Extension to non-Abelian symmetry

$(The\ Standard\ \ Model \quad SU(3)\otimes SU(2)\otimes U(1))$

$SU(2)$ local gauge invariance

Yang-Mills (+Shaw)

- **The Lorentz transformations form a group, G** $(g_1 g_2 \in G \quad if \quad g_1, g_2 \in G)$

Rotations

$$R(\theta) = e^{-i\mathbf{J}.\theta/\hbar}, \quad J_z = i\hbar(x\tfrac{\partial}{\partial y} - y\tfrac{\partial}{\partial x})..$$

Angular momentum operator

$$(c.f.\,\mathbf{J} = \mathbf{r}\times\mathbf{p})$$

$$[J_i, J_j] = i\hbar\sum_{k=1}^{3}\varepsilon_{ijk}J_k$$

$$SO(3) \quad (SU(2))$$

$$e.g. \quad Spin \quad R(\theta)\begin{pmatrix}1\\0\end{pmatrix} = e^{-i\alpha.\sigma/2\hbar}\begin{pmatrix}1\\0\end{pmatrix}$$

$$\sigma_1 = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} \quad \sigma_2 = \begin{pmatrix}0 & -i\\i & 0\end{pmatrix} \quad \sigma_3 = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$$

Extension to non-Abelian symmetry

Yang-Mills (+Shaw)

$(The\ Standard\ \ Model \quad SU(3)\otimes SU(2)\otimes U(1))$

$SU(2)$ local gauge invariance

$$Q=\begin{pmatrix}u\\d\end{pmatrix}_L, u_R, d_R$$

$$Q \rightarrow e^{ig_2 \vec{\alpha}(\xi).\frac{\vec{\sigma}}{2}} Q$$

$$\sigma_1 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \quad \sigma_2 = \begin{pmatrix}0 & i\\ -i & 0\end{pmatrix} \quad \sigma_3 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

$$\mathfrak{L} = i\overline{Q} D_\mu \gamma^\mu Q + i\overline{u}_R \partial_\mu \gamma^\mu u_R + i\overline{d}_R \partial_\mu \gamma^\mu d_R$$

$$\mathrm{D}_\mu = \partial_\mu + i g_2 \frac{\sigma_i}{2} W_\mu^i \qquad D_\mu Q \rightarrow e^{ig_2 \alpha(x).\frac{\sigma}{2}} D_\mu Q$$

where

$$W_{\mu,i} \rightarrow W_{\mu,i} - \partial_\mu \alpha_i - g_2 \varepsilon_{ijk} \alpha_j W_{\mu,k}$$

$$\left(\left[\frac{\sigma_i}{2}, \frac{\sigma_j}{2}\right] = i\varepsilon_{ijk}\frac{\sigma_k}{2}\right)$$

Extension to non-Abelian symmetry

(*The Standard Model* $SU(3) \otimes SU(2) \otimes U(1)$)

$SU(2)$ local gauge invariance

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$$D_\mu = \partial_\mu + ig_2 \frac{\sigma_i}{2} W_\mu^i$$

Need 3 gauge bosons

W^+, W^-, W^3

where

$$W_{\mu,i} \rightarrow W_{\mu,i} - \partial_\mu \alpha_i - g_2 \varepsilon_{ijk} \alpha_j W_{\mu,k}$$

$$\left(\left[\frac{\sigma_i}{2}, \frac{\sigma_j}{2} \right] = i\varepsilon_{ijk} \frac{\sigma_k}{2} \right)$$

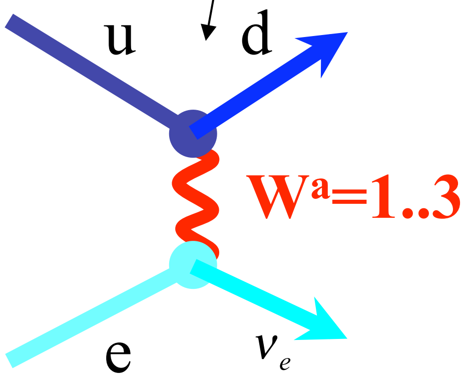
Weak Interactions

SU(2) local gauge theory

$$\begin{pmatrix} u \\ d \end{pmatrix}_L \begin{pmatrix} \nu \\ e \end{pmatrix}_L$$

$$u_R, d_R, e_R$$

Weak coupling, α_2



Gauge boson (J=1)

$W^{\pm}, Z$

Neutral currents

Symmetry :

Local conservation of 2 weak isospin charges

$$\Psi_a \rightarrow \left(e^{ig_2\alpha(x).\tau}\right)_b^a \Psi_a$$

$$W_\mu^r \rightarrow W_\mu^r - \partial_\mu \alpha^r - f^{rst}\alpha^s W_\mu^t$$

$$\overline{\Psi}(\partial_\mu - ig_2\lambda_r W_\mu^r)\gamma^\mu \Psi$$

A non-Abelian (SU(2)) local gauge field theory

$SU(3)$ local gauge invariance

$$\begin{pmatrix} q \\ q \\ q \end{pmatrix}$$

$$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_8 = \sqrt{\frac{1}{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ldots$$

