- **Relativistic QM - The Klein Gordon equation (1926)**

Scalar particle (field) (J=0) : $\phi(\mathbf{x})$

$$E^2 = \mathbf{p}^2 + m^2 \quad \Rightarrow \quad -\frac{\partial^2 \phi}{\partial t^2} + \nabla^2 \phi = m^2 \phi$$

(natural units)

Relativistic notation :

$$(\partial_\mu \partial^\mu + m^2)\phi = 0$$

4 vector notation

$$A^\mu = (A^0, \underline{A}), \quad B^\mu = (B^0, \underline{B})$$ contravariant

$$A_\mu = (A^0, \underline{-A}), \quad B_\mu = (B^0, -\underline{B})$$ covariant

$$A_\mu = g_{\mu\nu} A^\nu \qquad A^\mu = g^{\mu\nu} A_\nu$$

$$A.B = A_\mu B^\mu = A^\mu B_\mu = A^0 B^0 - \underline{A}.\underline{B}$$

4 vectors

$$(ct, \underline{x}) \equiv x^\mu \qquad (\frac{E}{c}, \underline{p}) \equiv p^\mu$$

$$\partial^\mu = (\frac{\partial}{c\partial t}, -\underline{\nabla}) \qquad \partial_\mu = (\frac{\partial}{c\partial t}, \underline{\nabla})$$

$$p^\mu \rightarrow i\hbar\partial^\mu \qquad p_\mu p^\mu = E^2 - \underline{p}^2 \quad \rightarrow \quad -\Box^2 \equiv \partial_\mu \partial^\mu$$

Physical interpretation of Quantum Mechanics

Schrödinger equation (S.E.)

$$i\frac{\partial\phi}{\partial t}+\frac{1}{2m}\nabla^2\phi=0$$

$$i\phi^*(S.E.)-i\phi(S.E.)^* \Rightarrow \frac{\partial\rho}{\partial t}+\nabla.\mathbf{j}=0 \text{ continuity eq.}$$

$$\rho=|\phi|^2$$

"probability density"

$$\mathbf{j}=-\frac{i}{2m}(\phi^*\nabla\phi-\phi\nabla\phi^*)$$

"probability current"

Klein Gordon equation

$$-\frac{\partial^2\phi}{\partial t^2}+\nabla^2\phi=m^2\phi$$

$$\rho=2E|N|^2$$

$$\rho=i\left(\phi^*\frac{\partial\phi}{\partial t}-\phi\frac{\partial\phi^*}{\partial t}\right)$$

$$\mathbf{j}=-i(\phi^*\nabla\phi-\phi\nabla\phi^*)$$

$$j^\mu=(\rho,\mathbf{j})$$

Negative probability?

$$\phi=Ne^{-ip.x},\quad \rho=2E|N|^2$$

$$\int_V\rho\,dV=\int\rho\,d^3x=2E$$

$$f_p^\pm=e^{\mp ip.x}\frac{1}{\sqrt{2p^0V}}$$

Normalised free particle solutions

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$$\rho=i(\phi^*\frac{\partial\phi}{\partial t}-\phi\frac{\partial\phi^*}{\partial t})$$

$$\mathbf{j}=-i(\phi^*\nabla\phi-\phi\nabla\phi^*)$$

$$j^\mu=(\rho,\mathbf{j})$$

Pauli and Weisskopf

$$j^\mu\rightarrow e(\rho,\mathbf{j})=j^\mu_{EM}$$

$$\frac{\partial\rho}{\partial t}+\nabla.\mathbf{j}=0=\partial^\mu j_\mu$$

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Klein Gordon equation

$$-\frac{\partial^2\phi}{\partial t^2}+\nabla^2\phi=m^2\phi$$

$$\rho_{EM}=ie(\phi^*\frac{\partial\phi}{\partial t}-\phi\frac{\partial\phi^*}{\partial t})$$

$$\mathbf{j}=-i(\phi^*\nabla\phi-\phi\nabla\phi^*)$$

$$j^\mu=(\rho,\mathbf{j})$$

Pauli and Weisskopf

$$j^\mu\to e(\rho,\mathbf{j})=j^\mu_{EM}$$

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Field theory of $\pi^{\pm}$

Scalar particle – satisfies KG equation

$$(\partial_\mu \partial^\mu + m^2)\phi = 0$$

- Classical electrodynamics, motion of charge –e in EM potential $\mathrm{A}^\mu = (\mathrm{A}^0, \mathbf{A})$

is obtained by the substitution : $p^\mu \to p^\mu + eA^\mu$

- Quantum mechanics : $i\partial^\mu \to i\partial^\mu + eA^\mu$

The Klein Gordon equation becomes:

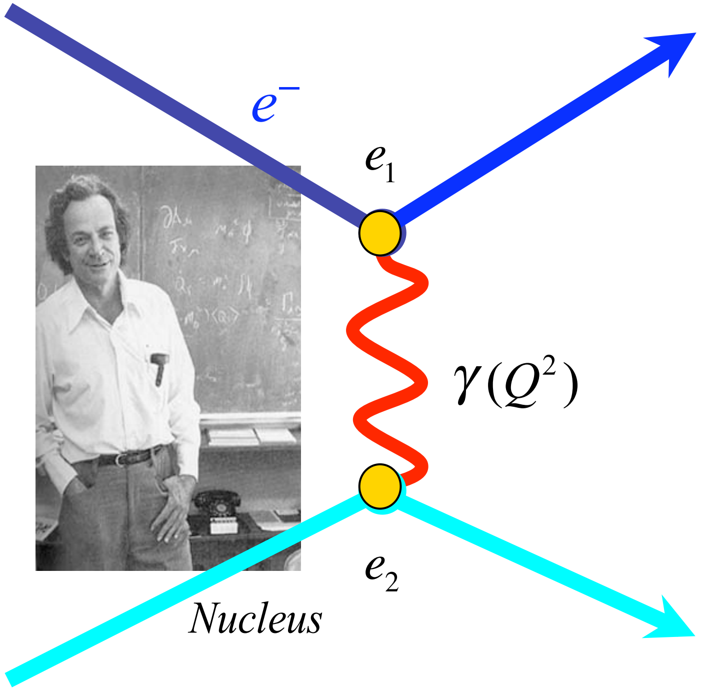
$$(\partial_\mu \partial^\mu + m^2)\phi = -V\phi \quad where \quad V = -ie(\partial_\mu A^\mu + A^\mu \partial_\mu) - e^2 A^2$$

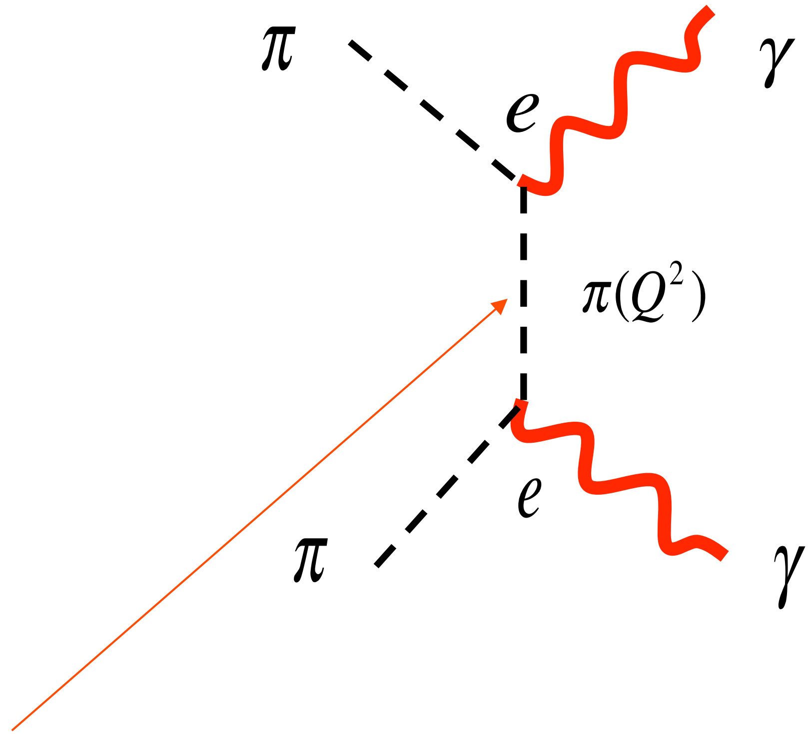
The smallness of the EM coupling, $\alpha_{em} = \frac{e^2}{4\pi} \sim \frac{1}{137}$, means that it is sensible to

