Approximation with respect to the spatial variable of the solution of a nonlinear dynamic beam problem

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1. Problem formulation

Let us consider the nonlinear equation

$$\frac{\partial^2 u}{\partial t^2}(x,t)+\frac{\partial^4 u}{\partial x^4}(x,t)-h\frac{\partial^2 u}{\partial x^2\partial t^2}(x,t)-\left(\lambda+\int_0^L\left(\frac{\partial u}{\partial \xi}(\xi,t)\right)^2 d\xi\right)\frac{\partial^2 u}{\partial x^2}=f(x,t), \qquad (1)$$

$$0<x<L, \qquad 0<t\le \mathrm{T}$$

With the initial boundary conditions

$$u(x,0)=u^0(x),\ \frac{\partial u}{\partial t}(x,0)=u^1(x),$$

$$u(0,t)=u(L,t)=0,\ \frac{\partial^2 u}{\partial x^2}(0,t)=\frac{\partial^2 u}{\partial x^2}(L,t)=0, \qquad (2)$$

where h and λ are some nonegative constants, $u^0(x)$, $u^1(x)$ and $f(x,t)$ are the given functions and u(x,t) is the function we want to obtain.

Equation (1) corresponds to the dynamic state of a beam. It is given in E. Henriques de brito [1] and belongs to the class of equations based on the Timoshenko theory [2]. For λ = 0, (1) is derived in [3] and [4] by passing to the limit in the one-dimensional version of the von karman system describing approximately the plane motion of a uniform prismatic beam. More precisely, based of the system of equations [5]

$$u_{tt}-\left(u_x+\frac{1}{2}w_x^2\right)_x=0,$$

$$w_{tt}+w_{xxxx}-hw_{xxtt}-\left[w_x\left(u_x+\frac{1}{2}w_x^2\right)\right]_x=f,$$

a coefficient $\varepsilon > 0$ is attached to the term v_{tt} and then the limit is taken respect $\varepsilon \longrightarrow 0$ provided that the condition v(0, t) = v(L, t) is fulfilled.

In [6], the equation for a beam is considered, in which equation (1) is generalized in a certain sense.

Another Timoshenko model for beam vibration has the form [7]

$$\frac{\partial^2 u}{\partial t^2} = \left(cd - a + b \int_0^L \left(\frac{\partial u}{\partial \xi}\right)^2 d\xi \right) \frac{\partial^2 u}{\partial x^2} - cd \frac{\partial \psi}{\partial x},$$

$$\frac{\partial^2 \psi}{\partial t^2} = c \frac{\partial^2 \psi}{\partial x^2} - c^2 d \left(\psi - \frac{\partial u}{\partial x}\right). \tag{3}$$

Comparing (1) and (3), we discover that both models have a common nonlinearity of the form

$$\left(\int_0^L \left(\frac{\partial u}{\partial \xi}\right)^2 d\xi \right) \frac{\partial^2 u}{\partial x^2},$$

which in 1876 Kirchhoff used for the first time in the equation for a string [5]

$$\frac{\partial^2 u}{\partial t^2} - \left(\alpha_0 + \alpha_1 \int_0^\pi \left(\frac{\partial w}{\partial \xi}\right)^2 d\xi \right) \frac{\partial^2 u}{\partial x^2} = 0\,, \tag{4}$$

and which was called the K-correction in [2]. Note that the K-correction is inherent in various models of beams and plates, and a lot of published works are devoted to the problem of solution existence and uniqueness as well as to a number of other problems for equation (4) and its generalizations, in particular for the equation

$$\frac{\partial^2 u}{\partial t^2} = \varphi \left(\int_0^\pi \left(\frac{\partial u}{\partial \xi}\right)^2 d\xi \right) \frac{\partial^2 u}{\partial x^2}. \tag{5}$$

There are several papers in which for the above equations the approximate solution algorithms are constructed and their error is estimated. The bibliography related to this topic is partly presented in [13].

The solvability of system (3) and a system differing from it in the presence of first derivatives of the unknown functions with respect to time is investigated in [15] and [1], respectively, while in [11], for (3) an approximation algorithm is constructed and its error is estimated. The question of numerical solution of the linearized Timoshenko system is dealt with in [7].