Symmetry :

Local conservation of
3 strong colour charges

$$\Psi_a \to \left(e^{ig_3\alpha(x).\lambda}\right)^a_b \Psi_a$$

$$G^r_\mu \to G^r_\mu - \partial_\mu \alpha^r - g_3 f^{rst} \alpha^s G^t_\mu$$

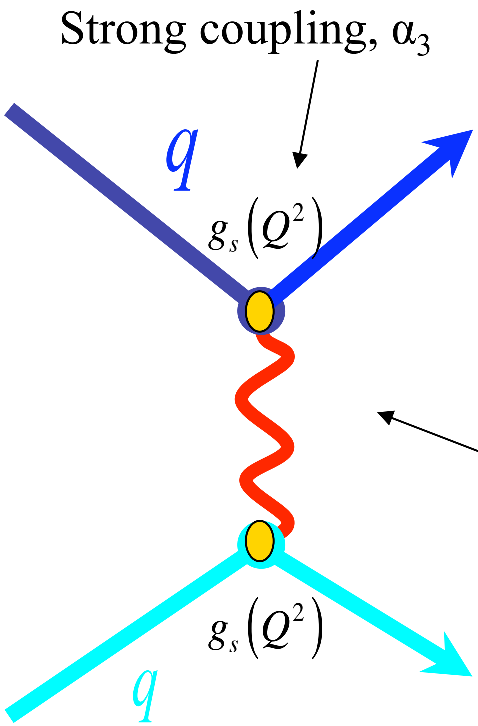
$$\overline{\Psi}(\partial_\mu - ig_3\lambda_r G^r_\mu)\gamma^\mu \Psi$$

QCD : a non-Abelian (SU(3))
local gauge field theory

The strong interactions

QCD Quantum Chromodynamics

SU(3) $\begin{pmatrix} q \\ q \\ q \end{pmatrix}$



Gauge boson
(J=1)
"Gluons"

G^a=1..8

Symmetry :

Local conservation of
3 strong colour charges

$$\Psi_a \to \left(e^{i\alpha(x).\lambda}\right)_b^a \Psi_a$$

$$G_\mu^r \to G_\mu^r - \frac{1}{g_3}\partial_\mu \alpha^r - f^{rst}\alpha^s G_\mu^t$$

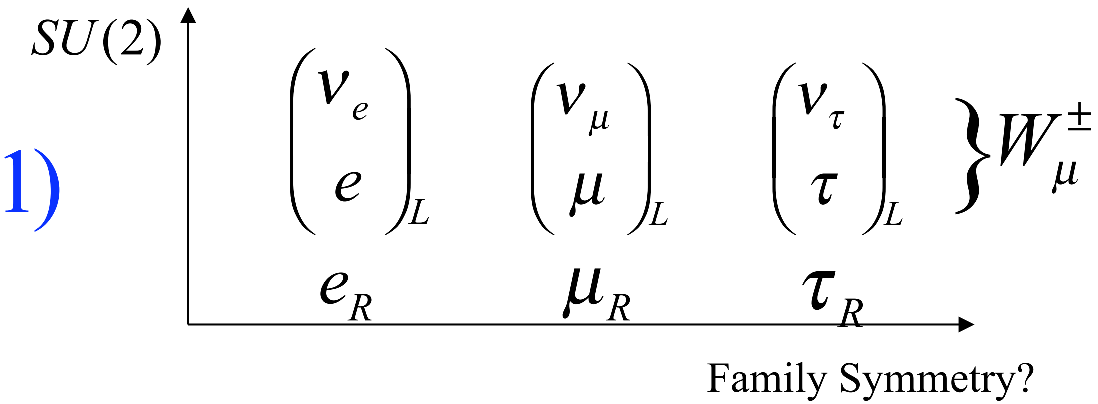
$$\overline{\Psi}(\partial_\mu - i g_3 \lambda_r G_\mu^r)\gamma^\mu \Psi$$