Make a "perturbation" expansion of V in powers of α_{em}

$$V \simeq -ie(\partial_\mu A^\mu + A^\mu \partial_\mu)$$

Exchange Force





Pion Propagator

The pion propagator

Want to solve :

$$(\partial_\mu \partial^\mu + m^2)\psi = -V\psi$$

Solution :

$$\psi(x) = \phi(x) - \int d^4x' \Delta_F(x'-x) V(x') \psi(x')$$

where

$$(\partial_\mu \partial^\mu + m^2)\phi = 0$$

and

$$(\partial_\mu \partial^\mu + m^2)\Delta_F(x'-x) = \delta^4(x'-x)$$

Feynman propagator — Dirac Delta function

$$\int d^4x' \delta^4(x'-x) f(x') = f(x)$$

The pion propagator

Want to solve : $$(\partial_\mu \partial^\mu + m^2)\psi = -V\psi$$

Solution : $$\psi(x) = \phi(x) - \int d^4x' \Delta_F(x'-x) V(x')\psi(x')$$

where $$(\partial_\mu \partial^\mu + m^2)\phi = 0$$

and $$(\partial_\mu \partial^\mu + m^2)\Delta_F(x'-x) = \delta^4(x'-x)$$

Simplest to solve for propagator in momentum space by taking Fourier transform

$$\frac{1}{(2\pi)^2}\int e^{-ip.(x'-x)} \quad (\partial_\mu \partial^\mu + m^2)\Delta_F(x'-x) d^4(x'-x) = \frac{1}{(2\pi)^2}\int e^{-ip.(x'-x)} \delta^4(x'-x) d^4(x'-x)$$

$$\Rightarrow \quad (-p^2 + m^2)\tilde{\Delta}_F(p) = \frac{1}{(2\pi)^2}$$

$$\tilde{\Delta}_F(p) = \frac{1}{(2\pi)^2}\frac{1}{-p^2 + m^2 + i\varepsilon}, \qquad \Delta_F(x) = -\frac{1}{(2\pi)^4}\int d^4p \, e^{-ip.x} \frac{1}{p^2 - m^2 - i\varepsilon}$$

The Born series

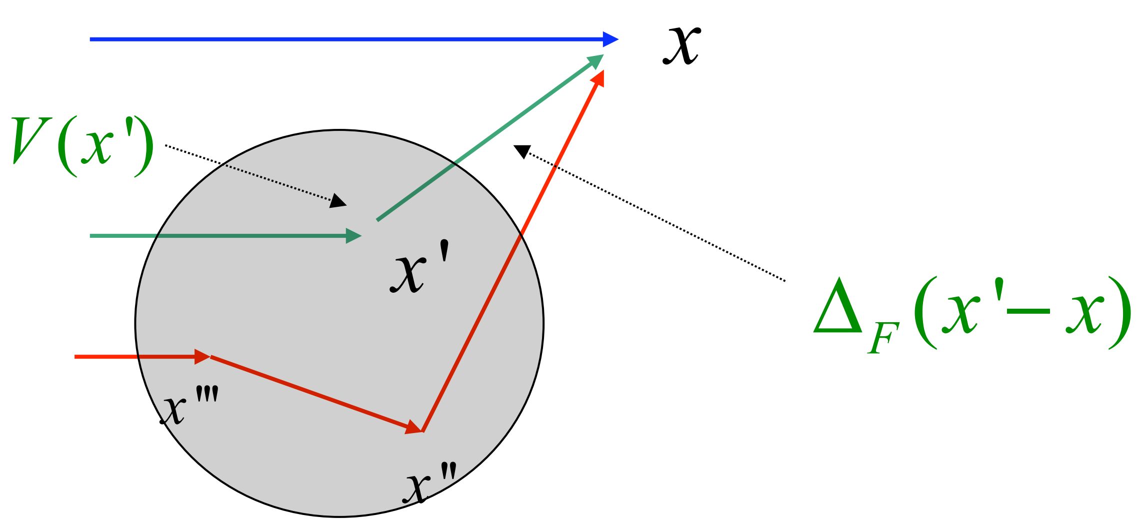
$$\psi(x) = \phi(x) - \int d^4x' \Delta_F(x'-x) V(x') \psi(x')$$

Since V(x) is small can solve this equation iteratively :

$$\psi(x) = \phi(x) - \int d^4x' \Delta_F(x'-x) V(x') \phi(x')$$

$$+ \int d^4x'' \int d^4x''' \Delta_F(x''-x) V(x'') \Delta_F(x'''-x'') V(x''') \phi(x''') + \ldots$$

Interpretation :



But energy eigenvalues $E = \pm(\mathbf{p}^2 + m^2)^{1/2}$???