As for the considered equation (1), note that in [3] the existence and uniqueness of a solution of a more general equation than (1) is proved. The present paper is partly devoted to the topic of construction of an approximate solution. Here we use the approach applied by us in [11]−[13] for system (3) and equations (4) and (5) as the first step on the path of approximate solution of the problem. It consists in approximation with respect to the variable x by the Galerkin method. The error estimate is derived by means of a priori inequalities and, moreover, all coefficients contained in it are expressed in explicit form through the initial data of the problem, which makes it possible to obtain a numerical value of the upper bound of the method error.

2. SOLVABILITY

In [3], the existence and uniqueness of a generalized solution of the Cauchy problem is proved for the equation

$$(I + hA)u'' + A^2 u + \left[\lambda + M\left(|A^{\frac{1}{2}}u|^2\right)\right] Au = f$$

a particular case of which is equation (1). Let us apply the result of [3] to our case.

The symbol $(\ ,\ )$ will be understood as a scalar product in $L_2(0, L)$. Let us present the result of that paper for our case. Let $\overset{\circ}{W}{}_2^2(0, L)$ consist of functions of the space $W_2^2(0, L)$ which vanish on the boundary.

Therefore, according to [3], if

$$u^0(x) \in \overset{\circ}{W}{}_2^2(0, L), \quad u^1(x) \in W_2^1(0, L), \tag{6}$$

then there is a unique function $u = u(x, t)$,

$$u \in L^\infty(0, T; \overset{\circ}{W}{}_2^2(0, L)), \quad \frac{\partial u}{\partial t} \in L^\infty(0, T; W_2^1(0, L)), \tag{7}$$

such that u is a weak solution of problem (1), (2), i.e. for every $v = v(x) \in \overset{\circ}{W}{}_2^2(0, L)$, u satisfies

$$\frac{d}{dt}\left[\left(\frac{\partial u}{\partial t}, v\right) + h\left(\frac{\partial^2 u}{\partial x \partial t}, \frac{\partial v}{\partial x}\right)\right] + \left(\frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 v}{\partial x^2}\right) + \left(\lambda + \int_0^L \left(\frac{\partial u}{\partial x}\right)^2 dx\right)\left(\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}\right) = (f, v) \tag{8}$$

and

$$u(x, 0) = u^0(x), \ \frac{\partial u}{\partial t}(x, 0) = u^1(x), \tag{9}$$

Let us make the conclusions needed in the sequel.

With (7) taken account we wrire the solution of the problem in the form

$$u(x, t) = \sum_{i=1}^{n} u_i\,(t) sin\frac{i\pi x}{L}, \tag{10}$$

where, as follows from (8), after replacing the function v by the function $\sin\frac{i\pi x}{L}$, i=1,2,…, n. the coefficients $u_i(t)$ satisfy the system of equations

$$\left(1 + h\left(\frac{\pi i}{L}\right)^2\right) u_i''(t) + \left(\frac{\pi i}{L}\right)^4 u_i(t) + \left(\lambda + \frac{L}{2}\sum_{j=1}^{\infty}\left(\frac{\pi j}{L}\right)^2 u_j^2(t)\right)\left(\frac{\pi i}{L}\right)^2 u_i(t) = f_i(t) \qquad (11)$$

$$i = 1, 2, \dots \;, \qquad 0 < t \leq T,$$

where

$$f_i(t) = \frac{2}{L}\int_0^L f(x,t) sin\frac{i\pi x}{L}\, dx \,,$$

To system (8) we add the initial conditions which follow from (7) and (6)

$$u_i(0) = a_i^{(0)}, u_i'(0) = a_i^{(1)}, \qquad i = 1, 2, \dots ,$$

Here

$$a_i^{(l)} = \frac{2}{L}\int_0^L u_i^{(l)}(x) sin\frac{i\pi x}{L}\, dx, \;\; l = 0, 1.$$

Finally, in view of (10) and (7) we conclude that

$$\text{the series } \sum_{i=1}^{\infty} i^4 u_i^2(t) \text{ and } \sum_{i=1}^{\infty} i^2 u_i'^2(t) \text{ converge.} \qquad (13)$$