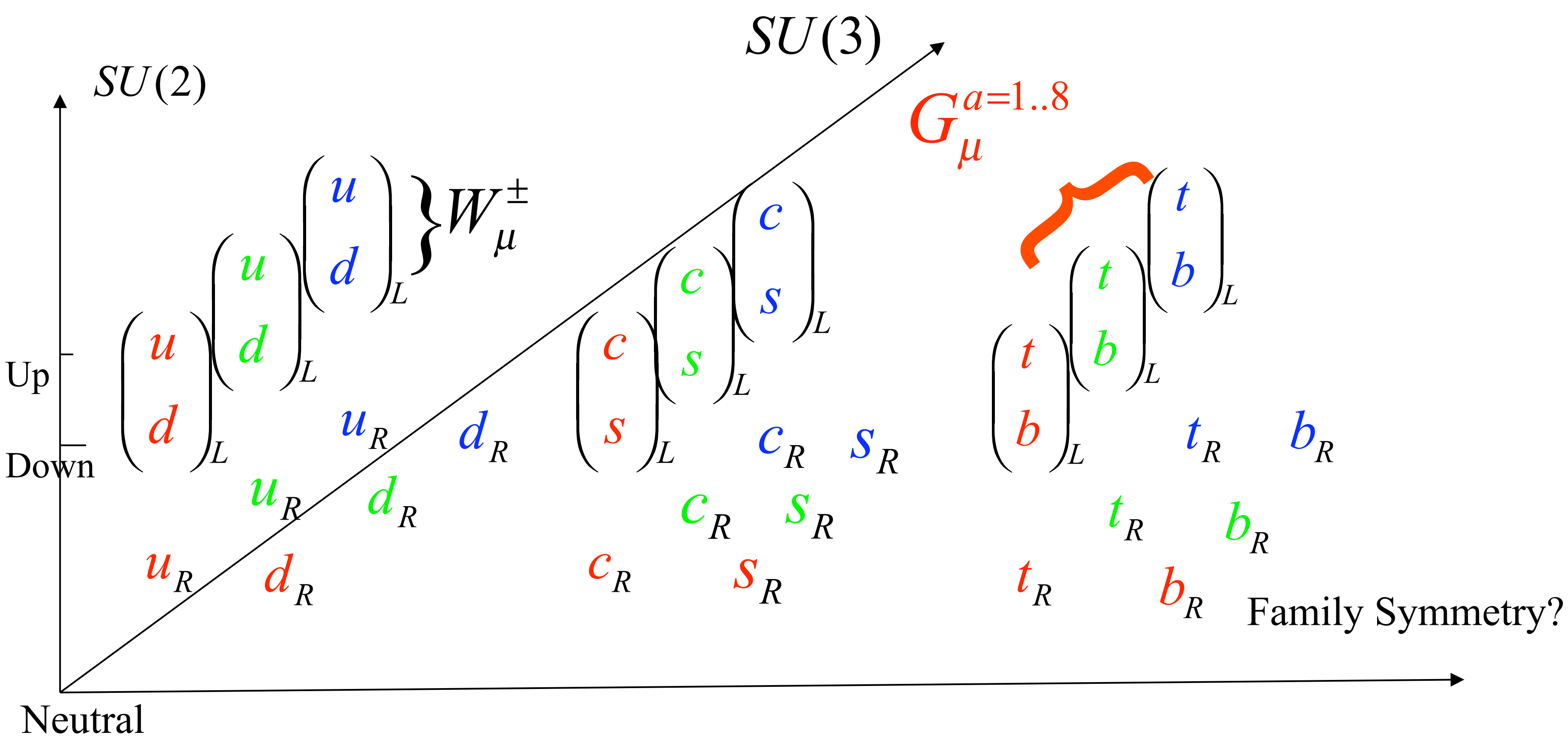
QCD : a non-Abelian (SU(3))
local gauge field theory

Partial Unification

$SU(3) \otimes SU(2) \otimes U(1)$

Matter Sector "chiral"





The inclusion of fermions – J=1/2 particles

Weyl spinors

$(\tfrac{1}{2},0)$ $(0,\tfrac{1}{2})$

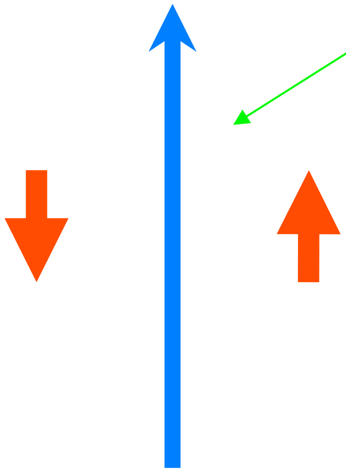
ψ_L ψ_{R}

2-component spinors of SU(2) $\begin{pmatrix} a(x) \\ b(x) \end{pmatrix}$

Dirac spinor

$$\psi = \begin{bmatrix} \psi_{\mathrm{L}} \\ \psi_{\mathrm{R}} \end{bmatrix}$$

velocity

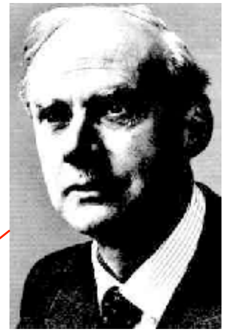


Left-handed

m=0

Right-handed

$$u_{e^-_L} \equiv u^c_{e^+_R}$$



The spinor structure follows from the representation structure of the Lorentz Group …