Feynman – Stuckelberg interpretation

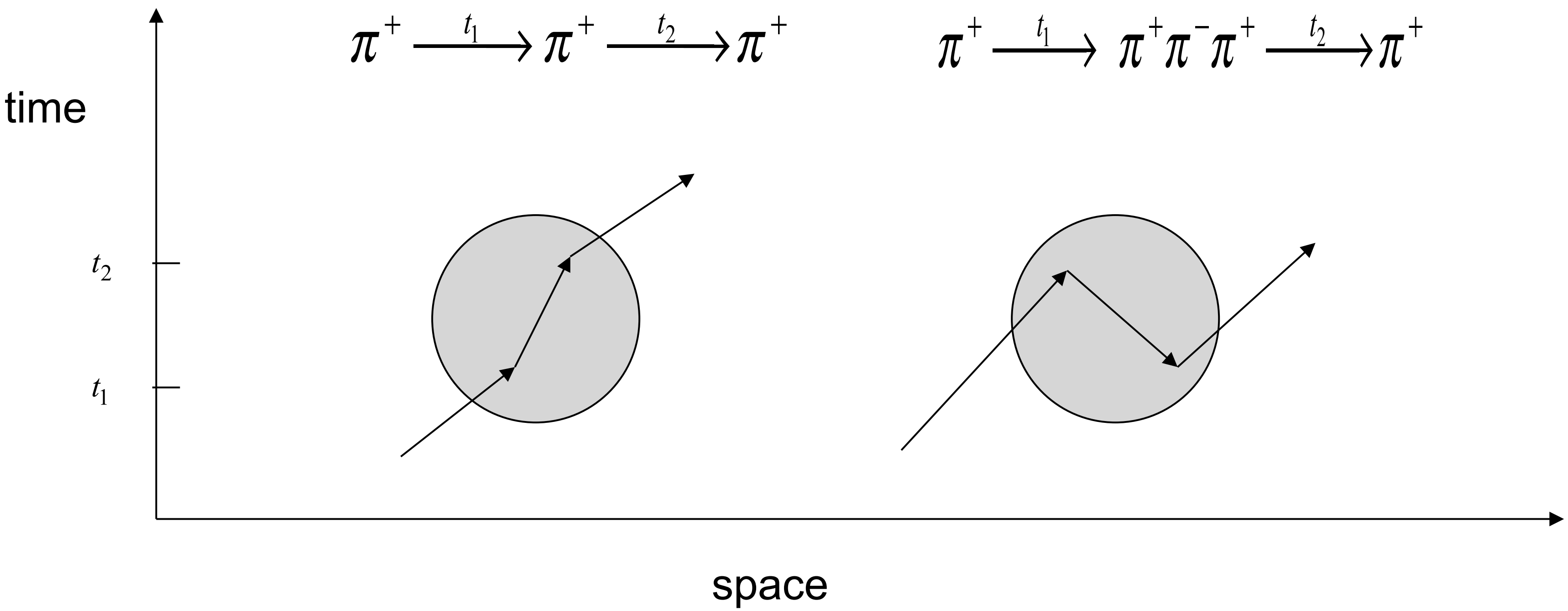
t

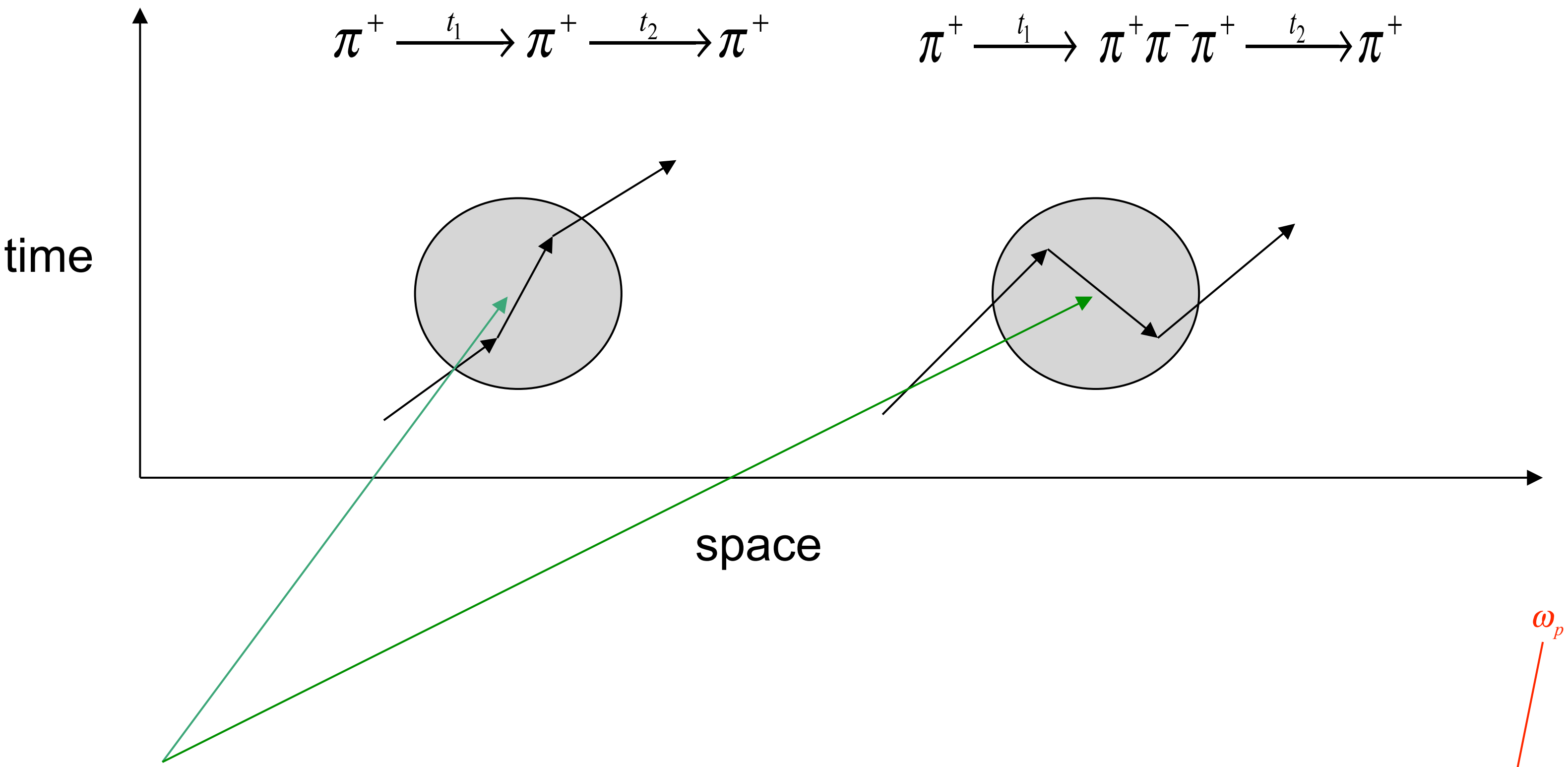
$\pi^+(E > 0)$ $\equiv$ $\pi^-(E < 0)$

e^{-iEt} $\qquad$ $e^{-i(-E)(-t)}$

Two different time orderings giving same observable event :

$\pi^+ \xrightarrow{t_1} \pi^+ \xrightarrow{t_2} \pi^+$ $\qquad$ $\pi^+ \xrightarrow{t_1} \pi^+\pi^-\pi^+ \xrightarrow{t_2} \pi^+$





$$\omega_p = \left(\underline{p}^2 + m^2\right)^{1/2}$$

$$\Delta_F(x'-x) = -\frac{1}{(2\pi)^4}\int d^4p\, e^{-ip.(x'-x)} \frac{1}{p^2 - m^2 - i\varepsilon} = -i\int \frac{d^3p}{(2\pi)^3\, 2\omega_p} e^{-i\omega_p|t'-t| - i\mathbf{p}.(\mathbf{x}'-\mathbf{x})}$$