3. ASSUMPTIONS

Assume that the initial functions are represented in the form

$$u^0(x) = \sum_{i=1}^{\infty} a_i^{(0)} \sin \frac{i\pi x}{L}, \quad u^1(x) = \sum_{i=1}^{\infty} a_i^{(1)} \sin \frac{i\pi x}{L}, \quad 0 \le x \le L, \tag{14}$$

and

$$a_i^{(0)^2} \le \frac{\omega_0}{i^{p_0+5}}, \quad a_i^{(1)^2} \le \frac{\omega_1}{i^{p_1+3}}, \quad i = 1, 2, \ldots, \tag{15}$$

where p_0, p_1, ω_0 and ω_1 are some positive numbers.

Note that the fulfillment of this condition implies the feasibility of (6). As we will see, inequalities (15) are introduced to facilitate the calculation of certain parameters. At the end of the paper, the case is considered, in which this restriction does not hold.

4. GALERKIN METHOD AND ITS ERROR

Let us perform approximation of the solution with respect to the variable x. For this we use the Galerkin method. A solution will be sought in the form of the finite series

$$u_n(x,t)=\sum_{i=1}^{n} u_{ni}(t)\sin\frac{i\pi x}{L}\,, \tag{16}$$

where the coefficients $u_{ni}(t)$ are a solution of the system of the differential equations

$$\left(1+h\left(\frac{\pi i}{L}\right)^2\right)u''_{ni}(t)+\left(\frac{\pi i}{L}\right)^4 u_{ni}(t)+\left(\lambda+\frac{L}{2}\sum_{j=1}^{\infty}\left(\frac{\pi j}{L}\right)^2 u_{nj}^2(t)\right)\left(\frac{\pi i}{L}\right)^2 u_{ni}(t)=f_i(t), \tag{17}$$

$$i=1,2,\dots,n,\qquad 0<t\le T,$$

with the initial conditions

$$u_{ni}(0)=a_i^{(0)},\quad u'_{ni}(0)=a_i^{(1)},\ \ i=1,2,\dots,n. \tag{18}$$

Now our aim is to estimate the error of the Galerkin method. Let us give the definition of this error. By the coefficients of decomposition (10) we form the function

$$\pi_n u(x,t)=\sum_{i=1}^{n} u_i(t)\sin\frac{i\pi x}{L}\,. \tag{19}$$

By the error of the Galerkin method we understood the difference between the functions $u_n(x,t)$ and $\pi_n u(x,t)$

$$\delta_n(x,t)=u_n(x,t)-\pi_n u(x,t). \tag{20}$$

From (20), (16) and (19) follows

$$\delta_n(x,t)=\sum_{i=1}^{n}\delta_{ni}(t)\,\sin\frac{i\pi x}{L}\,, \tag{21}$$

where

$$\delta_{ni}(t)=u_{ni}(t)-u_i(t). \tag{22}$$

Below, we will denote by || || the norm in the space $L_2(0, L)$. Let us derive the equations for $\delta_{ni}(t)$. Using the first n equations of system (11) and the first n equalities from each of the initial conditions (12), we subtract them from the corresponding equations of system (17) and conditions (18). As a result, applying (22), (16) and (19), we obtain the system of equations for $\delta_{ni}(t)$

$$\left(1+h\left(\frac{\pi i}{L}\right)^2\right)\delta''_{ni}(t)+\left(\left(\frac{\pi i}{L}\right)^4+(\lambda+\|u_{nx}(x,t)\|^2)\left(\frac{\pi i}{L}\right)^2\right)\delta_{ni}(t)$$
$$+\left(\|u_{nx}(x,t)\|^2-\|(\pi_n u)_x(x,t)\|^2\right)\left(\frac{\pi i}{L}\right)^2 u_i(t)=\frac{L}{2}\beta_n(t)\left(\frac{\pi i}{L}\right)^2 u_i(t), \qquad (23)$$
$$i=1,2,\dots,n,$$

With the initial conditions

$$\delta_{ni}(0)=0, \quad \delta'_{ni}(0)=0, \;\; i=1,2,\dots,n. \qquad (24)$$

Here

$$\beta_n(t)=\frac{2}{L}\|u_x(x,t)-(\pi_n u)_x(x,t)\|^2=\sum_{i=n+1}^{\infty}\left(\frac{\pi i}{L}\right)^2 u_i^2(t). \qquad (25)$$

System (23) and conditions (24) are the starting point of the investigation of the problem of method accuracy estimation.

we are going to speak about $f(x,t)=0$.

soon we will comlate the investigation when $f(x,t)\neq 0$.