The Lorentz group

Rotations J_i Boosts K_i

$$[J_i, J_j] = i\varepsilon_{ijk} J_k$$
$$[J_i, K_j] = i\varepsilon_{ijk} K_k$$
$$[K_i, K_j] = -i\varepsilon_{ijk} J_k$$

} Generate the group SO(3,1)

$$(M_{\rho\sigma} = i(x_\rho \tfrac{\partial}{\partial x^\sigma} - x_\sigma \tfrac{\partial}{\partial x^\rho}) \qquad J_i = \tfrac{1}{2}\varepsilon_{ijk} M_{jk} \qquad K_i = M_{0i})$$

To construct representations a more convenient (non-Hermitian) basis is

$$N_i = \tfrac{1}{2}(J_i + iK_i)$$

$$[N_i, N_j] = i\varepsilon_{ijk} N_k$$
$$[N_i^\dagger, N_j^\dagger] = i\varepsilon_{ijk} N_k^\dagger$$
$$[N_i, N_j^\dagger] = 0$$

} $SU(2) \otimes SU(2)$ *representation* (n, m)

The Lorentz group

Rotations $\mathrm{J_i}$ Boosts $\mathrm{K_i}$

$$[J_i, J_j] = i\varepsilon_{ijk} J_k$$
$$[J_i, K_j] = i\varepsilon_{ijk} K_k$$
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$$[N_i, N_j^\dagger] = 0$$

Representations $\quad J_i = N_i + N_i^\dagger$

$$(n, m) \quad J = n + m$$

$(0,0)$ scalar $J=0$

$(\frac{1}{2},0), (0,\frac{1}{2})$ LH and RH spinors $J=\frac{1}{2}$

$(\frac{1}{2},\frac{1}{2})$ vector $J=1$, etc

Weyl spinors

$(\frac{1}{2},0)$ $(0,\frac{1}{2})$

ψ_L ψ_R

2-component spinors of SU(2)

Rotations and Boosts $\psi_{L(R)} \rightarrow S_{L(R)}\,\psi_{L(R)}$

$S_{L(R)} = e^{i\frac{\alpha}{2}.\sigma}$: Rotations

$S_{L(R)} = e^{\pm\frac{\nu}{2}.\sigma}$: Boosts

Dirac spinor

Can combine $\psi_L,\ \psi_R$ to form a 4-component “Dirac” spinor $\psi = \begin{bmatrix} \psi_L \\ \psi_R \end{bmatrix}$

Lorentz transformations $\psi \rightarrow e^{i\omega\sigma}\psi, \quad \omega\sigma = \omega^{\mu\nu}\sigma_{\mu\nu} = \frac{i}{2}\left[\gamma_\mu, \gamma_\nu\right]\omega^{\mu\nu}$

where $\gamma_0 = \begin{bmatrix} 0 & -I \\ -I & 0 \end{bmatrix}, \quad \gamma_i = \begin{bmatrix} 0 & \sigma_i \\ -\sigma_i & o \end{bmatrix}, \quad \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$

Weyl basis

$\omega^{0i} \rightarrow boosts, \quad \omega^{ij} \rightarrow rotations \qquad i,j = 1,2,3$

Weyl spinors

$(\tfrac{1}{2},0)$ $(0,\tfrac{1}{2})$

ψ_L ψ_R

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$S_{L(R)} = e^{i\frac{\alpha}{2}.\sigma}$: Rotations

$S_{L(R)} = e^{\pm\frac{v}{2}.\sigma}$: Boosts

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Can combine ψ_L, ψ_R to form a 4-component "Dirac" spinor $\psi = \begin{bmatrix}\psi_L \\ \psi_R\end{bmatrix}$

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where $\gamma_0 = \begin{bmatrix}0 & -I \\ -I & 0\end{bmatrix},\quad \gamma_i = \begin{bmatrix}0 & \sigma_i \\ -\sigma_i & o\end{bmatrix},\quad \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{bmatrix}-I & 0 \\ 0 & I\end{bmatrix}$

(Dirac gamma matrices, …new 4-vector γ_μ)

$\{\gamma^\mu,\gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$

Note : $\psi_{R(L)} = \tfrac{1}{2}(1 \pm \gamma_5)\psi$

The Dirac equation

Fermions described by 4-cpt Dirac spinors ψ

- $\psi^\dagger \gamma^0 \psi \equiv \overline{\psi}\psi = \psi^*_L \psi_R + \psi^*_R \psi_L$ Lorentz invariant
- New 4-vector γ_μ

The Lagrangian

$$\mathfrak{L} = i\overline{\psi}\,\gamma_\mu \partial^\mu\,\psi - m\overline{\psi}\,\psi$$

Dimension?