(p^0 integral most conveniently evaluated using contour integration via Cauchy's theorem)

$$\Delta_F(x) = -\frac{1}{(2\pi)^4}\int d^4p\, e^{-ip.x}\,\frac{1}{p^2 - m^2 - i\varepsilon}$$

$$p^2 + m^2 - i\varepsilon \quad\Rightarrow\quad p_o^2 = \underline{p}^2 + m^2 - i\varepsilon \quad\Rightarrow\quad p_0 = \pm\left(\underline{p}^2 + m^2\right)^{1/2} \mp i\delta = \pm\omega_p \mp i\delta$$

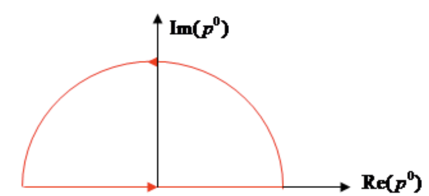
$$\Delta_F(x'-x) = -\frac{1}{(2\pi)^4}\int d^3p\, e^{-i\underline{p}.(\underline{x}'-\underline{x})}\int dp_0\,\frac{e^{-ip_0(t'-t)}}{\left(p_0 - (\omega_p - i\delta)\right)\underbrace{\left(p_0 - (-\omega_p + i\delta)\right)}_{I}}$$

