus, for the solution of Problem A, generalized extremum principle (2.5) is valid. □

It should be noted that the following results can be obtained from theorem 1.

Corollary 1. *If generalized functions* (*in the sense of Theorem* 1) *$u(x)$ and $v(x)$ are harmonic in D, continuous in $D' = \overline{D} \setminus \bigcup_{k=1}^{n} l_k$ and if $u(x) \leq v(x)$ on S', then $u(x) \leq v(x)$, $x \in D$.*

Indeed, function $v(x) - u(x)$ is continuous on S' and harmonic in D and $v(x) - u(x) \geq$ on S'. Due to Theorem 1 $v(x) - u(x) \geq 0 \;\; x \in D$ or $u(x) \leq v(x)$, $x \in D$.

Corollary 2. *If functions $u(x)$ and $v(x)$ are harmonic in D and continuous in D', and if $|u(x)| \leq v(x)$ on S', then $|u(x)| \leq v(x)$, $x \in D$.*

From the conditions it follows that $-v(x) \leq u(x) \leq v(x)$, $x \in S'$.

Applying twice Corollary 1, we have $-v(x) \leq u(x) \leq v(x)$, $x \in D$ or $|u(x)| \leq v(x)$, $x \in D$.

Corollary 3. *For function $u(x)$ which is harmonic in D and continuous in D' the inequality $|u(x)| \leq \max |u| \,|_{S'}$, $x \in D'$ is valid.*

In order to prove this we put $v = \max |u| \,|_{S'}$ and use Corollary 2.

Now theorem on uniqueness of solution of boundary Problem A can be easily proved.

Theorem 2. *The generalized spatial inner Dirichlet problem for Laplace equation cannot have two different solutions.*

Proof. Assume that there exist two different functions $u_1(x)$ and $u_2(x)$, satisfying conditions of the problem. Their difference $u(x) = u_1(x) - u_2(x)$ is harmonic in domain D, bounded in $\overline{D}$ and $u(x) = 0$, $x \in S^{'}$, From Theorem 1 it follows, that $u(x) = 0, x \in D$, i.e. $u_1(x) = u_2(x), x \in D$. The theorem is proved. □

Theorem 3. *The solution of generalized spatial inner Dirichlet problem for Laplace equation depends continuously on boundary data.*

Proof. It is known [2], that a problem is called physically definite (or stable), if a small change of conditions, determining problem solution(boundary conditions in the given case), causes small change of the solution itself.

Let $u_1(x)$ and $u_2(x)$ be generalized solutions of the problem under condition

$$|u_1(x) - u_2(x)| \leq \varepsilon, \quad x \in S^{'}, \tag{2.8}$$

then the same inequality is true in D. Indeed, the functions $u(x) = u_1(x) - u_2(x)$ and $v(x) = \varepsilon$ are harmonic in D and continuous in $D^{'}$, therefore due to Corollary 2 of Theorem 1, inequality (2.8) is valid in D.

Thus the theorem is proved. □

3. A Method of Probabilistic Solution

It is known [13] that a relation between the theory of probability and the Dirichlet problem for the Laplace's equation was noted long before general theory of Markove's processes arose (the works by G. Phillips and N. Wiener (1923), R. Courant, K. Fredrichs and Kh. Levi (1928)). This idea got a wide development in works of A.Ya. Khintchin (1933) and I.G. Petrovski (1934).

The mentioned idea obtained a finished form by E.B. Duenkin [13]. He obtained a formula which expresses the relation between a solution of Dirichlet ordinary(or generalized) boundary problem for the Laplace's equation and the Wiener (diffusion) process, when problem dimension $n \geq 2$.

In particular, E.B.Duenkin proved a general theorem which for $n = 3$ consists in the following.

Theorem 4. *If a finite domain $D \in E_3$ is bounded by piecewise smooth surface S and $g(y)$ is continuous (or discontinuous) bounded function on S, then the solution of the Dirichlet ordinary (or generalized) boundary problem for the Laplace equation at the fixed point $x \in D$ has the form*

$$u(x) = M_x g(x(t)). \tag{3.1}$$

In (3.1): $M_x g(x(t))$ is the mathematical expectation of the values of the boundary function $g(y)$ at the random intersection points of the Wiener process and the boundary S; t is the moment of first exit of the Wiener process $x(t) = (x_1(t), x_2(t), x_3(t))$ from the domain D. It is assumed that the starting point of the Wiener process is always $x(t_0) = (x_1(t_0), x_2(t_0), x_3(t_0)) \in D$, where the value of the desired function is being determined. If the number N of the random intersection points $y^i = (y_1^i, y_2^i, y_3^i) \in S$ $(i = 1, 2, \ldots, N)$ is sufficiently large, then according to the law of large numbers, from (3.1) we have

$$u(x) \approx u_N(x) = \frac{1}{N} \sum_{i=1}^{N} g(y^i) \tag{3.2}$$

or $u(x) = \lim u_N(x)$ for $N \to \infty$, in the probabilistic sense. Thus, in the presence of the Wiener process the approximate value of the probabilistic solution to Problem A at a point $x \in D$ is calculated by formula (3.2).

