On modeling of the turbulent movement

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Abstract. On investigation of the turbulent movement, the main problem is to give the velocity as the random process at any fixed point of the environment. According to our approach, the quantity of pulsations of the velocity at any fixed point is a Poison process; the random process of the pulsation part of the velocity is a Levy process. It follows that the velocity of the turbulent movement is a random element in suitable functional space. The theoretical achievements in development of the turbulent movement and experimental data is the foundations to give the covariance operator of this random element and the character of the randomness on the linear functionals of this random element. After preparing these necessity we can consider the corresponding stochastic differential equation of the trajectory in turbulent environment, the solution of this equation is the problem of future developments. Another problems are to give the approximative solution as a random process of this equation and to simulate in a computer the random process for various covariance operators to find the real, close to the objective account. Current advance of the computation technology and promised to our Institute high-capacity cluster give birth to hopes to receive an acceptable model of the turbulent movement.

Last unsolved problem of classical physics (`Is it possible to make a theoretical model to describe the statistics of a turbulent flow?') is the turbulence movement. Therefore, the interest of development of this is great. There are enormous quantity of literature concerning to this problem (see [2],[4],[8],[9]). The basic equations governing the motion of a fluid are known as the Navier –Stokes equations . The origin of the Navier-Stokes equations dates back to the late nineteenth century when Osborne Reinolds (1895) published results from his research on turbulence. Navier –Stokes equations are very complex due to the fact that turbulence is rotational, three-dimensional and time-dependant. The Navier-Stokes equations are the most general form of the laws governing fluid motion and contain all of the behavior which we can find in real problems. In practice are developed appropriate approximations which will let derive solutions for particular cases and thus find out something about the behavior of real turbulent systems. The concept of turbulence modeling is far less precise due to the complex nature of turbulent flow. Considering the enormous capacity of actual computers, it is possible to consider that high precision numerical simulations of the Navier-Stokes equations can solve the problem of turbulence. Unfortunately, with the current capacity of computing power, the attempts of direct numerical simulation of Navier-Stokes equations have been limited to low Reynolds numbers and simple geometries . Despite the current advance of the computation technology the possibility of using numerical simulation for flows with high Reynolds numbers in practical applications is still surely distant.

Before we describe our approach, we want lightly touch to analyze the history of development of the Brownian motion. The mathematical theory of the Brownian motion was produced by A. Einstein in 1905 [6]. According to Einstein, let $P(x,y,z;t)$ be the probability density of finding a Brownian particle at a point x,y,z at the time t. The density satisfies the diffusion equation

$$\frac{\partial P}{\partial t} = D\Delta P \,, \qquad (1)$$

Where Δ is the Laplacian $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ and D is a coefficient of diffusion.

There is another, purely probabilistic approach of description of the Brownian motion by the Gaussian process with independent increments. This approach is confirmed by the following point of view: let x_t be the axis coordinate of the particle for the moment of time t. Suppose that x_0 is equal to 0. As in the interval of time $[0,t]$ micro displacement of the particle is the sum of many almos independent parts (suppose that after collision the velocity vanishes rapidly because of viscosity), by the Central Limit theorem, it is natural to assume, that x_t be a Gaussian random variable; by the symmetry, the mean $Ex_t = 0$; by the homogeneity, the dispersion $E(x_{t+s} - x_t)^2 = f(s)$ does not depend on t, which gives $f(s) = cs$. Therefore, we receive the definition of a homogenous Brownian motion (Wiener process) $(W_t)_{t\in[0,T]}$ with the correlation , $EW_tW_s = c\min(t,s)$, $t,s \in [0,T]$, T is a

is continuance of the observation. In case $c = 1$, we have a standard Brownian motion (standard Wiener process). Wiener processes are widely used as in development of many fields of pure Mathematics, also in development of many applied sciences.

As in the Brownian motion case, there is pure probabilistic approach in development of turbulent movement. This approach has not a long history. In [9] is given the opinion that if the Brownian motion is described by the three dimensional random process, to describe the turbulent movement it is necessary to use the random process with values in a infinite-dimensional functional space. Here we want to mention that in a turbulent movement the displacement after the “collision” is sizable and before the next “collision” the velocity obtained after the last “collision” does not vanish. Therefore, in difference of the case of the Brownian motion, the trajectory of the turbulent movement may not be considered as a process with independent increments.