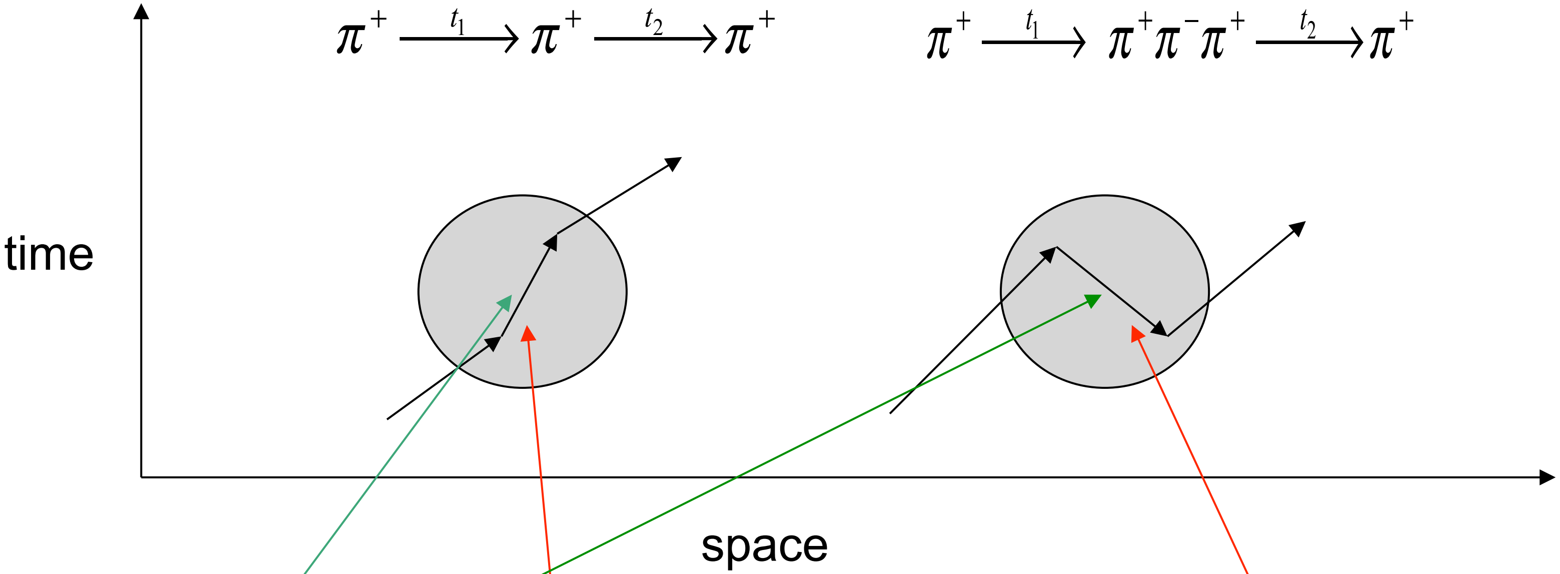
- If $t'-t > 0$, choose contour such that $p_0 = -ip_I \quad (p_I \text{ +ve}) \Rightarrow e^{-ip_0(t'-t)} = e^{-p_I(t'-t)}$

$$I = -\frac{\pi i}{\omega_p} e^{-i\omega_p(t'-t)}\theta(t'-t)$$

- If $t'-t < 0$, choose contour such that $p_0 = +ip_I \quad (p_I \text{ +ve}) \Rightarrow e^{+ip_0(t'-t)} = e^{-p_I(t'-t)}$

$$I = -\frac{\pi i}{\omega_p} e^{+i\omega_p(t'-t)}\theta(t-t')$$





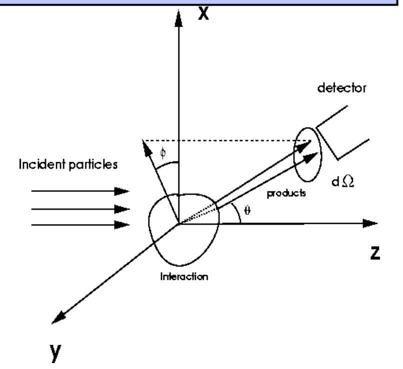
$$\Delta_F(x'-x) = -\frac{1}{(2\pi)^4}\int d^4p\, e^{-ip.(x'-x)}\,\frac{1}{p^2-m^2-i\varepsilon} = -i\int \frac{d^3p}{(2\pi)^3\, 2\omega_p}\, e^{-i\omega_p|t'-t|-ip.(x'-x)}$$

$$\Delta_F(x-x') = -i\int d^3p\, f_p^{+}(x')\, f_p^{+*}(x)\,\theta(t'-t) - i\int d^3p\, f_p^{-}(x')\, f_p^{-*}(x)\,\theta(t-t')$$

where $f_p^{\pm} = e^{\mp ip.x}\frac{1}{\sqrt{2p^0V}}$ are positive and negative energy solutions to free KG equation