Thus, on the basis of Theorem 4, existence of solution of Dirichlet generalized problem in the case of Laplace equation for a sufficiently wide class of domains is shown. Besides, we have also an explicit formula giving such solution.

Remark 2. If the finite domain D is bounded by several surfaces (or $S = \bigcup_{k=1}^{m} S^k$ and $S^k \cap S^j$ for $k \neq j$), then instead of formula (3.2) we will have the following formula

$$u(x) \approx u_N(x) = \frac{1}{N} \sum_{k=1}^{m} \sum_{i=1}^{N_k} g^k(y^{k,i}). \tag{3.3}$$

In (3.3): $N = N_1 + N_2 + \cdots + N_m$; $g^k(y)$ is boundary function on S^k; N_k is number of the intersection points $y^{k,i}$ ($k = \overline{1,m}$; $i = \overline{1,N_k}$) of the Wiener process and of the surface S^k. It is evident, in noted case it is not necessary for discontinuity curves to be situated on all S^k.

Analogously to the considered cases(see [14, 15, 16, 17, 18]), on the basis of Theorem 4, probabilistic solution of Problem A consists in realization of the Wiener process using the three-dimensional generator, which gives three independent values $w_1(t), w_2(t), w_3(t)$. In our case the Wiener process is realized by computer simulation. In particular, for the computer simulation of the Wiener process we use the following recursion relations

$$\begin{gathered} x_1(t_k) = x_1(t_{k-1}) + w_1(t_k)/kv, \\ x_2(t_k) = x_2(t_{k-1}) + w_2(t_k)/kv, \\ x_3(t_k) = x_3(t_{k-1}) + w_3(t_k)/kv, \\ (k = 1, 2, \dots), \end{gathered} \tag{3.4}$$

with the help of which coordinates of a current point $x(t_k) = (x_1(t_k), x_2(t_k), x_3(t_k))$ are being determined. In (3.4): $w_1(t_k), w_2(t_k), w_3(t_k)$ are three normally distributed independent random numbers for $k-th$ step, with zero means and variances one; kv is a number of the quantification and when $kv \to \infty$, then the discrete Wiener process approaches the continuous Wiener process. In the computer, the random process is simulated at each step of the walk and continues until it crosses the boundary .

It is known that there exist two principles of generating random numbers, physical and programmatic:

1. Physical principle of generation gives real random numbers but its realization is connected with heavy expenses of time and material (financial) resource, especially in the case of multidimensional case, and therefore its application is not reasonable.

2. In spite of a great number of methods of program generation of random numbers, they also have disadvantages which are contained in the generating principle itself. Firstly, they are pseudo-random, and not real random numbers. Besides, we can notice periodicity at generating such numbers of one dimensional series, in other words, in the case of a large array the sequence is repeated, in the case of generation of 2 and 3 dimensional pseudo-random sequences (vectors), a correlation between them is added to the mentioned above, which, as it is known, cannot be "extinguished" between pseudo-random numbers generators. In spite of this, when solving Dirichlet boundary problems for Laplace equation it is possible to use pseudo-random numbers. In our case computations and generation of pseudo-random numbers are realized in the environment of the MATLAB system.

4. Numerical Example

In the role of the numerical example we considered such simple case, which is considered in monographs [3, 4] and is solved by method of separation of variables. In particular, Problem A is considered for the finite right circular cylinder $D(0 \le r \le a, 0 \le x_3 \le h)$, in which $n = 2$ and l_1, l_2 are circles of bases of the cylinder. Besides, it is assumed that the boundary function $g(y)$ (potential) has the form

$$g(y) = \begin{cases} 0, & y \in S_1, \\ v = const, & y \in S_2, \\ 0, & y \in S_3, \end{cases} \tag{4.1}$$

where S_1, S_3 are bases and S_2 is the lateral surface of the cylinder, respectively. In [3] it is noted that fields of these type are in electron- optical apparatuses, and for application the potential of the electric field on the axis of cylinder represents a principle interest.

