SPECIAL RELATIVITY

(in 60 minutes ...)

With very strong emphasis on electrodynamics and accelerators

Werner Herr

Why Special Relativity ? because we have problems ...

- We have to deal with moving charges in accelerators
- Electromagnetism and fundamental laws of classical mechanics show inconsistencies
- Ad hoc introduction of Lorentz force
- Applied to moving bodies Maxwell's equations lead to asymmetries not shown in observations of electromagnetic phenomena
- Without relativity: no magnetic fields !

→ Essential to accelerator theory to sort this out

Principles of Relativity (Newton, Galilei)

Definition:

A frame moving at constant velocity is an (Inertial System)

Physical laws are invariant in all inertial systems

invariant:

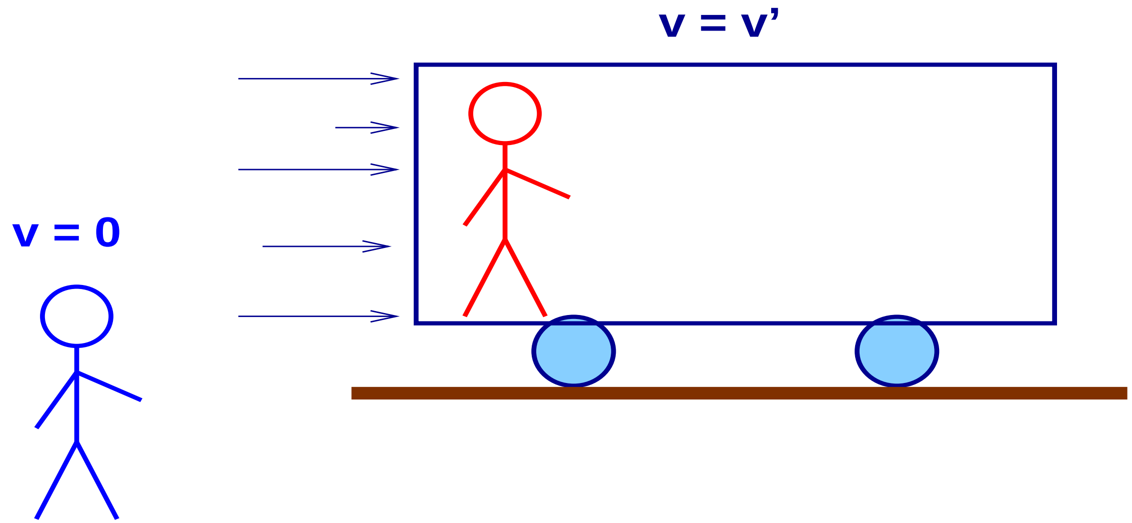
→ **the mathematical equations keep the same form**

Example (see example in Rende's lecture): we would like to have

$$\frac{d^2x}{dt^2} + kx = 0 \qquad \text{and} \qquad \frac{d^2x'}{dt'^2} + k'x' = 0$$

Example: two frames/system where something is happening ...

Assume a frame at rest (S) and another frame (S') moving in x-direction with velocity $\vec{v} = (v', 0, 0)$



- **Something is happening in the moving frame/car**
- **Passenger describes the observations <u>within</u> the frame**
- **Observer describes the observations <u>from</u> the rest frame**

Relativity: so how do we relate observations ?

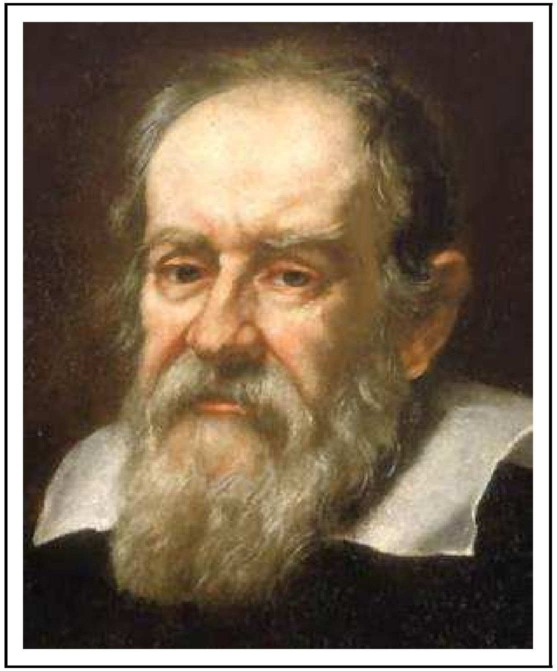
1. We have observed and described an event in rest frame S using coordinates (x, y, z) and time t

2. How can we describe it seen from a moving frame S' using coordinates (x', y', z') and t' ?

3. We need a transformation for:
   (x, y, z) and t $\rightarrow$ (x', y', z') and t'.

$\rightarrow$ Then laws <u>should</u> look the same, have the same form

Galilei transformation

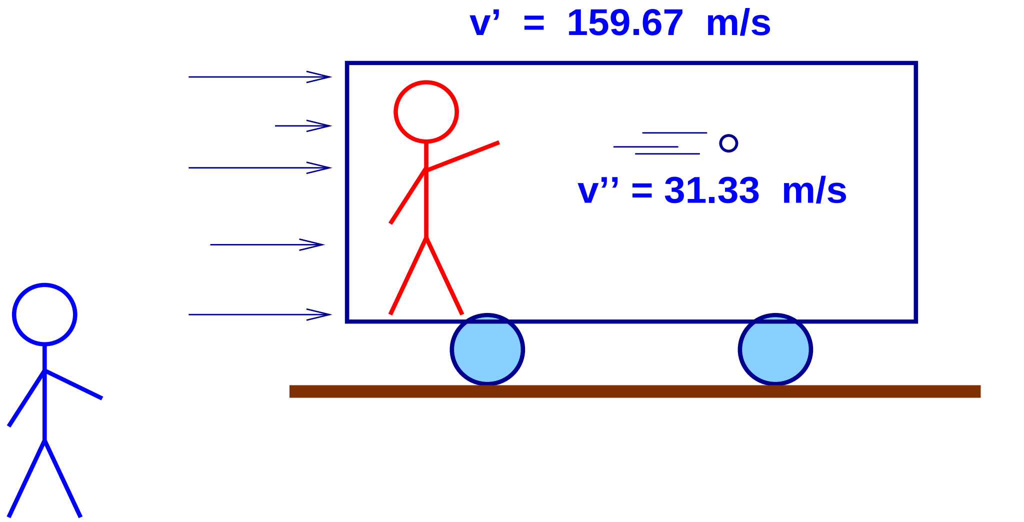


$$
\begin{aligned}
x' &= x - v_x t \\
y' &= y \\
z' &= z \\
t' &= t
\end{aligned}
$$

Galilei transformations relate observations in two frames moving relative to each other (here with constant velocity v_x in x-direction).

Only the position is changing, time is not changed

Consequence: velocities can be added



Fling a ball with 31.33 m/s in a frame moving with 159.67 m/s:

Observed from a non-moving frame: $v_{tot} = v' + v''$

speed of ping-pong ball: $v_{tot} = 191$ m/s

Problems with Galilei transformation

Maxwell describes light as waves, wave equation reads (see previous lecture):

$$\left(\frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2} + \frac{\partial^2}{\partial z'^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t'^2}\right)\Psi \quad = \quad 0$$

With Galilei transformation $x = x' - vt,\ y' = y,\ z' = z,\ t' = t$ **:**

$$\left(\left[1 - \frac{v^2}{c^2}\right]\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \frac{2v}{c^2}\frac{\partial^2}{\partial x \partial t} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\Psi \quad = \quad 0$$

... not quite the same appearance !