Lemma 1

The estimate

$$\left\|\frac{\partial^l}{\partial x^l}\pi_n u(x,t)\right\|^2 \le c_{l-1}, \;\; l=1,2, \qquad (26)$$

where c_0 *and* c_1 do not depend on n and t, is valid.

Proof

Multiply (11) by $2u_i(t)$ and sum the obtained expression over $i = 1, 2, \dots$. If we use (10), (13) and denote

$$\Phi(t) = \frac{2}{L}(\|u_t(x,t)\|^2 + h\|u_{xt}(x,t)\|^2 + \|u_{xx}(x,t)\|^2) + \frac{1}{L}(\lambda + \|u_x(x,t)\|^2)^2, \tag{27}$$

then the result is written as $\Phi'(t) = 0$, which means that for 0<t≤T

$$\Phi(t) = \Phi(0). \tag{28}$$

Applying (27), (10) and (19) in (28) we find

$$(\|u_t(x,t)\|^2 + h\|u_{xt}(x,t)\|^2 + \|u_{xx}(x,t)\|^2) + \frac{1}{L}(\lambda + \|u_x(x,t)\|^2)^2 \leq \Phi(0). \tag{29}$$

Let us calculate $\Phi(0)$. By (27) and (9), we have

$$\Phi(0)= \frac{2}{L}(\|u^1(x)\|^2 + h\|u^{1\prime}(x)\|^2 + \|u^{0\prime\prime}(x)\|)^2 + \frac{1}{L}(\lambda + \|u^{0\prime}(x)\|^2)^2.$$

From (29), first, taking into account that by virtue of (19) $\|(\pi_n u)_{xx}(x,t)\| \geq \frac{\pi}{L}\|(\pi_n u)_x(x,t)\|$, we obtain (26) for l =1, where

$$c_0 = \left(\left(\frac{\pi}{L}\right)^4 + 2\lambda\left(\frac{\pi}{L}\right)^2 + L\Phi(0)\right)^{\frac{1}{2}} - \left(\frac{\pi}{L}\right)^2 - \lambda\,. \tag{30}$$

and then verify the fulfillment of (26) for $l = 2$, where

$$c_1 = \frac{L}{2}\Phi(0). \tag{31}$$

The lemma is proved. ■

Lemma 2

The inequality

$$\|u_{nx}(x,t)\|^2 \leq c_2, \tag{31}$$

where the value c_2 does not depend on t, is valid.

Proof

Multiply (17) by $2u'_{ni}(t)$ and sum the obtained over $i = 1, 2, \ldots, n$. using (16), the result can be written in the form $\Phi'_n(t) = 0$, where

$$\Phi_n(t) = \frac{2}{L}\left(\|u_{nt}(x,t)\|^2 + h\|u_{nxt}(x,t)\|^2 + \|u_{nxx}(x,t)\|^2\right) + \frac{1}{L}\left(\lambda + \|u_{nx}(x,t)\|^2\right)^2. \quad (32)$$

Thus we get the equality

$$\Phi_n(t) = \Phi_n(0) \quad (33)$$

which together with (32) and the astimate $\|u_{nxx}(x,t)\| \geq \frac{\pi}{L}\|u_{nx}(x,t)\|$ which follows from (16) imply the fulfillment of (31) where

$$c_2 = \left(\left(\frac{\pi}{L}\right)^4 + 2\lambda\left(\frac{\pi}{L}\right)^2 + L\Phi_n(0)\right)^{\frac{1}{2}} - \left(\frac{\pi}{L}\right)^2 - \lambda\,. \quad (34)$$