From Euler Lagrange equation obtain the Dirac equation

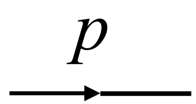
$$\frac{\partial \mathfrak{L}}{\partial \phi} - \partial^\mu \frac{\partial \mathfrak{L}}{\partial(\partial^\mu \phi)} = 0$$

$$(i\gamma_\mu \partial^\mu - m)\psi = 0$$

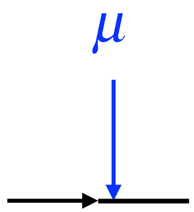
U(1) symmetry

$$\psi \rightarrow e^{i\alpha}\psi \quad \rightarrow \quad j^\mu = -e\overline{\psi}\,\gamma^\mu \psi$$

Feynman rules



$$\frac{i}{(p_\mu \gamma^\mu - m)}$$



$$ie\gamma^\mu$$

Fermion masses

$$\psi^{\dagger}\gamma^{0}\psi \equiv \overline{\psi}\psi = \psi_{L}^{*}\psi_{R} + \psi_{R}^{*}\psi_{L}$$ Lorentz invariant

$$\mathfrak{L} = m\left(e_{L}^{*}e_{R} + e_{R}^{*}e_{L}\right)$$ Dirac mass

$$\sigma_{2}\psi_{R}^{\dagger} \equiv \psi_{L} \quad \Rightarrow \quad \psi_{R}^{*} = \psi_{L}^{T}\sigma_{2}$$

$$\left(\psi_{L}^{T}\sigma_{2}\psi_{L} + hc\right)$$ Lorentz invariant

Only term allowed by charge conservation is

$$\mathfrak{L}' = m\left(\nu_{L}^{T}\sigma_{2}\nu_{L} + hc\right)$$ Majorana mass

Fermions and the weak interactions

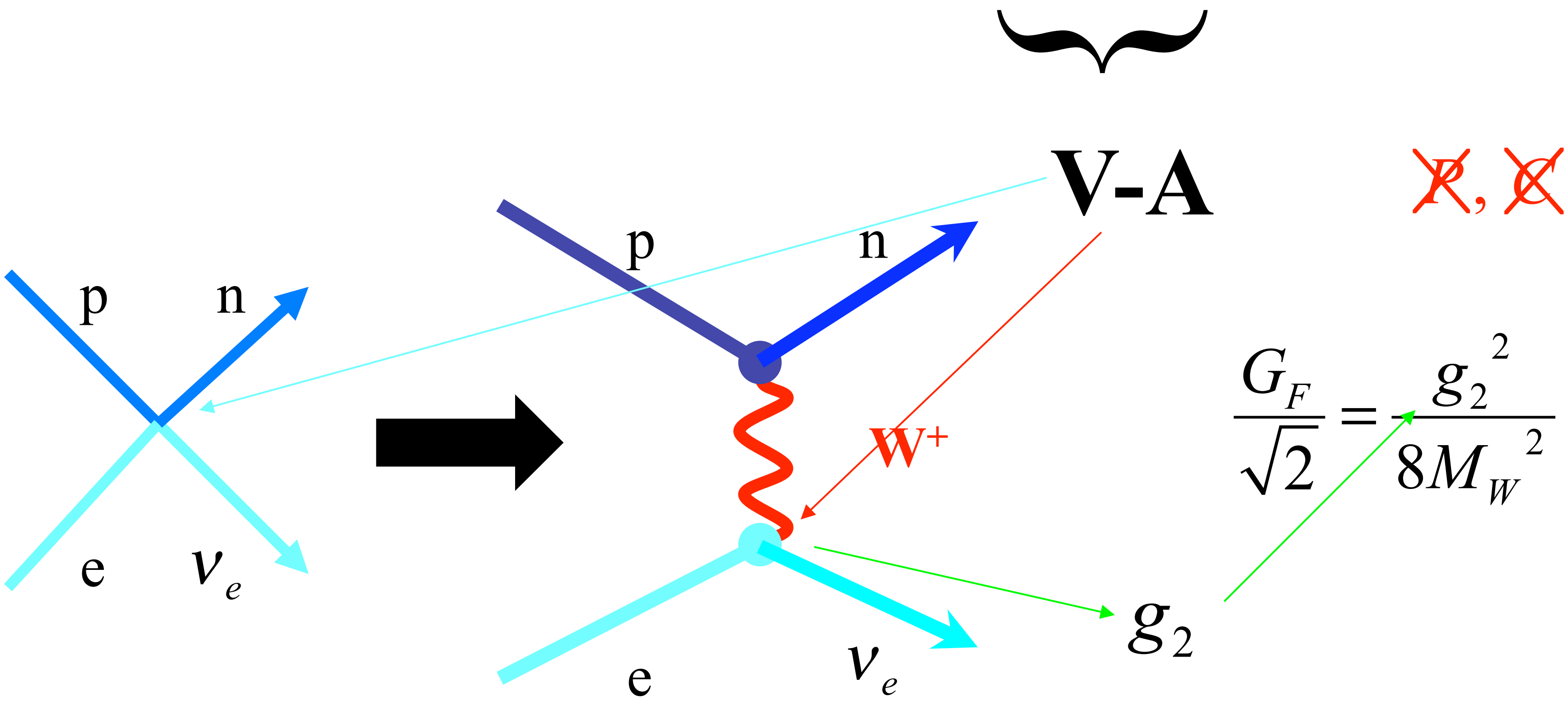
Fermi theory of β decay $\quad n \to p e \bar{\nu}_e$