Now we give the description of our approach: let us fix a point x in a turbulent fluid field. At this point, we have the forward (mean) motion with the velocity $V(t,x)$ at the moment t (we can assume that this value of velocity is given), the Brownian motion, which is negligible and the impulse of velocity $\xi(t,x)$, which arises from the turbulent “collision”. The main point of our approach consists to the assumption that the quantity of such impulses till the moment t is the integer-valued random process with independent increments, which we denote by $P(t,x)$. Consequently, in the point x of the turbulent environment, at the moment of time t, we have the velocity $V(t,x) + \xi(t,x)$. The impulse $\xi(t,x)$ at the fixed point x is a random variable, as well as the quantity of such impulses in the time interval $(0,t)$ is a random variable. Moreover, these quantities in nonintersecting time intervals are independent . According to the theory of the random processes with independent increments, $P(t,x)$ is a Poisson process. Among the pulsations of impulses, there are many weak pulsations. The values of the impulses of such pulsations of the velocity rapidly vanish by the reason of viscosity. Denote by ε the maximal value of such impulses in the pulsation of velocity. The value of ε depends on the viscosity of the turbulent environment. Decompose the random process ($P(t,x))_{t\le T}$ by the following two component: $P(t,x) = P(t,x)(\le \varepsilon) + P(t,x)(> \varepsilon)$, where $P(t,x)(\le \varepsilon)$ is the quantity of such pulsations of the vector of velocity, the value of which is less or equal to ε. $P(t,x)(> \varepsilon)$ is the quantity of the rest pulsations. Just such pulsations generate the turbulence. Let us consider them one by one.

The velocity of the first type pulsations quickly vanishes by the reason of viscosity; therefore, the displacement of the particle, caused by the velocity of first type, is the sum of many almost independent parts. Therefore, the sum of such displacements up to the moment t, as the sum of independent, identically distributed random variables, is a Gaussian random variable. Likewise the displacements for the disjoint time intervals one may consider to be independents. Thus, the displacement caused by the small ($P(t,x)(\le \varepsilon)$) pulsations may be described by the random process of the Brownian movement (Wiener process) or, rather, by the stochastic integral with respect to the Wiener process $\int_0^t \sigma(\tau,x)dW_\tau$, where $\sigma(t,x):[0,T]\times S \to R^3$ depends on the properties of the environment at the point x and time t. Likewise this value depends on the scale. This is the well known model to describe the molecular diffusion.

Let now consider the pulsations, the quantity of which is $P(t,x)(> \varepsilon)$. As we mentioned above, such pulsations generate turbulence. Consider the set $U \equiv \{x : x \in R^3, \|x\| > \varepsilon\}$. Let $\mathrm{B}(U)$ -be the Borel $\sigma-$ algebra on U. Denote by $\nu(x,t,A)$, $A \in B(U)$, the quantity of pulsations of the velocity with values in the set A up to the moment of time t. $\nu(x,t,A)$ is a Poisson process (see [13], Teor. 1 of par.14). In particular, $\nu(x,t,U) = P(t,x)(> \varepsilon)$. $\nu(x,t,A)$ is characterized by the parameter (mean) $\pi(x,t,A)$. In the case when $A = U$, we will use $\pi(t,x)$ instead of $\pi(t,x,U)$. The quantity of $\pi(x,t,A)$ may be estimated by statistical observations. Denote $\eta_t(x) \equiv \int_{\|y\|>\varepsilon} y\nu(x,t,dy)$. $(\eta_t(x))_{t\le T}$ is a process with independent increments (see [13], par. 13). It is the sum of the independent random pulsations of the velocity the quantity of which is $P(t,x)(> \varepsilon)$. The value $\eta_t(x) - \eta_s(x)$ is the sum of values of pulsations of the velocity in the time interval (s,t) at the point x. The value $d\eta_\tau(x)$ is the value of pulsation of the velocity at the moment of time τ, at the point x of the turbulent environment.

Consequently, we have a formula of the velocity at the point x of the turbulent environment at the moment of time t: $u(t,x) = V(t,x) + \sigma(t,x)dW_t + d\eta_t(x)$ The value of pulsation of the velocity, which appears at the random moment of time, is a random vector in R^3. This random vector depends on the mean velocity $V(t,x)$ at the point x in the moment t, and we can receive it by the statistical observation. For different values of x and t the values of the velocity are in certain correlation to each other. If two points are sufficiently close to each other, then at the same moment of time the coefficient of the correlation of values of the velocity of these points is near to one. If the coefficients of the correlations is near to one of the points of the large mass of the turbulence environment, the turbulence effect with high Reinolds number is impressive. To give these correlations for all points of the turbulent environment for any time (and different times) is the main problem in developing the turbulent movement. Otherwise, it is the problem to receive the random element with values in certain functional space.