Reason: Waves are required to move in a medium (ether !) which travels along in a fixed reference frame, observed from another frame the speed is different ...

Incompatible with experiments:

- Speed of light in vacuum is independent of the motion of the source, i.e. $v_{tot} = c + v' = c$
- Speed of light in vacuum c is the maximum speed and cannot be exceeded

  c = 299792458.000 m/s

- There is no ether, light is not a wave

Postulates of Special Relativity (Einstein)

All physical laws in inertial frames must have equivalent forms

Speed of light in vacuum c must be the same in all frames

(Implied: energy and momentum conservation)

Need Transformations (not Galileaen) which make ALL physics laws look the same !

Coordinates must be transformed differently

Front of a moving light wave in S and S':

$$\mathsf{S}: \quad x^2 + y^2 + z^2 - c^2t^2 = 0$$

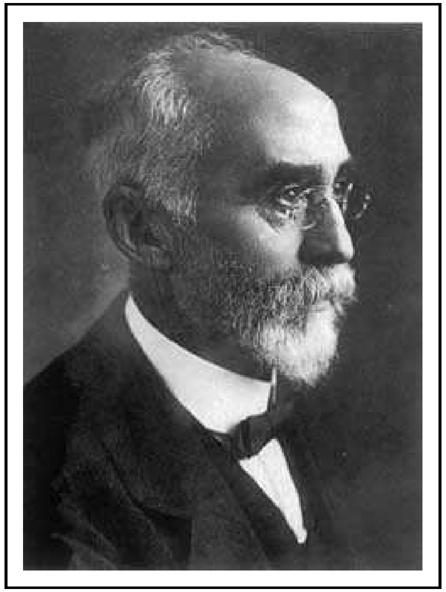
$$\mathsf{S}': \quad x'^2 + y'^2 + z'^2 - c'^2t'^2 = 0$$

Constant speed of light requires $c = c'$

- **To fulfill this condition, time must be changed by transformation as well as space coordinates**
- **Transform** $(x, y, z),\ t \quad \rightarrow \quad (x', y', z'),\ t'$

➡ **After some standard mathematics: Lorentz transformation**

Lorentz transformation



$$x' = \frac{x - vt}{\sqrt{(1 - \frac{v^2}{c^2})}} = \gamma \cdot (x - vt)$$

$$y' = y$$

$$z' = z$$

$$t' = \frac{t - \frac{v \cdot x}{c^2}}{\sqrt{(1 - \frac{v^2}{c^2})}} = \gamma \cdot (t - \frac{v \cdot x}{c^2})$$

Transformation for constant velocity v along x-axis

Time is now also transformed

Note: for $v \ll c$ it reduces to a Galilei transformation !

Definitions: relativistic factors

$$\beta_r = \frac{v}{c}$$

$$\gamma = \frac{1}{\sqrt{(1 - \frac{v^2}{c^2})}} = \frac{1}{\sqrt{(1 - \beta_r^2)}}$$

β_r **relativistic speed:** β_r **= [0, 1]**

γ **Lorentz factor:** γ **= [1,** ∞**)**

Consequences of Einstein's interpretation

- Space and time are <u>NOT</u> independent quantities
- There are no absolute time and space, no absolute motion
- Relativistic phenomena (with relevance for accelerators):
  - No speed of moving objects can exceed speed of light
  - (Non-) Simultaneity of events in independent frames
  - Lorentz contraction
  - Time dilation
  - Relativistic Doppler effect
- Formalism with four-vectors saves the day (see later)

- Simultaneity -

(or: what is observed by different observers ..)

Simultaneity between moving frames

Assume two events in frame S at (different) positions x_1 and x_2 happen simultaneously at times $t_1 = t_2$

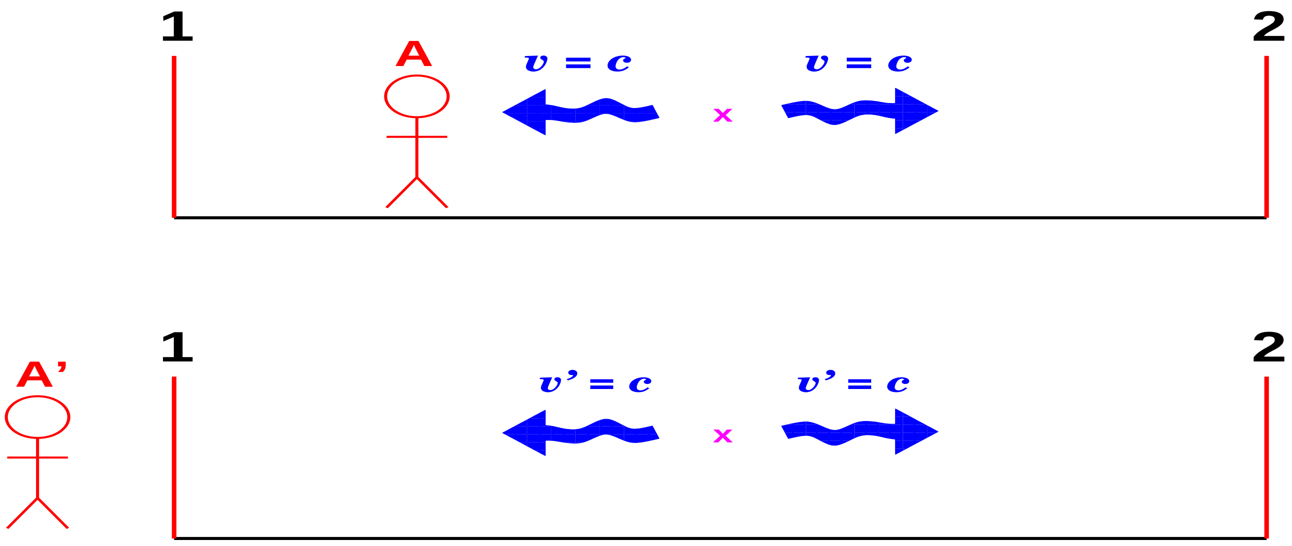
The times t'_1 and t'_2 in S' we get from:

$$t'_1 = \frac{t_1 - \frac{v \cdot x_1}{c^2}}{\sqrt{(1 - \frac{v^2}{c^2})}} \quad \text{and} \quad t'_2 = \frac{t_2 - \frac{v \cdot x_2}{c^2}}{\sqrt{(1 - \frac{v^2}{c^2})}}$$

$x_1 \neq x_2$ **in** S **implies that** $t'_1 \neq t'_2$ **in frame** S' **!!**

$$\boxed{\text{Two events simultaneous at (different) positions } x_1 \text{ and } x_2 \text{ in } S \text{ are not simultaneous in } S'}$$