a). In the conditions (4.1) the "exact" solution to Problem A obtained by G.Grinberg and W.R. Smythe has the following form (in cylindrical coordinates)

$$W(r, x_3) = \frac{4v}{\pi} \sum_{k=0}^{\infty} \frac{I_o[\frac{(2k+1)\pi r}{h}]}{I_o[\frac{(2k+1)\pi a}{h}]} \frac{\sin\frac{(2k+1)\pi x_3}{h}}{2k+1} \equiv \sum_{k=0}^{\infty} \omega_k(r, x_3), \tag{4.2}$$

Table 1:

(r,x_3) $r=0$	$w_m(r,x_3)$ $m=10$	(r,x_3) $r=0.5$	$w_m(r,x_3)$ $m=1000$	$w_m(r,x_3)$ $m=4000$
0.0001	0.000252825	0.0001	0.391704	1.13433
0.0005	0.00126412	0.0005	1.17898	0.94994
0.001	0.00252824	0.001	0.902825	0.974749
0.005	0.0126403	0.005	0.979786	0.994937
0.01	0.0252753	0.01	0.989891	0.997472
0.05	0.125528	0.05	0.998065	0.999516
0.1	0.245902	0.1	0.999167	0.999792
0.2	0.454543	0.2	0.999832	0.999958
0.3	0.604858	0.3	1.00012	1.00003
0.4	0.69272	0.4	1.00027	1.00007
0.5	0.721326	0.5	1.00032	1.00008

where $r=(x_1^2+x_2^2)^{\frac{1}{2}}$, h is a height of the cylinder, and a is a radius of the bases. In (4.2) $I_0(x)$ is Bessel's function of order zero from pure imaginary argument. Namely,

$$\begin{gathered} I_0(x)\equiv J_0(ix)=\sum_{n=0}^{\infty}\frac{(\frac{x}{2})^{2n}}{(n!)^2}, \quad \text{where} \quad x\in R, \\ I_0(0)=1 \quad \text{and} \quad I_0(x)\to\frac{e^x}{\sqrt{2\pi x}} \quad \text{for} \quad x\to\infty. \end{gathered} \tag{4.3}$$

It is evident, that for the solution $W(r,x)$ boundary conditions are satisfied on the bases S_1 and S_3 or $W(r,0)=W(r,h)=0$, where $0\le r\le a$.

On the basis of (4.3), it is easy to see that series (4.2) converges rapidly for all points $(r,x_3)\in D$, when $0\le r<a$, especially for $r=0$. If $r=a$, then the rate of convergence becomes worse on S_2 , especially in the neighborhood of curves l_1 and l_2 (i. e., when $(r,x_3)\in S_2$ and $x_3\to 0$ or $x_3\to h$). In particular, the convergence is very slow and consequently, the accuracy of satisfiability of boundary condition on S_2 is very low. This circumstance is caused with the fact that, when $x_3\to 0$ or $x_3\to h$, all terms of series (4.2) tend to zero.

Besides, it should be noted that the methods which are considered in [3, 4], can be applied to solution of Problem A only when discontinuity curves are circles of bases of the cylinder. In particular, if $n=2$, then l_1,l_2 are circles of bases of the cylinder, and if $n=1$, then l_1 is one of noted circles.

Since boundary condition (4.1) is independent of an angle of rotation with respect to ox_3 and symmetric with respect to the plane $x_3=\frac{h}{2}$, the potential has the same properties. In numerical experiments we took: $v=1$, $h=1$, $a=0.5$.

In Table 1 the results of calculations for the sum of the first $m+1$ terms of the series (4.2) (which is denoted by $w_m(r,x_3)$) are given.

In Table 1, because of the above- mentioned, $w_m(r,x_3)$ is calculated at the points (r,x_3) ($r=0,\ 0.5$ and $0<x_3\le 0.5$) which represent a certain interest for us. The numerical calculations have shown that practically $w_{10}(0,x_3)=w_m(0,x_3)$ when $m>10$, therefore in Table 1 the results of calculations are given only for $m=10$. For example, $\omega_{11}(0;0.0001)\approx 0.1*10^{-17}$, and $\omega_{101}(0;0.5)\approx -0.7*10^{-139}$ (see (4.2)).

It should be noted that in spite of the low accuracy of the solution $W(r,x_3)$ (on the basis of extremum principle and condition (4.1)) $|u(x)-W(r,x_3)|$ is minimal on axis, where $u(x)$ is the exact solution of Problem A.

b). In order to determine the intersection points $y^i=(y_1^i,y_2^i,y_3^i)$ $(i=1,2,\dots,N)$ of the Wiener process and of the surface S, we operate in the following way. During the realization of the Wiener process, for each current point $x(t_k)$, defined from (3.4), its location with respect to S is checked. In particular: if $x(t_k)\in D$ then the Wiener process is continued by (3.4); if $x(t_k)\in S$ then $y^i=x(t_k)$, in this case, if $y^i\in l_1$ or $y^i\in l_2$ then we always assume that $y^i\in S_1$ or $y^i\in S_2$, respectively.