Suppose, that the turbulent environment is a compact set S in R^3. As the velocity at any fix point of S is a right continuous function, which has the left limit by the argument t, it is natural to take as a corresponding working space the space $D([0,T],C(S))$, where $C(S)$ is the separable Banach space of continuous functions from S to R^3. (it is clear to assume the continuity by x of the value of the impulse of velocity for every fixed moment of time t. Recall that, in general, for any Banach space X, $(D([0,T],X)$ is the space of right-continuous functions defined in $[0,T]$ with values in X, which have left limits). In the space $D[0,T]=(D[0,T],R^1)$ A. V. Schorokhod introduced a special metric, as it is impossible to introduce any natural norm in it to safe separability of the space . But if we fix a countable number of points Q in $[0,T]$, where (only) we can have discontinuity, the space $D_Q([0,T],C(S))$ with the norm $\|f\|=\sup_{t\in[0,T]}\|f(t)\|$ is a separable Banach space. We observe a turbulent movement in rational moments of time; therefore, it is natural to consider the Banach space of the right-continuous vector functions with left limits, which may have a discontinuity only in the rational points. Denote this space by $D_Q([0,T],C(S))$, where Q is the set of rational numbers in $[0,T]$. Let $D^*{}_Q([0,T],C(S))$ be the conjugate space of the separable Banach space $D_Q([0,T],C(S))$. We will consider the subset Γ of the conjugate space $D^*{}_Q([0,T],C(S))$, where for all $f\in D_Q([0,T],C(S))$, $t\in[0,T]$ and $x\in C(S)$, $\langle f,\delta_{t,x}\rangle = f(t,x)$; the symbol $\langle .,. \rangle$ denotes dual pairing. Γ is a total subset of the space $D^*{}_Q([0,T],C(S))$ (if $\langle f,\delta_{t,x}\rangle = f(t,x)$=0 for all $t\in[0,T]$ and $x\in C(S)$, then $f=0\in$ $D_Q([0,T],C(S))$).

Remark. The symbol $\langle .,. \rangle$ is not here dual pairing symbol in ordinary sense as $\langle f,\delta_{t,x}\rangle = f(t,x)$ is not real valued function because $f(t,x)\in R^3$. Further we will use this symbol in both---ordinary and above mentioned sense according to the context of the sentence.

Consider now the last member of the equality (4) ---$\eta_t(x), t\in[0,T], x\in S$. For all fixed $x\in S$, $\eta_t(x)$ may be form from the compound Poisson process (see [1], example 2) indeed: let $Y:\Omega\to C(S)$ be a random element with the law μ on the Borel σ-algebra on $C(S)$, such that $\langle Y,\delta_x\rangle$ is a random vector with $\|\langle Y,\delta_x\rangle\|>\varepsilon$, where $\delta_x\in C(S)^*$ is a linear continuous functional on $C(S)$, $\langle f,\delta_x\rangle = f(x)$ for all $f\in C(S)$ and $x\in S$. $\langle Y,\delta_x\rangle$ is a random pulsation of velocity at the point x, which promotes turbulence. To construct and develop such a random element is one of the main problems in the development of the turbulent movement. Experimental results, as well as the theoretical achievements like Kolmogorov's theory and other advances (see e.g. [7]) will be used to construct the random element Y.

The existence of such a random element is another problem from the field of probability distributions on linear spaces. Let now $Y_1, Y_2, \ldots,$ be the independent copies of the random element Y. Denote by $n(t,x)$ the quantity $P(t,x)(>\varepsilon)$ of the pulsations of velocity, $n(t,x) := P(t,x)(>\varepsilon)$. Consider the following random process

$$L_t(x) := \langle L_t, \delta_x \rangle := \langle Y_1, \delta_x \rangle + \langle Y_2, \delta_x \rangle + \cdots +$$

$$+ \langle Y_{n(t,x)}, \delta_x \rangle .$$

For all fixed $x \in S$, $L_t(x)$ is a compound R^3 valued Poisson process. It is easy to show that we have the following equality: $\eta_t(x) = L_t(x) = \langle Y_1, \delta_x \rangle + \langle Y_2, \delta_x \rangle + \cdots + \langle Y_{n(t,x)}, \delta_x \rangle,$ where $n(t,x)$ is the random variable distributed by Poisson law with mean $\pi(t,x)$. $\eta_t(x)$ has no physical sense, but