Using light sources to judge time order of events

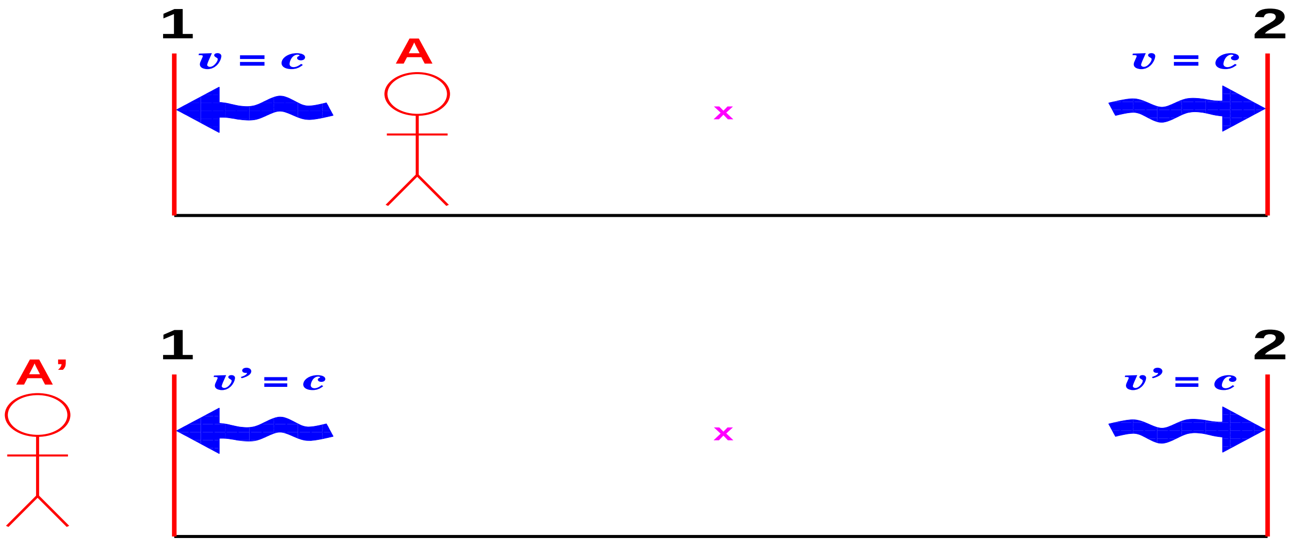


- System with a light source (x) and detectors (1, 2) and
- flashes moving from light source towards detectors

  Observer (A) inside this frame

  Observer (A') outside

After some time:

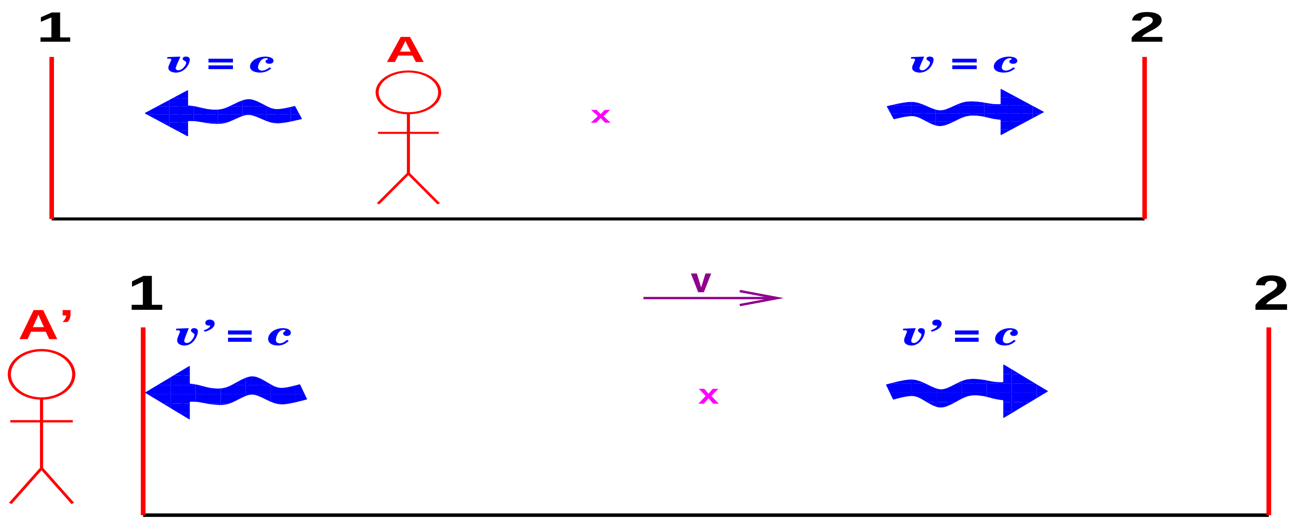


Observed by A: both flashes arrive simultaneously at 1 and 2

Observed by A': both flashes arrive simultaneously at 1 and 2

What if the frame is moving relative to observer A' ?

Now one frame is moving with speed v:



Observed by A: both flashes arrive simultaneously in 1,2

Observed by A': flash arrives first in 1, later in 2

A simultaneous event in S is not simultaneous in S'

Why do we care ??

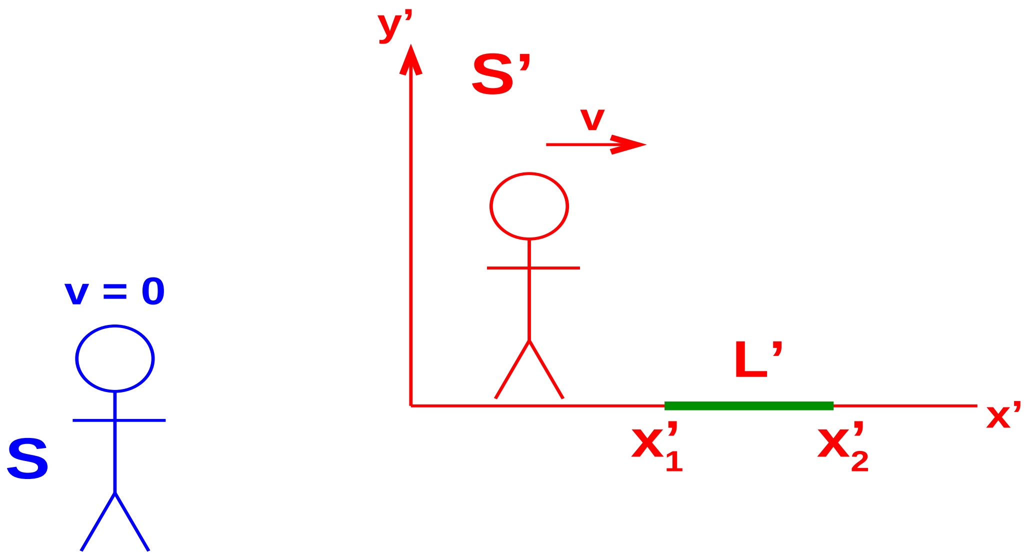
Why care about simultaneity ?

- Simultaneity is not frame independent
- It plays the key role in special relativity
- Almost all paradoxes are explained by that !
- Important role for measurements !
- Different observers see a different reality, in particular the sequence of events can change !

  - For $t_1 \; < \; t_2$ we may find (not always !) a frame where $t_1 \; > \; t_2$ (concept of before and after depends on the observer)

- Lorentz contraction -

Length L: difference between two positions



Have to measure positions (e.g. ends of a rod) simultaneously !

Length of a rod in S' is $L' = x'_2 - x'_1$, measured simultaneously at a fixed time t' in frame S',

What is the length L measured from S ??