Theory confronts experiment - Cross sections and decay rates

Scattering in Quantum Mechanics



- Prepare state at $t = -\infty$ $\qquad |\psi_{in}(t=-\infty)> = |i>$
- Time evolution (possibly scattering) $\qquad |\psi_{in}(t=+\infty)> = S|\psi_{in}(t=-\infty)>$
- Observe resulting system in state $\qquad |\psi_{out}(t=+\infty)> = |f>$

QM : probability amplitude :

$$<\psi_{out}(t=+\infty)|\psi_{in}(t=+\infty)> = <\psi_{out}(t=+\infty)|S|\psi_{in}(t=-\infty)>$$
$$= <f|S|i> = S_{fi}$$

$$S_{fi} = \delta_{fi} + iT_{fi}$$

S matrix for Klein Gordon scattering

Lorentz invariant probability density

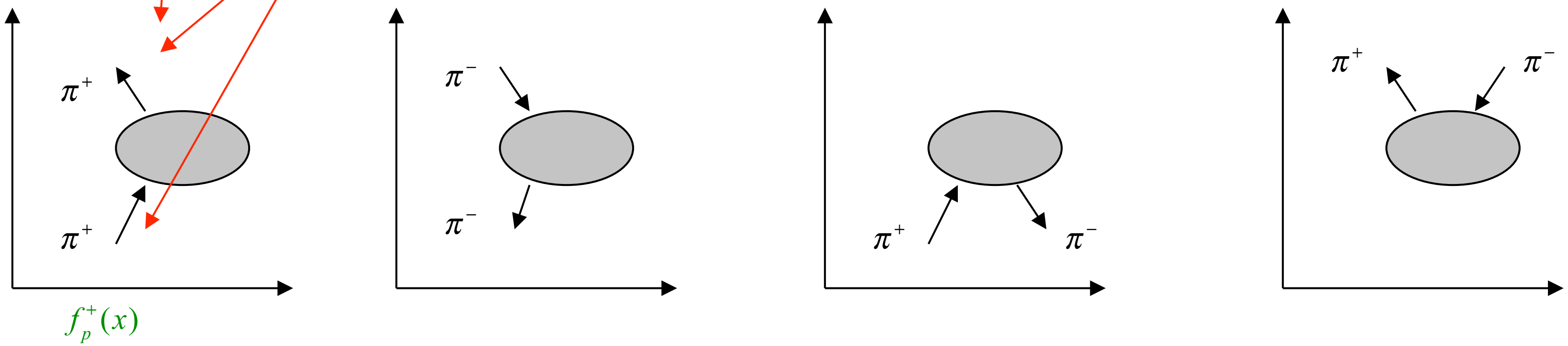
c.f. KG derivation

$$S_{\mathbf{p}'+,\mathbf{p}+} = < \psi_{out}(t=+\infty) \mid \psi_{in}(t=+\infty) > \;\; = \lim_{t\to\infty} \int d^3x \; f^{+*}_{p'+} i\partial_0 \psi(x)$$

$$\psi(x) = \phi(x) - \int d^4x' \Delta_F(x-x') V(x')\phi(x') + \ldots$$

$$\Delta_F(x'-x) = -i\int d^3p \; f^+_p(x') f^{+*}_p(x)\theta(t'-t) - i\int d^3p \; f^-_p(x') f^{-*}_p(x)\theta(t-t')$$

$$S_{\mathbf{p}'+,\mathbf{p}+} = \delta^3(\mathbf{p}'_+ - \mathbf{p}_+) - i\int d^4x' \; f^{+*}_{p+}(x') \, V(x') \, f^+_{p+}(x') + \ldots$$