$\lim_{s \to 0} \eta_{t+s}(x) - \eta_t(x) := d\eta_t(x)$ is the value of the velocity impulse of the particle in the point x, at the moment of time t. Therefore, we can rewrite the equality (4) in the following way:

$$u(t,x) = V(t,x) + \sigma(t,x) dW_t + dL_t(x). \tag{5}$$

For simplicity, the Wiener process we can take one dimensional and consider the stochastic integral from Banach space valued function by the one dimensional Wiener process (see [11]). We can consider the process $u(t,x)$ as a Levy process in a separable Banach space $C(S)$ with the characters $(-\pi(t,x)\int_{B_1} x\mu(dx)\mu, R_\sigma, \pi(t,x)\mu)$ where R_σ is a covariance operator (see [14]), that is, the positive and symmetric linear operator $R_\sigma : C(S)^* \to C(S)$, $\langle R_\sigma \delta_x, \delta_y \rangle = \sigma(t,x)\sigma(t,y)$, B_1 is the ball in $C(S)$ with the radius 1(see [1]).

Remark 1. . Above, In our verbal proof, we receive independence of random variables $\langle Y_1, \delta_x \rangle, \langle Y_2, \delta_x \rangle, \cdots \langle Y_{n(t,x)}, \delta_x \rangle$ for all fixed $x \in S$, which does not give independence of the random elements $Y_1, Y_2, \ldots$ (see [3]). In real turbulent environment we have dependences in a probability sense (high correlations) of random variables $\langle Y_i, \delta_x \rangle$ and $\langle Y_i, \delta_y \rangle$ for all $x, y \in S$, close to each other. As well as, we have dependence of the random variables $dL_t(x)$ and $dL_{t+\Delta t}(y)$. Therefore, in our situation, to give the correlation operators (see definition in[14]) of these random elements is the one of the main problems to construct the model of the turbulent movement. Therefore, The independent copies of the random element Y we considered above only for comprehensibility of the model as we have independence of the random variables $\langle Y_1, \delta_x \rangle, \langle Y_2, \delta_x \rangle, \cdots \langle Y_{n(t,x)}, \delta_x \rangle$.

Note that there is considered in the paper [15] the model of one dimensional turbulent movement, where the Poisson process is used to describe the pulsations of velocity. . It is proposed in [12] a stochastic differential equation framework for modeling the timewise dynamics of the main component of the velocity .

Now, let us try to describe the trajectory of the particle, moving in a turbulent fluid field. If the particle is in the point y at the moment of time t, the pulsation of the velocity , received the particle at the moment of time τ , $(\tau < t)$, vanishes for the moment of time t. This occurrence we can describe by the function $\exp[-\alpha(t-\tau)]$. The coefficient α characterizes the viscosity of the environment. That is, the value $\exp[-\alpha(t-\tau)]\; d\eta_\tau(x)$ is the part of the velocity at the moment of time t, which the particle obtained at the moment of time τ, when it appeared at the point x. Let the particle in the turbulent environment at the time moment $t=0$ be at the point x_0 and X_t be the position of the particle at the moment t, then, according to the above mentioned assertion, we have the following stochastic differential equation for the trajectory X_t:

$$dX_t = V(t,X_t)dt + \sigma(t,X_t)dW_t + \int_0^t \exp[-\alpha(t-\tau)]d\eta_\tau(X_\tau) \,, \qquad (6)$$

with the initial condition $X_0 = x_0$. Where $V(\cdot,\cdot) \in C([0,T],C(S)) \subset D_Q([0,T],C(S))$,

$\sigma(\cdot,\cdot) \in C([0,T],C(S)) \subset D_Q([0,T],C(S))$, , $\eta_\cdot(\cdot)(\cdot):\Omega \to D([0,T],C(S))$ and $(W_t)_{t\in[0,T]}$ is a standard one dimensional Wiener process. The corresponding integral form of the equation (5) gives the formula of the trajectory of the particle:

$$X_t = x_0 + \int_0^t V(s,X_s)ds + \int_0^t \sigma(s,X_s)dW_s +$$

$$\int_0^t\int_0^s \exp[-\alpha(s-\tau)]d\eta_\tau(X_\tau)ds \qquad (7)$$

The stochastic integral $\int_0^t V(s,\cdot)dW_s$ is a $C(S)$-valued random process; the member of the equation (7)-- $\int_0^t V(s,X_s)dW_s$ we can consider as a generalized stochastic integral $\int_0^t \langle V(s,\cdot),\delta_{X_S}\rangle dW_s$ (see [11]). As for integral $\int_0^t\int_0^s \exp[-\alpha(s-\tau)d\eta_\tau(X_\tau)ds$, close to this stochastic integral is considered in [1]. The theoretical aspects of the stochastic differential equation (7) is the problem for future developments. As well as we will consider the approximate methods of the solution of this equation.

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