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SYMBOLIC ESTIMATION OF DISTANCES BETWEEN EIGENVALUES OF HERMITIAN OPERATOR

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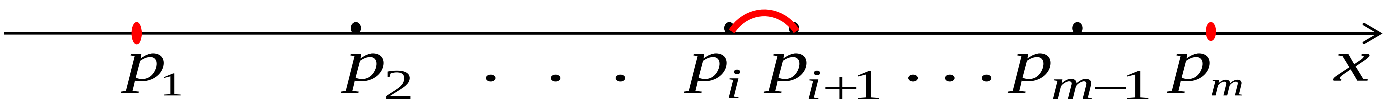
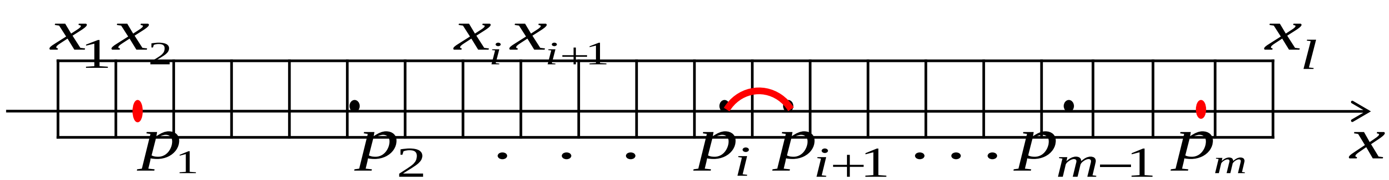
1) $P_n(x) = x^n + c_1 x^{n-1} + \ldots + c_{n-1} x + c_n = 0.$ Real roots only.

$p_1, p_2, \ldots, p_m$, $m \le n$, $p_i \ne p_j$. Roots have multiplicities $r_i \ge 1$, $\sum_{i=1}^{m} r_i = n.$



We can express p_1, p_m, $\mu \equiv \min(p_{i+1} - p_i)$ with any precision by coefficients $\{c_i\}$.



$x_{i+1} - x_i < \mu$

If $P_m(x_i) P_m(x_{i+1}) < 0$, then there is one root $p_k \in (x_i, x_{i+1})$.

If $P_m(x_i) P_m(x_{i+1}) > 0$, then there is no roots inside (x_i, x_{i+1}).

If $P_m(x_i) = 0$ (or $P_{m+1}(x_i) = 0$), x_i (or x_{i+1}) is a root and neighbor cells are empty.

2) Minimal polynomial, which has only simple roots ($r_i = 1$)

$p_1, p_2, \ldots, p_m, \quad p_i \neq p_j$:

$$P_m(x) = D_m^{-1} \det \begin{bmatrix} t_0 & t_1 & \ldots & t_{m-1} & t_m \\ t_1 & t_2 & \ldots & t_m & t_{m+1} \\ \ldots & & \ldots & \ldots & \ldots \\ t_{m-1} & t_m & \ldots & t_{2m-2} & t_{2m-1} \\ 1 & x & \ldots & x^{m-1} & x^m \end{bmatrix},$$

where

$$t_k = \sum_{l=1}^{m} r_l p_l^k, \; k = 0,1,2,\ldots, \quad H_k = \left[h_{ij}\right]_1^k = \left[t_{i+j-2}\right]_1^k, \quad D_m = \det H_m \neq 0.$$

All parameteres can be rationally expressed by $\{c_i\}$.

I. Lomidze. *Criteria of Unitary Equivalence of Hermitian Operators with Degenerate Spectrum***.** **Georgian Math. Journal.** http//www.jeomj.rmi.acnet.ge/GMJ/ **v.3, #2, 1996, p.141-152**.

3) As we have used only symmetric functions

$$\sum_{i=1}^{m} p_i, \quad \sum_{1\le i<j\le m} (p_i + p_j)^2, \quad \sum_{1\le i<j\le m} (p_i - p_j)^2, \ldots,$$

all our results (may be, except the limits $\lim_{k\to\infty} \varepsilon_k$, $\lim_{k\to\infty} \alpha_k$ and $\lim_{k\to\infty} \beta_k$) can be expressed (rationally) by the coefficients of the minimal polynomial $P_m(x)$, which form a basis in the space of symmetric functions. These coefficients, in theirs turn, can be rationally expressed by the coefficients of the given polynomial $P_n(x)$.