$$\frac{G_F}{\sqrt{2}}\left[\bar{u}_n \gamma^\sigma (1-\gamma_5) u_p\right]\left[\bar{u}_{\nu_e} \gamma_\sigma (1-\gamma_5) u_e\right]$$

Fermions and the weak interactions

Fermi theory of β decay $\quad n \to p e \bar{\nu}_e$

$$L = \frac{G_F}{\sqrt{2}} \left[\bar{u}_n \gamma^\sigma (1-\gamma_5) u_p \right] \left[\bar{u}_{\nu_e} \gamma_\sigma (1-\gamma_5) u_e \right]$$



Massive vector propagator (W, Z bosons)

$$\left(g^{\nu\mu}(\partial^2 + M^2) - \partial^\nu \partial_\mu\right) B^\mu = j^\nu$$

$$(-g^{\mu\nu}(-p^2 + M^2) + p_\mu p_\nu)^{-1} = \frac{i(-g^{\mu\nu} + p^\mu p^\nu / M^2)}{p^2 - M^2}$$

$$B_\mu = \varepsilon_\mu e^{ip.x}$$

Free particle solution

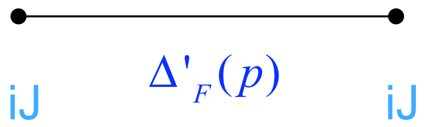
$$\varepsilon^{(\lambda=\pm 1)} = (0,1,\pm i,0)/\sqrt{2}$$

$$\varepsilon^{(\lambda=0)} = (|\mathbf{p}|,0,0,E)/M$$

Helicity polarisation vectors

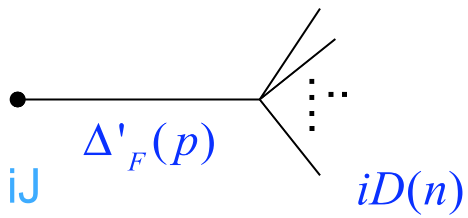
$$\sum_\lambda \varepsilon_\mu^{(\lambda)*} \varepsilon_\nu^{(\lambda)} = -g_{\mu\nu} + \frac{p_\mu p_\nu}{M^2}$$

Propagation of unstable scalar particle



$$= -iJ^2 \Delta'_F(p)$$

No decay



$$= -iJ \Delta'_F(p) D(n)$$

Particle decays into final state n

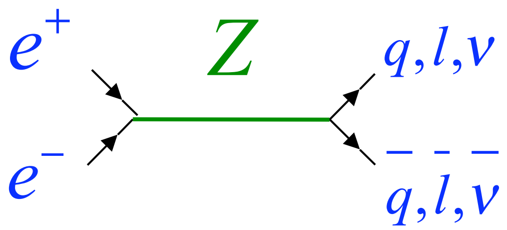
Optical theorem – conservation of probability, time evolution is unitary

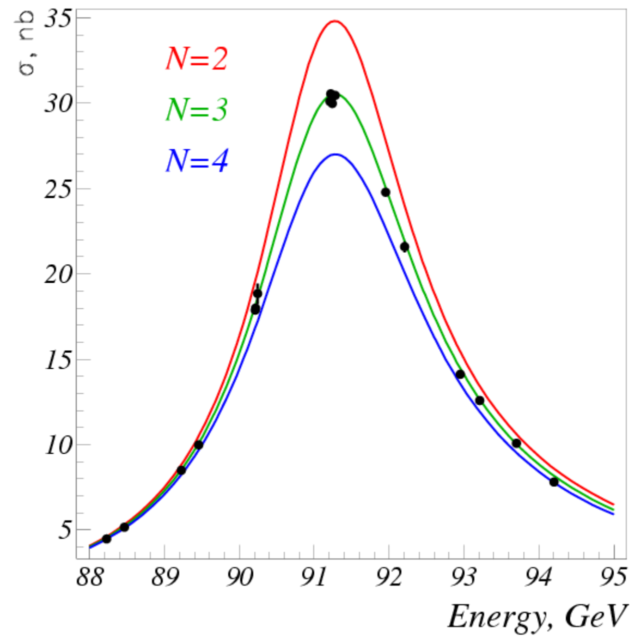
$$S^\dagger S = S S^\dagger = 1 \qquad S_{fi} = \delta_{fi} + iT_{fi}$$