The lemma is proved. ∎

If it is required to calculate or estimate c_2, we can use the following relations for $\Phi_n(0)$

$$\begin{aligned}
\Phi_n(0) &= \sum_{i=1}^{n}\left(1 + h\left(\frac{\pi i}{L}\right)^2\right) a_i^{(1)^2} + \sum_{i=1}^{n}\left(\frac{\pi i}{L}\right)^4 a_i^{(0)^2} \\
&\qquad + \frac{1}{L}\left(\lambda + \frac{L}{2}\sum_{i=1}^{n}\left(\frac{\pi i}{L}\right)^2 a_i^{(0)^2}\right)^2 \leq \Phi(0), \\
\Phi_n(0) &\leq \frac{1}{L}\left[\lambda + \frac{L}{2}\left(\frac{\pi}{L}\right)^2 \omega_0 \sum_{l=0}^{1}\left(\frac{1}{p_0+2}\left(1 - \frac{1}{n^{p_0+2}}\right)\right)^l\right]^2 \\
&\qquad + \sum_{l=0}^{1}\sum_{m=0}^{l} \omega_l h^{l(1-m)}\left(\frac{\pi}{L}\right)^{4-2(l+m)}\left[1 + \frac{1}{p_l+2m}\left(1 - \frac{1}{n^{p_l+2m}}\right)\right],
\end{aligned} \quad (35)$$

which are the result of the application of (32), (16), (18) together with (27), (10), (12), (15) and the integral test for the convergence of series.

Comparing (30) and (34) and applying (35) we observe that

$$c_2 \leq c_0. \quad (36)$$

Let us estimate $\beta_n(t)$ defined by (25).

Lemma 3

The inequality

$$\beta_n(t) \leq c_3\left(\frac{1}{n^{p_0}}\left(c_4 + \frac{c_5}{n^2}\right) + \frac{1}{n^{p_1}}\left(c_6 + \frac{c_7}{n^2}\right)\right), \quad (37)$$

where the values c_l, $l = 3, 4, \ldots, 7$, do not depend on n and t, is valid.

Proof

Using (13), let us introduce into the consideration the function

$$\Psi_n(t) = \sum_{i=n+1}^{\infty} \left(1 + h\left(\frac{\pi i}{L}\right)^2\right) u_i'^2(t) + \sum_{i=n+1}^{\infty} \left(\frac{\pi i}{L}\right)^4 u_i^2(t) + \left(\lambda + \|u_x(x,t)\|^2\right) \beta_n(t). \quad (38)$$

we need to estimate its value for t=0. By (12), (9) and (25), we get

$$\Psi_n(0) = \sum_{i=n+1}^{\infty} \left(1 + h\left(\frac{\pi i}{L}\right)^2\right) a_i^{(1)^2} + \sum_{i=n+1}^{\infty} \left(\frac{\pi i}{L}\right)^4 a_i^{(0)^2} + \left(\lambda + \|u^{0\prime}(x)\|^2\right) \sum_{i=n+1}^{\infty} \left(\frac{\pi i}{L}\right)^2 a_i^{(0)^2}. \quad (39)$$

The application of (15) and the integral test of the convergence of series gives

$$\begin{aligned}\Psi_n(0) \le \sum_{i=n+1}^{\infty} \Bigg\{ & \omega_1 \left(1 + h\left(\frac{\pi i}{L}\right)^2\right) \frac{1}{i^{p_1+3}} \\ & + \omega_0 \left(\frac{\pi}{L}\right)^2 \left[\left(\frac{\pi}{L}\right)^2 \frac{1}{i^{p_0+1}} + \left(\lambda + \|u^{0\prime}(x)\|^2\right) \frac{1}{i^{p_0+3}}\right]\Bigg\} \\ \le & \ \omega_1 \frac{1}{n^{p_1}} \left(\frac{1}{(p_1+2)n^2} + h\left(\frac{\pi}{L}\right)^2 \frac{1}{p_1}\right) \\ & + \omega_0 \left(\frac{\pi}{L}\right)^2 \frac{1}{n^{p_0}} \left[\left(\frac{\pi}{L}\right)^2 \frac{1}{p_0} + \left(\lambda + \|u^{0\prime}(x)\|^2\right) \frac{1}{(p_0+2)n^2}\right]. \end{aligned} \quad (40)$$