Consequences: length measurement

We have to measure simultaneously (!) the ends of the rod at a fixed time t in frame F, i.e.: $L = x_2 - x_1$

Lorentz transformation of "rod coordinates" into rest frame:

$$x'_1 = \gamma \cdot (x_1 - vt) \quad \text{and} \quad x'_2 = \gamma \cdot (x_2 - vt)$$

$$L' = x'_2 - x'_1 = \gamma \cdot (x_2 - x_1) = \gamma \cdot L$$

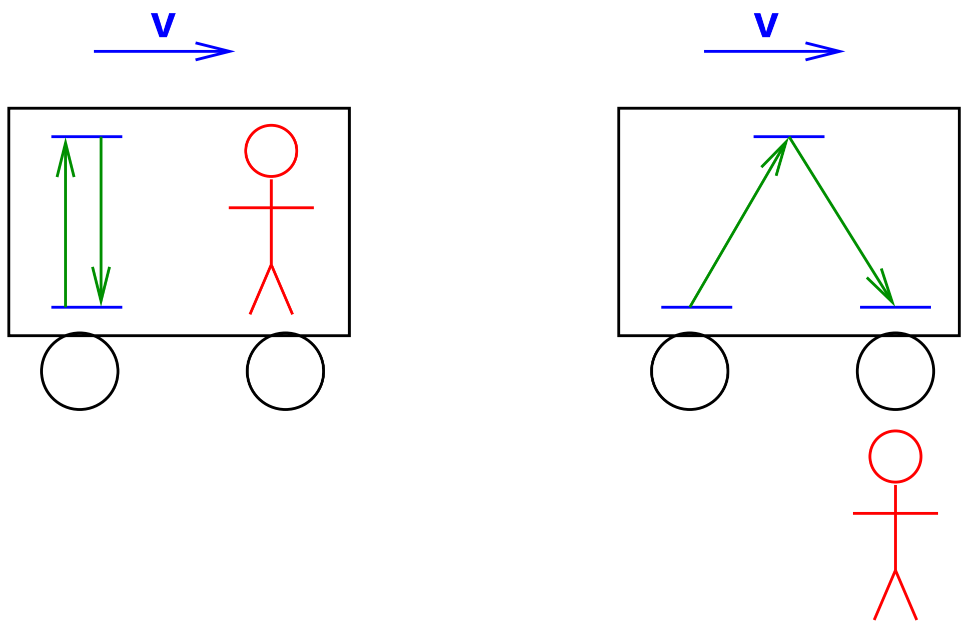
$$\longrightarrow \boxed{L = L'/\gamma}$$

In accelerators: bunch length, electromagnetic fields, magnets, ...

- Time dilation -

Time dilation - schematic

Reflection of light between 2 mirrors seen inside moving frame and from outside

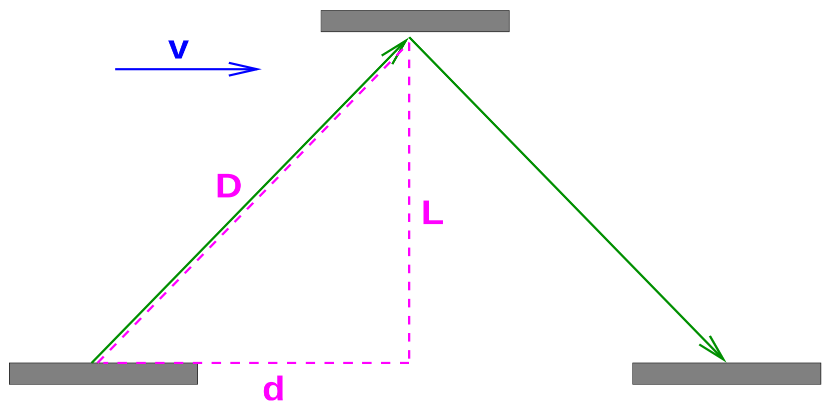


Frame moving with velocity v

Car has moved during the up-down

Seen from outside the path is longer, but c must be the same ..

Time dilation - derivation - just geometry



In frame S': light travels L in time $\Delta t'$

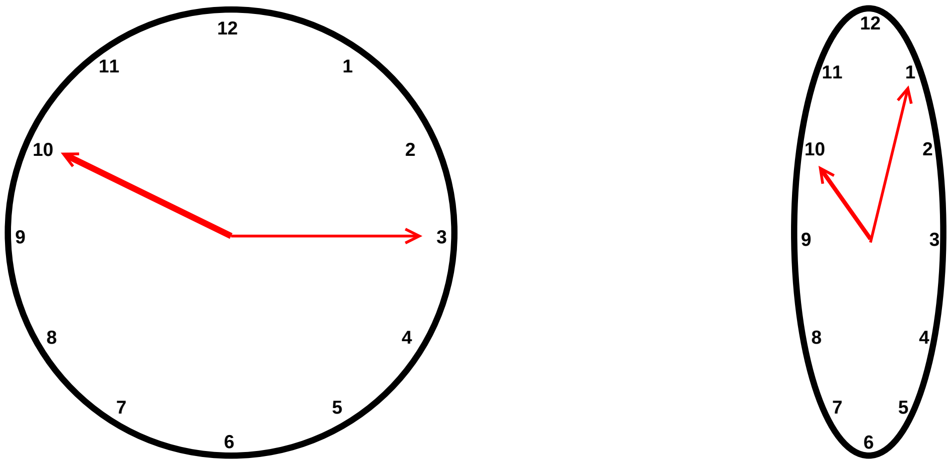
In frame S: light travels D in time Δt

system moves d in time Δt

$$L = c \cdot \Delta t' \qquad D = c \cdot \Delta t \qquad d = v \cdot \Delta t$$

$$(c \cdot \Delta t)^2 = (c \cdot \Delta t')^2 + (v \cdot \Delta t)^2$$

$$\rightarrow \quad \boxed{\Delta t = \gamma \cdot \Delta t'}$$

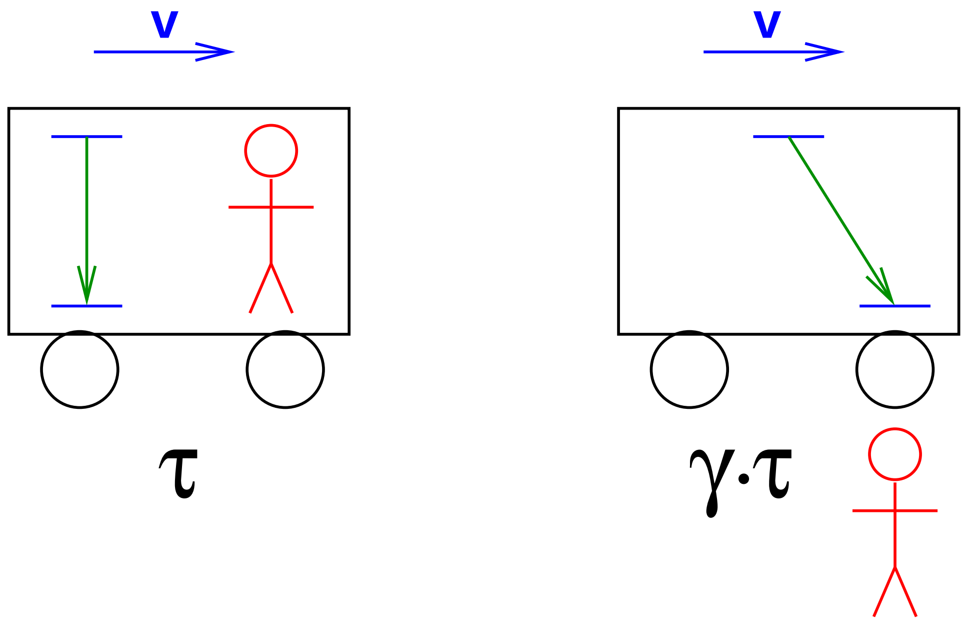


Resting clock and a clock moving with $\gamma = 3$

Seen by the stationary observer:

- **Contracted clock**
- **Slowed down**

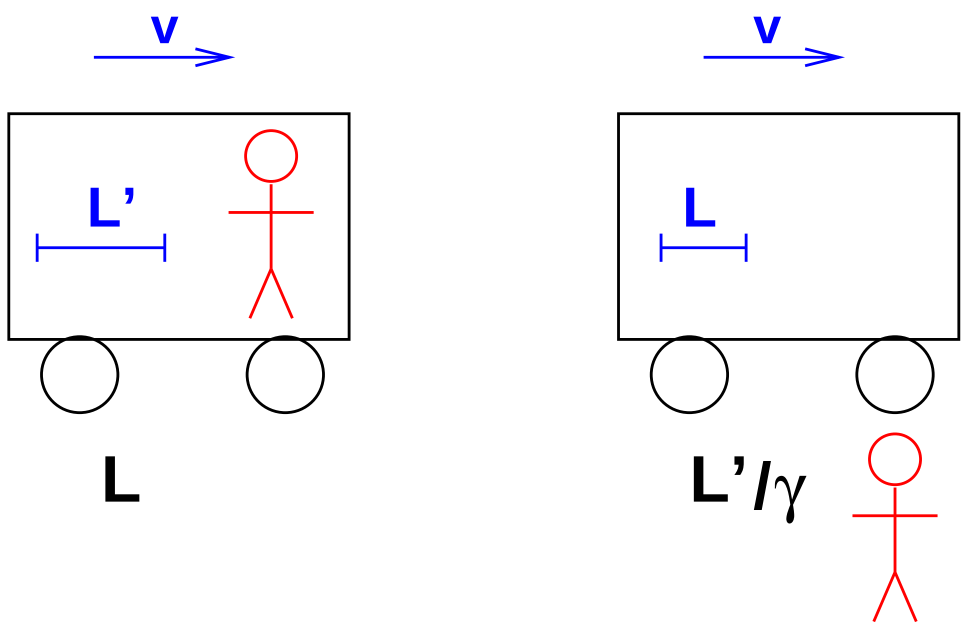
Times measured in moving and from rest frame:



Observer in car <u>always</u> measures the <u>same</u> time τ, independent of the motion/speed of the car

This time is called "proper time" τ
(from German: Eigenzeit = inherent time)

Length measured in moving and from rest frame:



Observer in car <u>always</u> measures the <u>same</u> length L, independent of the motion/speed of the car

This length is called "proper length" L

Other 'propers' ...

Not surprisingly, there are:

- **Proper Time**
- **Proper Length**
- **Proper Mass**
- **Proper Velocity**
- **Proper Acceleration (not often mentioned, but relevant for accelerated objects and in particle physics)**

Example: moving electrons and protons

Electron with E = 1 GeV: v = 99.999987 % of c ($\gamma \approx$ 1960)
Proton with E = 1 GeV: v $\approx$ 25 % of c ($\gamma \approx$ 1.07)

Important consequences for accelerators:*)

Bunch length:

In lab frame: σ_z In frame of electron/proton: $\gamma \cdot \sigma_z$

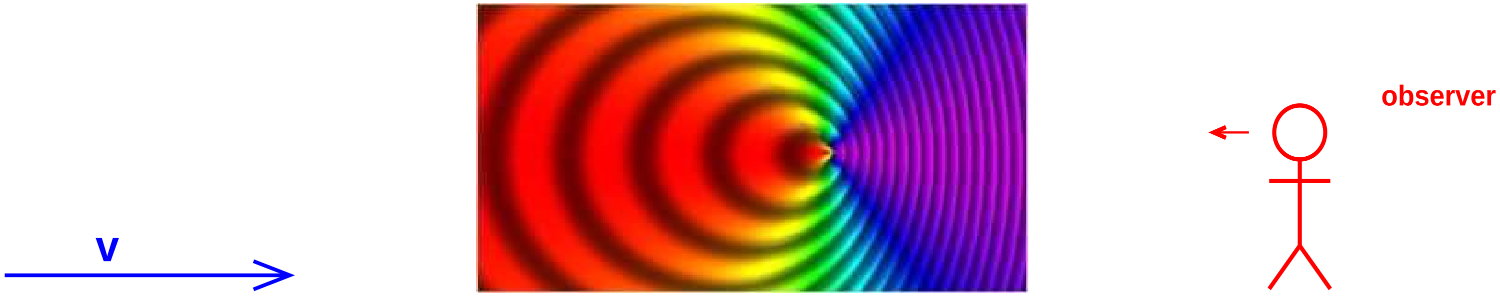
Length of an object (e.g. magnet, distance between magnets !):

In lab frame: L In frame of electron/proton: L/γ

Electrons and protons live in a very different world ..

*) (see e.g. collective effects, light sources, FEL, ..)

Example: moving (white) light source with speed $\mathbf{v} \approx \mathbf{c}$



Unlike sound: no medium of propagation

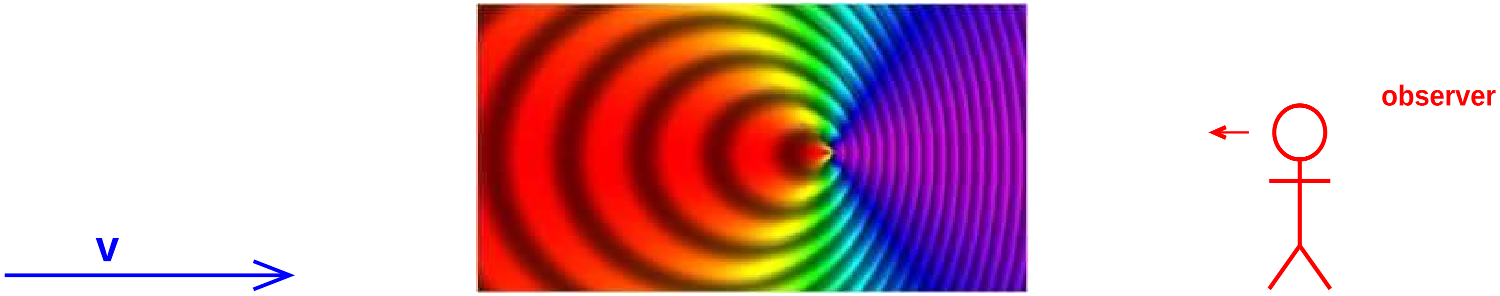
Relativistic Doppler effect: mostly due to time dilation

Observed frequency depends on observation angle θ

➡ frequency is changed: $\nu = \nu_0 \cdot \gamma \cdot (1 - \beta_r \cos(\theta))$

Very important and needed for Free Electron Lasers (FEL) !

Example: moving (white) light source with speed $\mathbf{v} \approx \mathbf{c}$



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→ frequency is changed: $\nu = \nu_0 \cdot \gamma \cdot (1 - \beta_\mathrm{r} \cos(\theta))$

Travelling at $v \approx c$ through space can damage your health !

Moving clocks appear to go slower:

Travel by airplane (you age a bit slower compared to ground): tested experimentally with atomic clocks (1971 and 1977)

Assume regular airplane cruising at $\approx$ 900 km/h

On a flight from Montreal to Geneva, the time is slower by 25 - 30 ns (considering only special relativity) !

Not a strong effect, what about other examples ?

Every day example (GPS satellite):

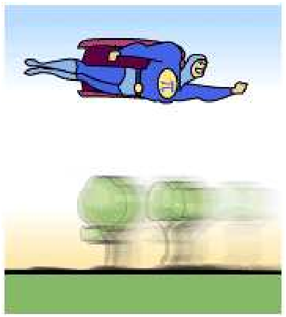
- 20000 km above ground
- Orbital speed 14000 km/h (i.e. relative to observer on earth)
- On-board clock accuracy $\leq$ 1 ns
- Relative precision of satellite orbit $\leq 10^{-8}$
- At GPS receiver, for 5 m need clock accuracy $\approx$ 10 ns

Do we have to correct for relativistic effects ?

Do the math or look it up in the backup slides (and be surprised)..

Of course ALL inertial frames are equivalent

- **Length contraction observed in F' from F is the same as observed in F from F'**
- **Time dilation observed in F' from F is the same as observed in F from F'**



- **No contradiction: the same reality can look very different from different perspectives**

To make it clear:

Key to understand relativity

- **Lorentz contraction:**
  - It is not the matter that is compressed
  - It is the space that is modified
- **Time dilation:**
  - It is not the clock that is changed
  - It is the time that is modified

What about the mass m ?