$$\mathrm{Im}(T_{kk}) = \tfrac{1}{2} \sum_n |T_{nk}|^2$$

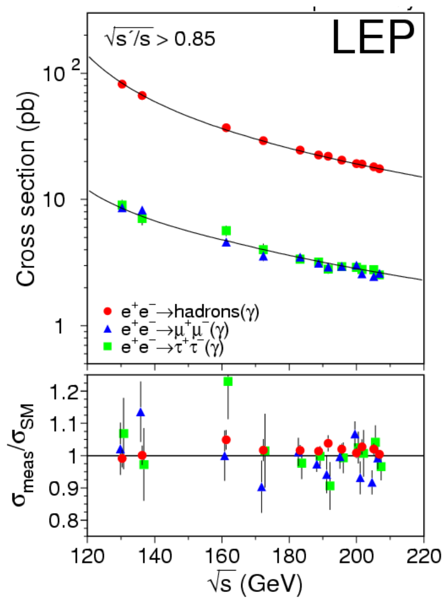
$$-J^2\, \mathrm{Im}(\Delta'_F(p)) = \tfrac{1}{2} \sum_n \left| -iJ \Delta'_F(p) D(n) \right|^2 = J^2 \left| \Delta'_F(p) \right|^2 \overbrace{\int \tfrac{1}{2} \sum_n |D(n)|^2 \, dQ}^{m\Gamma_{tot}}$$

$$\Rightarrow \quad \boxed{\Delta'_F(p) = \frac{1}{p^2 - m^2 + im\Gamma_{tot}}} \qquad (\Delta_F(x) \propto e^{-m\Gamma_{tot} t})$$

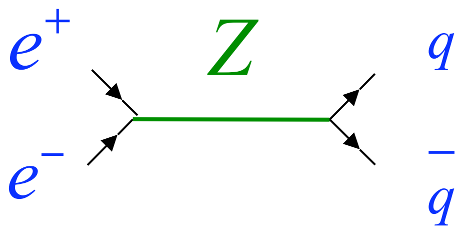


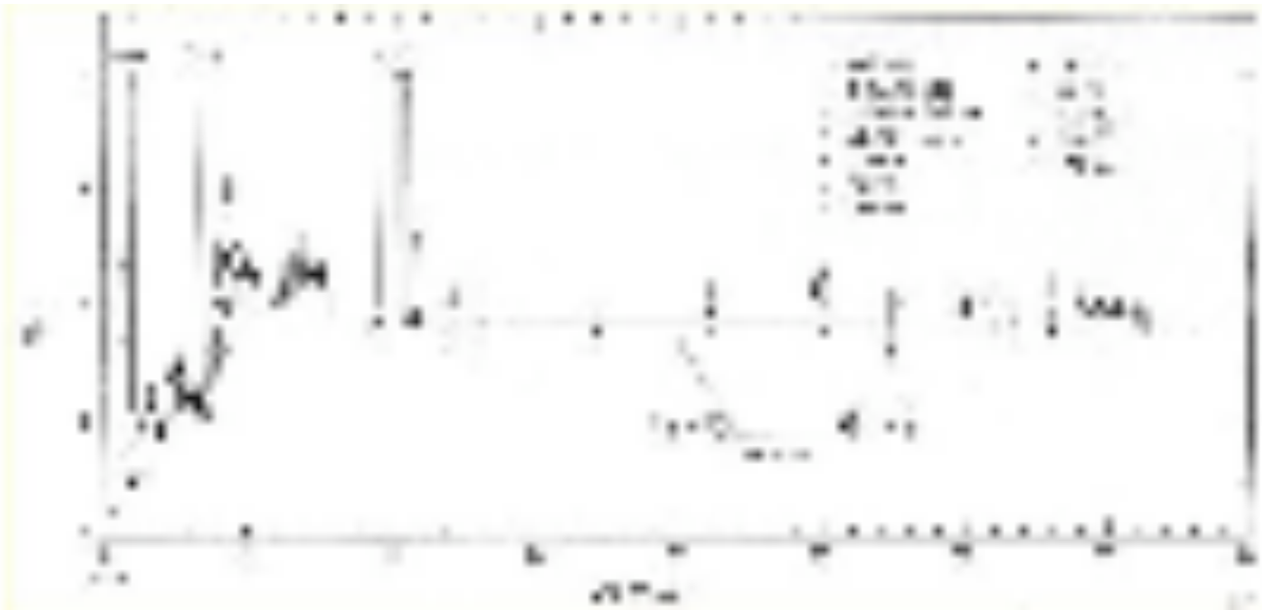


(N is no. of light - wrt M_Z - neutrinos)