Further, comparing formulas (25) and (38), we conclude that

$$\beta_n(t) \le \left(\lambda + \left(\frac{\pi}{L}\right)^2\right)^{-1} \Psi_n(t). \quad (41)$$

Now, let us estimate the function $\psi_n(t)$. After multiplying (11) by $2u_i'(t)$, summing the resulting equality over $i = n+1, n+2, \ldots$, and using (38) and (25), we obtain

$$\Psi_n'(t) = \left(\|u_x(x,t)\|^2\right)_t \beta_n(t). \quad (42)$$

By (10), (27) and (28), we have

$$\left|\left(\|u_x(x,t)\|^2\right)_t\right| \le \frac{2}{L}\,\alpha\left[(\|u_t(x,t)\|^2 + \|u_{xx}(x,t)\|^2) + (h\|u_{xt}(x,t)\|^2 + 2\lambda\|u_x(x,t)\|^2)\right] \le \alpha\Phi(t) = \alpha\Phi(0), \quad (43)$$

where

$$\alpha = \frac{L}{2}\left(1 + (2\lambda h)^{\frac{1}{2}}\right)^{-1}.$$

By virtue of (41) – (43), (38) and the Gronwall inequality

$$\Psi_n(t) \le \Psi_n(0)\exp\left(\alpha\left(\lambda + \left(\frac{\pi}{L}\right)^2\right)^{-1}\Phi(0)t\right). \quad (44)$$

Applying relations (41)-(44) and (40) successively, we see that (37) is fulfilled and also that

$$c_3 = \left(\lambda + \left(\frac{\pi}{L}\right)^2\right)^{-1}\exp\left(\alpha\left(\lambda + \left(\frac{\pi}{L}\right)^2\right)^{-1}\Phi(0)T\right), \quad c_4 = \frac{1}{p_0}\,\omega_0\left(\frac{\pi}{L}\right)^4,$$
$$c_5 = \frac{1}{p_0+2}\,\omega_0\left(\frac{\pi}{L}\right)^2\left(\lambda + \|u^{0\prime}(x)\|^2\right), \quad c_6 = \frac{1}{p_1}\,\omega_1 h\left(\frac{\pi}{L}\right)^2, \quad c_7 = \frac{1}{p_1+2}\,\omega_1. \quad (45)$$

The lemma is proved. □

Let us formulate the main result.

Theorem

The inequality

$$\left(\|\delta_{nt}(x,t)\|^2 + h\|\delta_{nxt}(x,t)\|^2 + \|\delta_{nxx}(x,t)\|^2 + \lambda\|\delta_{nx}(x,t)\|^2\right)^{\frac{1}{2}} \le c(t)\left(\frac{1}{n^{p_0}}\left(c_4 + \frac{c_5}{n^2}\right) + \frac{1}{n^{p_1}}\left(c_6 + \frac{c_7}{n^2}\right)\right), \quad (46)$$

where $c(t)$ is defined below, is fulfilled for the error of the Galerkin method.

Proof

Multiplying (23) by $L\delta'_{ni}(t)$, performing summation over $i = 1, 2, \ldots, n$, and taking into account (21), we obtain

$$F_n'(t) = \left(\|u_{nx}(x,t)\|^2\right)_t \|\delta_{nx}(x,t)\|^2 + \left[L\left(\|(\pi_n u)_x(x,t)\|^2 - \|u_{nx}(x,t)\|^2\right) + \frac{L^2}{2}\beta_n(t)\right]\sum_{i=1}^{n}\left(\frac{\pi i}{L}\right)^2 u_i(t)\delta_{ni}'(t), \quad (47)$$

where

$$F_n(t) = \|\delta_{nt}(x,t)\|^2 + h\|\delta_{nxt}(x,t)\|^2 + \|\delta_{nxx}(x,t)\|^2 + \left(\lambda + \|u_{nx}(x,t)\|^2\right)\|\delta_{nx}(x,t)\|^2. \quad (48)$$

Let us estimate terms in the right-hand part of relation (47).