4) For μ: $\varepsilon_0 < \varepsilon_1 < \ldots < \mu$:

$$\varepsilon_{k+1}^2 = \varepsilon_k^2 + \left(\sum_{1 \le i < j \le m} \frac{1}{(p_i - p_j)^2 - \varepsilon_k^2} \right)^{-1}, \quad \varepsilon_0 = \left(\sum_{1 \le i < j \le m} \frac{1}{(p_i - p_j)^2} \right)^{-1}.$$

$$\varepsilon_0 < \mu \quad \Rightarrow \quad \varepsilon_k < \varepsilon_{k+1} < \mu, \quad k = 0, 1, 2, \ldots$$

$$\lim_{k \to \infty} \varepsilon_k = \mu$$

$$\varepsilon_\infty^2 = \varepsilon_\infty^2 + \left(\sum_{1 \le i < j \le m} \frac{1}{(p_i - p_j)^2 - \varepsilon_\infty^2} \right)^{-1} \quad \Rightarrow \quad \left(\sum_{1 \le i < j \le m} \frac{1}{(p_i - p_j)^2 - \varepsilon_\infty^2} \right)^{-1} = 0$$

$$\Rightarrow \quad \frac{1}{(p_{i_0} - p_{j_0})^2 - \varepsilon_\infty^2} \to \infty \quad \Rightarrow \quad \min\left\{ (p_i - p_j)^2 \right\} \equiv \mu = \varepsilon_\infty^2$$

5) For p_1: $\alpha_0 < \alpha_1 < \ldots < p_1 - a,$

$$\alpha_{k+1}^2 = \alpha_k^2 + \left(\sum_{i=1}^{m} \frac{1}{(p_i - a)^2 - \alpha_k^2} \right)^{-1}, \quad \alpha_0^2 = \left(\sum_{i=1}^{m} \frac{1}{(p_i - a)^2} \right)^{-1}.$$

For p_m: $\beta_0 < \beta_1 < \ldots < b - p_m,$

$$\beta_{k+1}^2 = \beta_k^2 + \left(\sum_{i=1}^{m} \frac{1}{(p_i - b)^2 - \beta_k^2} \right)^{-1}, \quad \beta_0^2 = \left(\sum_{i=1}^{m} \frac{1}{(b - p_i)^2} \right)^{-1}.$$

$$a = \frac{1}{m} \sum_{i=1}^{m} p_i - \sqrt{\sum_{1 \le i < j \le m} (p_i - p_j)^2} < p_1,$$

$$b = \frac{1}{m} \sum_{i=1}^{m} p_i + \sqrt{\sum_{1 \le i < j \le m} (p_i - p_j)^2} > p_m.$$

$$\lim_{k \to \infty} \alpha_k = p_1 - a, \qquad \lim_{k \to \infty} \beta_k = b - p_m.$$

6) Number of steps needed to find ε_k, if $0 < \mu - \varepsilon_k < \delta$.

$$k < \frac{\ln\left\{\frac{\tilde{\mu}}{\delta}\left[1-\left[m\psi'(1)-m\psi'(m)+\psi(1)-\psi(m)\right]^{-1/2}\right]\right\}}{\ln\left(1+(L+m-2)^{-1}\right)},$$

where

$$\tilde{\mu} = \frac{(b-\beta_r)-(a+\alpha_i)}{m-1} \approx \frac{p_m - p_1}{m-1} \quad \text{(for any } r \text{ and } i\text{)}, \quad \psi(z) = \frac{\Gamma'(z)}{\Gamma(z)},$$

$$L = \sum_{l=2}^{m} \frac{m-l}{l^2-1} = \frac{3}{4}(m+1) + \psi(1) + \frac{1}{2}\left[(m-1)\psi(m-1) - (m+1)\psi(m+1)\right].$$

If $m >> 1$, then

$$k < \frac{7}{4} m \ln \frac{\tilde{\mu}}{\delta}.$$

7) Number of steps needed to find α_k (or β_k)

$0 < (p_1 - a) - \alpha_k < \delta$ (or $0 < (b - p_m) - \beta_k < \delta$):

$$k < \frac{\ln\left(\frac{\tilde{\mu}}{\delta}\left\{\rho - \left[\psi'(\rho) - \psi'(m+\rho)\right]^{-1/2}\right\}\right)}{\ln\left(1 + F^{-1}\right)},$$

where

$$\rho = m\sqrt{\frac{m^2 - 1}{12}} - \frac{m-1}{2},$$

$$F = \frac{\left[\rho^2 - \left[\psi'(\rho) - \psi'(m+\rho)\right]^{-1}\right]\left[\psi(m) - \psi(1) - \psi(m+2\rho) + \psi(1+2\rho)\right]}{2\rho}.$$

If $m >> 1$, then

$$k < \frac{m^2 \ln m}{4\sqrt{3}}\left(2\ln m + \ln\frac{\tilde{\mu}}{\delta}\right).$$

Thank you

for attention