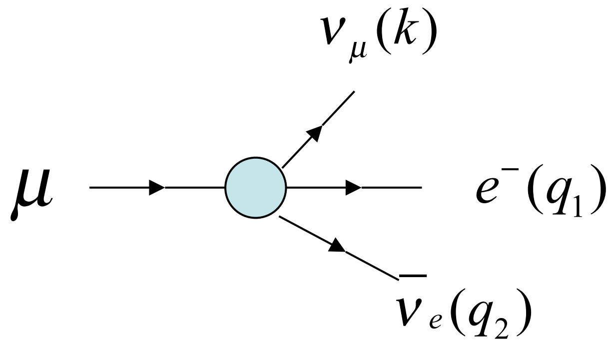
$$\sigma(e^+e^- \to X) = \frac{12\pi\, \Gamma(Z \to ee)\Gamma(Z \to X)}{(E^2 - M_Z^2)^2 + M_Z^2 \Gamma_Z^2}$$





$$\frac{\sigma(e^+e^- \to hadrons)}{\sigma(e^+e^- \to \mu^+\mu^-)} = 3\sum_{f=1}^{n_f} Q_f^2$$

μ *decay*



Fermi theory ('40s)

$$M = \frac{G_F}{\sqrt{2}} \bar{u}(k)\gamma^\mu(1-\gamma_5)u(p)\,\bar{u}(q\ )\gamma_\mu(1-\gamma_5)v(p)$$

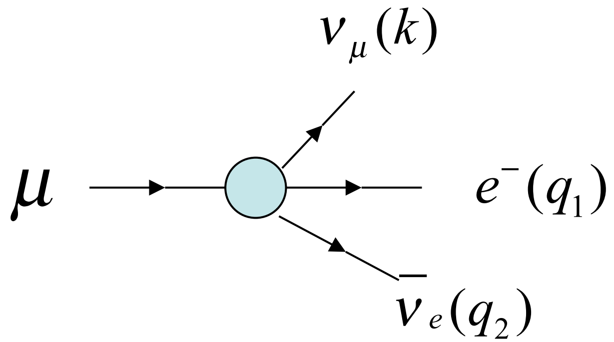
Dimensional analysis

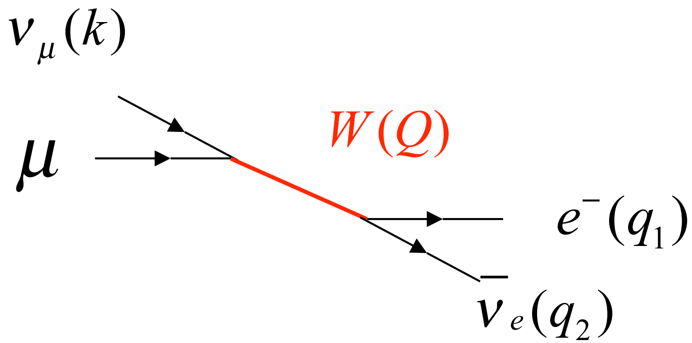
$$\Gamma_{tot} = \frac{1}{192\pi^3} m_\mu^5 G_F^2$$

The hard part!

$$\tau_\mu^{\mathrm{expt}} = \frac{1}{\Gamma} = 2.19703(4)\,10^{-6}\ \mathrm{sec} \quad \Rightarrow \quad G_F = 1.16637(1)\,10^{-5}\ GeV^{-2}$$

μ decay





$$\frac{Q^\mu Q^\nu}{M_W^2} \approx \frac{m_\mu m_e}{M_W^2} \approx 0$$

$$M = ig_W \, \bar{u}(k)\gamma^\mu(1-\gamma_5)u(p) \quad \frac{g_{\mu\nu} - \frac{Q_\mu Q_\nu}{M_W^2}}{Q^2 - M_W^2 + i\varepsilon} \quad g_W \bar{u}(q\;)\gamma_\mu(1-\gamma_5)v(p)$$

In μ decay $\quad Q^2 \le O(m_\mu^2) << M_W^2$

$$G_F = 1.16637(1)\,10^{-5}\ GeV^{-2}$$

$$\Rightarrow \quad \frac{g_W^2}{Q^2 - M_W^2} \approx \frac{-g_W^2}{M_W^2} \quad \Rightarrow \quad \frac{g_W^2}{M_W^2} = \frac{G_F}{\sqrt{2}}$$

$$\approx 80 GeV$$

Fundamental principles of particle physics

- Introduction - Fundamental particles and interactions
- Symmetries I - Relativity
- Quantum field theory - Quantum Mechanics + relativity
- Theory confronts experiment - Cross sections and decay rates
- Symmetries II – Gauge symmetries, the Standard Model
- Fermions and the weak interactions

⇨ The Standard Model and Beyond

Have Fun!