Let $x(t_{k-1})\in D$ for the moment $t=t_{k-1}$ and $x(t_k)\overline{\in}\overline{D}$ for the moment $t=t_k$. In this case, for an approximate determination of the point y^i, an equation of a line l passing through the points $x(t_{k-1})$ and $x(t_k)$ is written firstly. For

Table 2:

$u_N(0,0,x_3)$				
$(0,0,x_3)$	$kv = 200$ $N = 50000$	$kv = 200$ $N = 100000$	$kv = 200$ $N = 200000$	$kv = 400$ $N = 200000$
$(0,0,0.0001)$	0.0096	0.0090	0.0092	0.0041
$(0,0,0.0005)$	0.0097	0.0095	0.0098	0.0058
$(0,0,0.001)$	0.0110	0.0108	0.0106	0.0064
$(0,0,0.005)$	0.0200	0.0197	0.0197	0.0157
$(0,0,0.01)$	0.0325	0.0322	0.0324	0.0285
$(0,0,0.05)$	0.1313	0.1322	0.1312	0.1287
$(0,0,0.1)$	0.2511	0.2515	0.2514	0.2477
$(0,0,0.2)$	0.4586	0.4587	0.4576	0.4554
$(0,0,0.3)$	0.6078	0.6037	0.6063	0.6055
$(0,0,0.4)$	0.6914	0.6961	0.6940	0.6908
$(0,0,0.5)$	0.7222	0.7194	0.7197	0.7210

Table 3:

$u_N(0,0,x_3),\ \ N = 200000$				
$(0,0,x_3)$	$kv = 1000$	$kv = 2000$	$kv = 4000$	$kv = 8000$
$(0,0,0.0001)$	0.0019	0.0010	0.0005	0.00034
$(0,0,0.0005)$	0.0028	0.0018	0.0016	0.0015
$(0,0,0.001)$	0.0037	0.0033	0.0031	0.0027
$(0,0,0.005)$	0.0141	0.0133	0.0121	
$(0,0,0.01)$	0.0268	0.0254	0.0253	
$(0,0,0.05)$	0.1262			
$(0,0,0.1)$	0.2455			

the intersection point y^i we have three cases: 1) $y^i = l \cap S_1$; 2) $y^i = l \cap S_3$; 3) $y^i = l \cap S_2$, in this case, if we have two intersection points x^* and x^{**} of the line l and the surface S_2, then in the role of the point y^i we choose the one (from x^* and x^{**}) for which $|x(t_k) - x|$ is minimal.

The results of the probabilistic solution to Problem A for cylinder D with boundary function (4.1) (calculated by formula (3.2)) are given in Tables 2 and 3. The numerical solutions $u_N(0,0,x_3)$ are found at the same points of axis for various N and kv, where N is the Wiener process realization number, and kv is the number of the quantification.

The analysis of the results of numerical experiments show the following (see Tables 2 and 3): if the point $x(t_o)$ (at which the approximate solution of Problem A must be determined) is situated at a small distance from surface S, then current point $x(t_k)$ must be under condition of a random walk in D until it crosses S. To get this, the number kv must be taken sufficiently large.

Though, we have solved Problem A for $n = 2$, its solution under condition (2.4) is not difficult. Indeed, after finding the intersection point y^i of Wiener process and surface S, it is easy to establish the part of S in which the point y^i is situated. Moreover, in general, we can solve Problem A for all such locations of discontinuity curves, which give possibility to establish the part of surface S where the intersection point is located.

From Tables 1,2,3 and the above mentioned it is clear that the results obtained by probabilistic method are reliable, and this method is effective for numerical solution of problems of type A. In particular, the algorithm is sufficiently simple for numerical implementation.

It should be noted that if we apply the method of parallel programming to probabilistic solution of Problem A, then we will avoid those difficulties which are noted in point 2 of section 3. Consequently, significantly less time will be needed for numerical realization and besides the accuracy of the obtained results will increase.

5. Concluding Remarks on Application of Probability Method

1. The method is suitable for approximate solution of both ordinary and generalized Dirichlet problems for rather a wide class of domains, in the case of Laplace equation. The results obtained using this method are reliable and characterized by an accuracy which is sufficient for many problems (see [14, 15, 16, 17, 18]).

2. The method is very simple and does not require high professional qualification in numerical methods and programming. Accordingly, it satisfies modern requests to numerical methods and algorithms.

3. In the presence of modern computing technology, it can be noted that the probability method represent one of the universal method especially for numerical solution of problems of type A.

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