Using (16), (32) and (33), by analogy with (43) we get

$$\left|\left(\|u_{nx}(x,t)\|^2\right)_t\right| \le \alpha\,\Phi_n(t) = \alpha\,\Phi_n(0). \quad (49)$$

Further, by virtue of (19), (16), (21), (22), (26) and (31), we see that

$$\left|\,\|(\pi_n u)_x(x,t)\|^2 - \|u_{nx}(x,t)\|^2\right| \le \frac{L}{2}\sum_{i=1}^{n}\left(\frac{\pi i}{L}\right)^2\left|u_i^2(t) - u_{ni}^2(t)\right| \le \left(\|(\pi_n u)_x(x,t)\| + \|u_{nx}(x,t)\|\right)\|\delta_{nx}(x,t)\| \le (c_0 + c_2)\|\delta_{nx}(x,t)\|. \quad (50)$$

Finally, again using (19), (21) and (26), we find

$$\frac{L}{2}\left|\sum_{i=1}^{n}\left(\frac{\pi i}{L}\right)^2 u_i(t)\delta_{ni}'(t)\right| \le \|(\pi_n u)_{xx}(x,t)\|\,\|\delta_{nt}(x,t)\| \le c_1\|\delta_{nt}(x,t)\|. \quad (51)$$

Relations (47)-(51) together with (21), (24), (37) and the inequalities

$\|\delta_{nxt}(x,t)\| \ge \frac{\pi}{L}\|\delta_{nt}(x,t)\|$, $\|\delta_{nxx}(x,t)\| \ge \frac{\pi}{L}\|\delta_{nx}(x,t)\|$, allow us to infer that

$$F_n(t) \le \int_0^t |F_n'(\tau)|\,d\tau \le c_3^2 T\left[\sum_{l=0}^{1}\left(\frac{1}{n^{p_l}}\left(c_{2l+4} + \frac{c_{2l+5}}{n^2}\right)\right)\right]^2 + \max(c_8, c_9)\int_0^t F_n(\tau)\,d\tau,$$

where

$$c_8 = \left(1 + h\left(\frac{\pi}{L}\right)^2\right)^{-1}\left(c_{10} + \left(c_1\frac{L}{2}\right)^2\right), \quad c_9 = \left(\lambda + \left(\frac{\pi}{L}\right)^2\right)^{-1}\left(c_{10} + \alpha\,\Phi_n(0)\right), \quad c_{10} = c_1(c_0 + c_2). \quad (52)$$

Applying the Gronwall inequality and definition (48), we obtain estimate (46) together with the formula for the coefficient c(t)

$$c(t) = c_3\sqrt{t e^{\max(c_8, c_9)t}}.$$

The theorem is proved. ■

Note that if we weaken the accuracy requirement, relations (52) can be simplified. By virtue of (35) and (36), $\Phi_n(0)$ and c_2 in (52) can be replaced by $\Phi(0)$ and c_0.

The fulfillment of inequalities (15) is not obligatory. We will obtain the method error estimate in the general case where the initial functions satisfy only (6) and (14). For this, only one change should be made in the proof of the theorem. Instead of (37), the result of using of relations (44), (39), (45) in (41) should be used as the estimate of $B_n(t)$. Then, instead of (46) we will have

$$\left(\|\delta_{nt}(x,t)\|^2 + h\|\delta_{nxt}(x,t)\|^2 + \|\delta_{nxx}(x,t)\|^2 + \lambda\|\delta_{nx}(x,t)\|^2\right)^{\frac{1}{2}} \\ \leq c(t)\Bigg[\sum_{i=n+1}^{\infty}\left(1 + h\left(\frac{\pi i}{L}\right)^2\right)a_i^{(1)2} + \sum_{i=n+1}^{\infty}\left(\frac{\pi i}{L}\right)^4 a_i^{(0)2} \\ + \left(\lambda + \|u^{0\prime}(x)\|^2\right)\sum_{i=n+1}^{\infty}\left(\frac{\pi i}{L}\right)^2 a_i^{(0)2}\Bigg]. \quad (53)$$

The right-hand side of (53) tends to zero as n→ ∞, which means that Galerkin discretization is convergent. Moreover, using (53) one can obtain estimates analogous to (46) provided that a_i^0 and a_i^1 change by a rule different from (15).

We have obtained corresponding calculating formulas for all coefficients taking part in this investigation.

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