Momentum conservation: $\vec{p} = \vec{p'}$

To simplify the computation:

Object inside moving frame S' moves with $\vec{u}' = (0, u'_y, 0)$

Transverse momentum must be conserved:

$$p_y = p'_y$$

$$m \cdot \ u_y = m' \cdot \ u'_y$$

velocity u_y **transformed:**

$$m \cdot \ u'_y/\gamma = m' \cdot \ u'_y$$

implies:

$$m = \gamma \cdot m'$$

For momentum conservation: <u>mass must also be transformed</u> !

Implications:

Assume m_0 is the mass of an object at rest (i.e. the "proper mass"):

$$m = \gamma \cdot m_0 = \frac{m_0}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}$$

and for small speeds we can very well approximate:

$$m \cong m_0 + \frac{1}{2} m_0 v^2 \left(\frac{1}{c^2}\right)$$

and multiplied by c^2:

$$mc^2 \cong m_0 c^2 + \frac{1}{2} m_0 v^2 = m_0 c^2 + T$$

Relativistic energy

Interpretation:

$$E = mc^2 = m_0c^2 + T$$

- **Kinetic energy of particle is** T
- **Energy of particle at rest is** $E_0 = m_0c^2$ **(new concept)**
- **Total energy** E **of a particle "in motion" is** $E = mc^2$
  (Sum of kinetic energy plus rest energy)

Always true: $$\boxed{E = m \cdot c^2 = \gamma m_0 \cdot c^2}$$

using the definition of relativistic mass again: $m = \gamma m_0$

Interpretation of relativistic energy

- **For any object, $m \cdot c^2$ is the total energy**
- **Follows directly from momentum conservations**
  - m **is the mass (energy) of the object "in motion"**
  - m_0 **is the mass (energy) of the object "at rest"**
- **The mass m is not the same in all inertial systems, the rest mass m_0 is ... !**

Relativistic momentum

Classically:

$$p \;=\; m\,v$$

with $m = \gamma m_0$:

$$p \;=\; \gamma \cdot m_0\, v \;=\; \gamma \cdot \beta \cdot c \cdot m_0$$

with the previous equations:

$$E^2 \;=\; (m_0 c^2)^2 + (pc)^2 \quad \Rightarrow \quad \boxed{\frac{E}{c} \;=\; \sqrt{(m_0 c)^2 \;+\; p^2}}$$

Rather important formula in practice, e.g. accelerators ..

Practical and impractical units

Standard units are not very convenient, easier to use:

$$[\mathbf{E}] = \mathbf{eV} \qquad [\mathbf{p}] = \mathbf{eV}/c \qquad [\mathbf{m}] = \mathbf{eV}/c^2$$

then: $\boxed{E^2 \ = \ m_0^2 \ + \ p^2}$

Mass of a proton: $m_p = 1.672 \ \cdot \ 10^{-27}$ **Kg**
Energy(at rest): $m_p c^2 = 938$ **MeV** $= 0.15$ **nJ**

Ping-pong ball: $m_{pp} = 2.7 \ \cdot \ 10^{-3}$ **Kg** **(** $\approx \ 1.6 \ 10^{24}$ **protons)**
Energy(at rest): $m_{pp} c^2 = 1.5 \ \cdot \ 10^{27}$ **MeV** $= 2.4 \cdot 10^{14}$ **J**

$\approx$ **750000 times the full LHC beam**

$\approx$ **60 kilotons of TNT**

Relativistic mass

The mass of a fast moving particle is increasing like:

$$m = \gamma m_0 = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

- **Why do we care ?**
  - **Particles cannot go faster than c !**
  - **What happens when we accelerate ?**

When we accelerate:

- **For v $\ll$ c:**
  - **E, m, p, v increase ...**
- **For v $\approx$ c:**
  - **E, m, p increase, but v does not !**

$$T = m_0(\gamma - 1)c^2 \quad \rightarrow \quad \gamma = 1 + \frac{T}{m_0 c^2} \quad \rightarrow \quad \beta = \sqrt{1 - \frac{1}{\gamma^2}}$$

Why do we care ??

E (GeV)	v (km/s)	γ	β	T (LHC)
450	299791.82	479.74	0.99999787	88.92465 μs
7000	299792.455	7462.7	0.99999999	88.92446 μs

- For identical circumference very small change in revolution time
- If path for faster particle slightly longer, the faster particle arrives later !

→ Concept of transition energy (CPS, SPS)

Kinematic relations (just a collection for everyday use)

We have already seen a few, e.g.:

- $T = E - E_0 = (\gamma - 1)E_0$
- $E = \gamma \cdot E_0$
- $E_0 = \sqrt{E^2 - c^2p^2}$
- **etc. ...**

Very useful for everyday calculations →

Kinematic relations

	cp	**T**	**E**	γ
$\beta =$	$\frac{1}{\sqrt{(\frac{E_0}{cp})^2+1}}$	$\sqrt{1-\frac{1}{(1+\frac{T}{E_0})^2}}$	$\sqrt{1-(\frac{E_0}{E})^2}$	$\sqrt{1-\gamma^{-2}}$
cp =	cp	$\sqrt{T(2E_0+T)}$	$\sqrt{E^2-E_0^2}$	$E_0\sqrt{\gamma^2-1}$
$E_0 =$	$\frac{cp}{\sqrt{\gamma^2-1}}$	$T/(\gamma-1)$	$\sqrt{E^2-c^2p^2}$	E/γ
T =	$cp\sqrt{\frac{\gamma-1}{\gamma+1}}$	**T**	$E-E_0$	$E_0(\gamma-1)$
$\gamma =$	$cp/E_0\beta$	$1+T/E_0$	E/E_0	γ

Kinematic relations - logarithmic derivatives

	$\frac{d\beta}{\beta}$	$\frac{dp}{p}$	$\frac{dT}{T}$	$\frac{dE}{E} = \frac{d\gamma}{\gamma}$
$\frac{d\beta}{\beta} =$	$\frac{d\beta}{\beta}$	$\frac{1}{\gamma^2}\frac{dp}{p}$	$\frac{1}{\gamma(\gamma+1)}\frac{dT}{T}$	$\frac{1}{(\beta\gamma)^2}\frac{d\gamma}{\gamma}$
$\frac{dp}{p} =$	$\gamma^2\frac{d\beta}{\beta}$	$\frac{dp}{p}$	$[\gamma/(\gamma+1)]\frac{dT}{T}$	$\frac{1}{\beta^2}\frac{d\gamma}{\gamma}$
$\frac{dT}{T} =$	$\gamma(\gamma+1)\frac{d\beta}{\beta}$	$(1+\frac{1}{\gamma})\frac{dp}{p}$	$\frac{dT}{T}$	$\frac{\gamma}{(\gamma-1)}\frac{d\gamma}{\gamma}$
$\frac{dE}{E} =$	$(\beta\gamma)^2\frac{d\beta}{\beta}$	$\beta^2\frac{dp}{p}$	$(1-\frac{1}{\gamma})\frac{dT}{T}$	$\frac{d\gamma}{\gamma}$
$\frac{d\gamma}{\gamma} =$	$(\gamma^2-1)\frac{d\beta}{\beta}$	$\frac{dp}{p} - \frac{d\beta}{\beta}$	$(1-\frac{1}{\gamma})\frac{dT}{T}$	$\frac{d\gamma}{\gamma}$

Example LHC (7 TeV): $\frac{\Delta p}{p} \approx 10^{-4} \Rightarrow \frac{\Delta\beta}{\beta} = \frac{\Delta v}{v} \approx 2 \cdot 10^{-12}$

Example LEP (0.1 TeV): $\frac{\Delta p}{p} \approx 10^{-4} \Rightarrow \frac{\Delta\beta}{\beta} = \frac{\Delta v}{v} \approx 2 \cdot 10^{-15}$

Key takeaways - first summary

- Physics laws the same in all inertial frames ...