Feynman rules

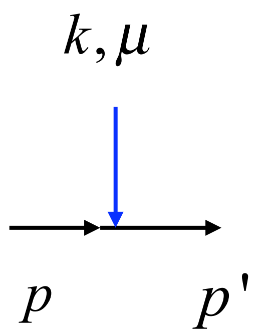
$$iT_{fi} = -i\int d^4y\, f^{+*}_{p+}(y)\, V(y)\, f^{+}_{p+}(y) = i\int d^4y\, f^{+*}_{p+}(y)\, ie(A^\mu\partial_\mu + \partial_\mu A^\mu)\, f^{+}_{p+}(y)$$

$$f^{\pm}_{p} = e^{(-,+)ip.x}\frac{1}{\sqrt{2p^0V}}$$

$$= -i\int d^4y\, j^{fi}_{\mu} A^\mu$$

$$j^{fi}_{\mu} = -ie\left(f^{+*}_{p+}(y)\left[\partial_\mu f^{+}_{p+}(y)\right] - \left[\partial_\mu f^{*+}_{p+}(y)\right] f^{+}_{p+}(y)\right)$$

$$= -e(p_f + p_i)_\mu e^{i(p_f - p_i).y}$$



$$-ie(p_\mu + p'_\mu)$$

Feynman rule associated with Feynman diagram

Fundamental experimental objects

$\Gamma(a_1 \to b_1 b_2 \ldots b_n)$ **Decay width = 1/lifetime** (Dimension 1/T=M)

$\sigma(a_1 a_2 \to b_1 b_2 \ldots b_n)$ **Cross section** (Dimension L^2=M^{-2})

Units "barn"

$$1\,barn = 10^2\,fm^2$$
$$1\,mb \quad = 10^{-1}\,fm^2 \quad "milli"$$
$$1\,\mu b \quad = 10^{-4}\,fm^2 \quad "micro"$$
$$1\,nb \quad = 10^{-7}\,fm^2 \quad "nano"$$
$$1\,pb \quad = 10^{-10}\,fm^2 \quad "pico"$$
$$1\,fb \quad = 10^{-13}\,fm^2 \quad "fempto"$$

(Natural Units $1 GeV^{-2} = 0.39 mb$)

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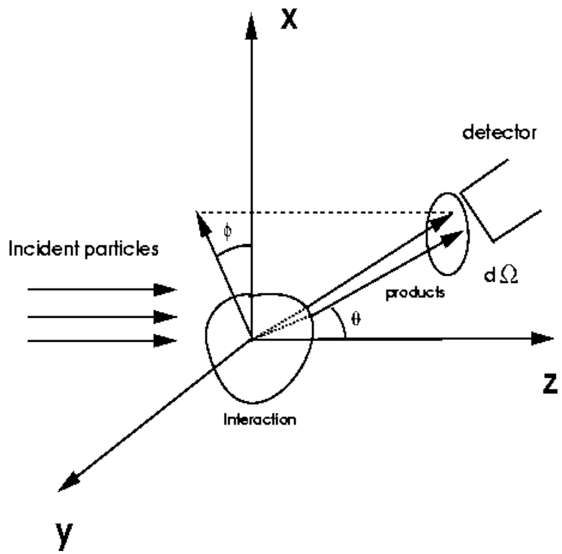


$$\text{Cross section} = \frac{\text{Transition rate x Number of final states}}{\text{Initial flux}}$$

Fundamental experimental objects

$\Gamma(a_1 \to b_1 b_2 ... b_n)$ **Decay width = 1/lifetime** (Dimension 1/T=M)

$\sigma(a_1 a_2 \to b_1 b_2 ... b_n)$ **Cross section** (Dimension L^2=M^{-2})

Momenta of final state forms phase space

Cross section = $\dfrac{\text{Transition rate x Number of final states}}{\text{Initial flux}}$

For a single particle the number of final states in volume V with momenta in element $d^3 p$ is $\frac{V d^3 p}{(2\pi)^3}$ ⇒ $\Pi_{i=1}^{n} \frac{V d^3 p}{(2\pi)^3}$

Fundamental experimental objects

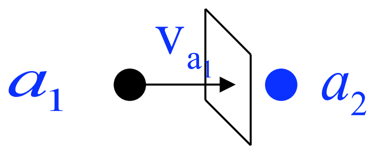
$\Gamma(a_1 \to b_1 b_2 \ldots b_n)$ **Decay width = 1/lifetime**

$\sigma(a_1 a_2 \to b_1 b_2 \ldots b_n)$ **Cross section**

$$\Pi_{i=1}^{n} \frac{V d^3 p}{(2\pi)^3}$$

$$\text{Cross section} = \frac{\text{Transition rate x Number of final states}}{\text{Initial flux}}$$

$$\frac{|\mathrm{v}_{a_1}|}{V} \times \frac{1}{V}$$



(Lab frame)

particles passing through unit area in unit time

target particles per unit volume