→ Speed of light in vacuum c is the same in all frames and requires Lorentz transformation

- Moving objects appear shorter
- Moving clocks appear to go slower
- Mass is not independent of motion ($m = \gamma \cdot m_0$) and total energy is $E = m \cdot c^2$
- No absolute space or time: where it happens and when it happens is not independent

Different observers have a different perception of space and time !

→ Next: how to calculate something and applications ...

Introducing four-vectors

Since space and time are not independent, must reformulate physics taking both into account:

$t, \quad \vec{a} = (x, y, z)$ → Replace by one vector including the time

We have a temporal and a spatial part
(time t multiplied by c to get the same units)

We get a four-vector (here position four-vector):

$$X = (ct, x, y, z)$$

A short selection of important four-vectors:

$$
\begin{aligned}
&\text{Coordinates :} && X = (ct, x, y, z) = (ct, \vec{x}) \\
&\text{Velocities :} && U = \tfrac{dX}{d\tau} \;=\; \gamma(c, \dot{\vec{x}}) = \gamma(c, \vec{u}) \\
&\text{Momenta :} && P = mU = m\gamma(c, \vec{u}) = \gamma(mc, \vec{p}) \\
&\text{Force :} && F = \tfrac{dP}{d\tau} = \gamma \tfrac{d}{d\tau}(mc, \vec{p})
\end{aligned}
$$

Life becomes really simple ➡

Lorentz transformation can be written in matrix form:

$$X' = \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = X$$

$$X' = \Lambda \circ X \qquad (\Lambda \quad \text{for "Lorentz"})$$

<u>ALL</u> four-vectors A transform like:

$$A' = \Lambda \circ A$$

Scalar products revisited for four-vectors

Cartesian Scalar Product (Euclidean metric):

$$\vec{x} \cdot \vec{y} \;=\; (x_a, y_a, z_a) \;\cdot\; (x_b, y_b, z_b) = (x_a \cdot x_b \; + y_a \cdot y_b \; + \; z_a \cdot z_b)$$

Space-time four-vectors like:

$A \;=\; (ct_a, x_a, y_a, z_a) \qquad B \;=\; (ct_b, x_b, y_b, z_b)$

The four-vector scalar product is <u>not</u>:

$$AB = (ct_a \cdot ct_b \; + \; x_a \cdot x_b \; + \; y_a \cdot y_b \; + \; z_a \cdot z_b)$$

but:

$$AB = (ct_a \cdot ct_b \; \underbrace{- \; x_a \cdot x_b \; - \; y_a \cdot y_b \; - \; z_a \cdot z_b}_{\text{comes with a negative sign}})$$

Why bother about four-vectors ?

- We want invariant laws of physics in different frames
- The solution: write the laws of physics in terms of four vectors and use Lorentz transformation
- Without proof*): any four-vector (scalar) product ZZ has the same value in all inertial frames:

$$ZZ \quad = \quad Z'Z'$$

All scalar products of any four-vectors are invariant !

*) The proof is extremely simple !

A special invariant

From the velocity four-vector V:

$$U \;=\; \gamma(c, \vec{u})$$

we get the scalar product:

$$UU = \gamma^2(c^2 - \vec{u}^2) = c^2 \;\; !!$$

- c **is an invariant, has the same value in all inertial frames**

$$UU = U'U' = c^2$$

- **The invariant of the velocity four-vector** U **is the speed of light** c**, i.e. it is the same in ALL frames !**

Gives us invariants - an important one:

Momentum four-vector P:

$$P = m_0 U = m_0 \gamma(c, \vec{u}) = (mc, \vec{p}) \;=\; (\frac{E}{c}, \vec{p})$$

$$P' = m_0 U' = m_0 \gamma(c, \vec{u'}) = (mc, \vec{p'}) \;=\; (\frac{E'}{c}, \vec{p'})$$

We can get an important invariant:

$$PP \;=\; P'P' \;=\; m_0^2 c^2$$

Invariant of the four-momentum vector is the mass m_0

→ The rest mass is the same in all frames !
(otherwise we could tell whether we are moving or not !!)

Note: m_0 is the "proper mass"

Four vectors

- Use of four-vectors simplify calculations significantly
- Follow the rules and look for invariants
- In particular kinematic of particles:
  - Relationship between kinetic parameters
  - Particle collisions and particle decay

Particle collisions

Fixed target:

Collider:

What is the available (i.e. useful) collision energy ?

Collider: Assume identical particles and energies

The four momentum vectors are:

$$P1 = (E, \vec{p}) \qquad P2 = (E, -\vec{p})$$

The four momentum vector in centre of mass system is:

$$P^* = P1 + P2 = (E + E, \vec{p} - \vec{p}) = (2E, \vec{0})$$

$$E_{cm} = \sqrt{P^* P^*} = 2E$$

i.e. in a (symmetric) collider the total energy is twice the beam energy

Fixed target: Assume identical particles

The four momentum vectors are:

$$P1 = (E, \vec{p}) \qquad P2 = (m_0, \vec{0})$$

The four momentum vector in centre of mass system is:

$$P^* = P1 + P2 = (E + m_0, \vec{p})$$

$$E_{cm} = \sqrt{P^* P^*} = \sqrt{E^2 - \vec{p}^2 + 2m_0 E + m_0^2}$$

$$E_{cm} \approx \sqrt{2m_0 E}$$

Particle collisions - fixed target

Examples:

collision	beam energy	$\sqrt{s}$ (collider)	$\sqrt{s}$ (fixed target)
pp	315 (GeV)	630 (GeV)	24.3 (GeV)
pp	7000 (GeV)	14000 (GeV)	114.6 (GeV)
e^+e^-	100 (GeV)	200 (GeV)	0.320 (GeV)

Back to Electrodynamics - Strategy:

1. Apply the concepts (i.e. four-vectors) to classical electrodynamics
2. This leads to reformulation and new interpretation of electromagnetic fields

   ➡ can handle moving charges

- **Expect that life becomes much easier with four-vectors ..**

- **Strategy: one $+$ three**

Write potentials and currents as four-vectors:

$$\Phi,\ \vec{A} \quad \Rightarrow \quad A^{\mu} \ = \ (\frac{\Phi}{c},\ \vec{A})$$

$$\rho,\ \vec{j} \quad \Rightarrow \quad J^{\mu} \ = \ (\rho \cdot c,\ \vec{j})$$

What about the transformation of current and potentials ?

Transform the four-current like:

$$\begin{pmatrix} \rho' c \\ j'_x \\ j'_y \\ j'_z \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \rho c \\ j_x \\ j_y \\ j_z \end{pmatrix}$$

It transforms via: $J'^{\mu} = \Lambda \, J^{\mu}$ **(always the same** Λ**)**

Ditto for: $A'^{\mu} = \Lambda \, A^{\mu}$ **(always the same** Λ**)**

Electromagnetic fields using potentials (see lecture on EM theory):

Magnetic field: $\vec{B} = \nabla \times \vec{A}$

e.g. the x-component:

$$B_x = \frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}$$

Electric field: $\vec{E} = -\nabla\Phi - \frac{\partial \vec{A}}{\partial t}$

e.g. for the x-component:

$$E_x = -\frac{\partial A_0}{\partial x} - \frac{\partial A_1}{\partial t} = -\frac{\partial A_t}{\partial x} - \frac{\partial A_x}{\partial t}$$

→ **after getting all combinations ($E_x, B_x, E_y, ..$)**

Electromagnetic fields described by field-tensor $F^{\mu\nu}$:

$$F^{\mu\nu} = \begin{pmatrix} 0 & \dfrac{-E_x}{c} & \dfrac{-E_y}{c} & \dfrac{-E_z}{c} \\ \dfrac{E_x}{c} & 0 & -B_z & B_y \\ \dfrac{E_y}{c} & B_z & 0 & -B_x \\ \dfrac{E_z}{c} & -B_y & B_x & 0 \end{pmatrix}$$

It transforms via: $F'^{\mu\nu} = \Lambda \; F^{\mu\nu} \; \Lambda^T$ **(same Λ as before)**

Transformation of fields into a moving frame (x-direction):

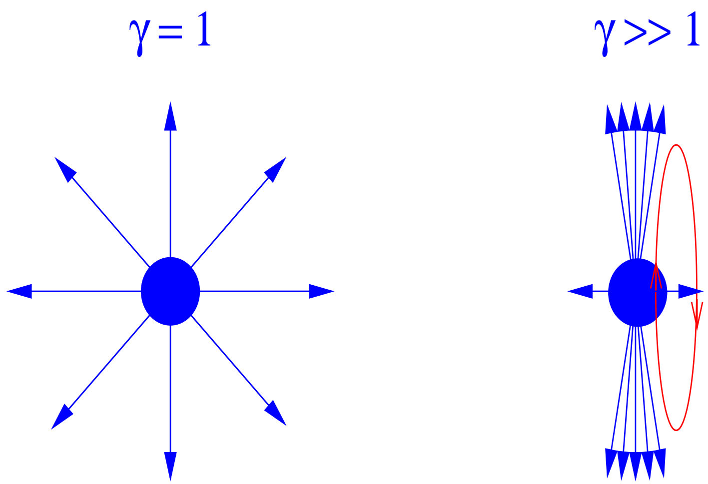
Use Lorentz transformation of $F^{\mu\nu}$ and write for components:

$$
\begin{aligned}
E'_x &= E_x & B'_x &= B_x \\
E'_y &= \gamma(E_y - v \cdot B_z) & B'_y &= \gamma(B_y + \tfrac{v}{c^2} \cdot E_z) \\
E'_z &= \gamma(E_z + v \cdot B_y) & B'_z &= \gamma(B_z - \tfrac{v}{c^2} \cdot E_y)
\end{aligned}
$$

Fields perpendicular to movement are transformed

Electric and magnetic fields are not independent quantities, depend on the relative motion

Example Coulomb field: (a charge moving with constant speed)



- **In rest frame purely electrostatic forces**
- **In moving frame $\vec{E}$ transformed and $\vec{B}$ appears**

Magnetic fields are <u>always</u> the consequence of moving charges

An important consequence - field of a moving charge:

$$
\begin{aligned}
E'_x &= E_x & B'_x &= B_x \\
E'_y &= \gamma(E_y - v \cdot B_z) & B'_y &= \gamma(B_y + \tfrac{v}{c^2} \cdot E_z) \\
E'_z &= \gamma(E_z + v \cdot B_y) & B'_z &= \gamma(B_z - \tfrac{v}{c^2} \cdot E_y)
\end{aligned}
$$

Assuming that $\vec{B}' = 0$, we get for the <u>transverse</u> forces:

$$\boxed{\vec{F}_{mag} = -\beta^2 \cdot \vec{F}_{el}}$$

For β = 1, Electric and Magnetic <u>forces</u> cancel, plenty of consequences, e.g. Space Charge

Most important for stability of beams (so watch out for $\beta \ll 1$) !

Quote Einstein (1905):

For a charge moving in an electromagnetic field, the force experienced by the charge is equal to the electric force, transformed into the rest frame of the charge

There is no mystic, velocity dependent coupling between a charge and a magnetic field !

It is just a consequence of two reference frames: the magnetic force is the electric force seen by a moving charged particle !!

How far we've come ?

- Can to deal with moving charges in accelerators
- Electromagnetism and fundamental laws of classical mechanics now consistent
- Ad hoc introduction of Lorentz force not necessary, comes out automatically

Summary I (things to understand)

Special Relativity is relatively[*)] simple, few basic principles

- Physics laws are the same in all inertial systems
- Speed of light in vacuum the same in all inertial systems

Everyday phenomena lose their meaning (do not ask what is "real"):

- Only union of space and time preserve an independent reality: space-time

Electric and Magnetic fields do not exist as independent quantities

- Just different aspects of a single electromagnetic field
- Its manifestation, i.e. division into electric $\vec{E}$ and magnetic $\vec{B}$ components, depends on the chosen reference frame

*) No pun intended ..

Summary II (accelerators - things to remember)

Write everything as four-vectors, blindly follow the rules and you get it all easily, in particular transformation of fields etc.

- **Relativistic effects in accelerators (used in later lectures)**
  - **Lorentz contraction and Time dilation (e.g. FEL, ..)**
  - **Relativistic Doppler effect (e.g. FEL, ..)**
  - **Relativistic mass effects and dynamics (everywhere, ...)**
  - **New interpretation of electric and magnetic fields (collective effects, space charge, beam-beam, ..)**

- BACKUP SLIDES -

Side note: Relativistic Space Travel

The formula for time dilation also holds for an accelerated system !
Assuming constant acceleration $g = 9.81 \, \frac{m}{s^2}$ **(without proof):**

$$\begin{aligned}
&\text{Time on earth/space :} && t \\
&\text{Time in space ship :} && t_p \quad (\text{your 'proper' time}) \\
&\text{Speed :} && \beta \; = \; \tanh\left(\frac{g \cdot t_p}{c}\right) \\
&\text{Distance from earth :} && d \; = \; \left(\frac{c^2}{g}\right) \cdot \left[\cosh\left(\frac{g \cdot t_p}{c}\right) \; - \; 1\right]
\end{aligned}$$

➡ **After** $t_p = 12$ **years on board:** $d \approx 120000$ **light years !**
This is the diameter of the milky way !

(... but there is - at least - one problem !)

What about forces ??

(Four-)force is the time derivative of the four-momentum:

$$\mathcal{F}_L^{\mu} \;=\; \frac{\partial P^{\mu}}{\partial \tau} \;=\; \gamma \frac{\partial P^{\mu}}{\partial t}$$

We get the four-vector for the Lorentz force, with the well known expression in the spatial part:

$$\mathcal{F}_L^{\mu} \;=\; \gamma\; q\; (\frac{\partial (mc)}{\partial \tau},\; \vec{E} \;+\; \vec{u} \times \vec{B}) \;=\; q \cdot F^{\mu\nu} U_{\nu}$$

Remember: $U_{\nu} \;=\; (\gamma c, -\gamma v_x, -\gamma v_y, -\gamma v_z)$

Quote Einstein (1905):

For a charge moving in an electromagnetic field, the force experienced by the charge is equal to the electric force, transformed into the rest frame of the charge

There is no mystic, velocity dependent coupling between a charge and a magnetic field !

It is just a consequence of two reference frames

No more inconsistencies:

Mechanisms are the same, physics laws are the same:

- Formulated in an invariant form and transformed with Lorentz transformation
- Different reference frames are expected to result in different <u>observations</u>

→ In an accelerator we have <u>always</u> at least two reference frames

Interesting, but not treated here:

- Principles of Special Relativity apply to inertial (non-accelerated) systems
- Is it conceivable that the principle applies to accelerated systems ?
- Introduces General Relativity, with consequences:
  - Space and time are dynamical entities:

    → space and time change in the presence of matter
  - Explanation of gravity (sort of ..)
  - Black holes, worm holes, Gravitational Waves, ...
  - Time depends on gravitational potential, different at different heights (Airplanes, GPS !)

A last word ...

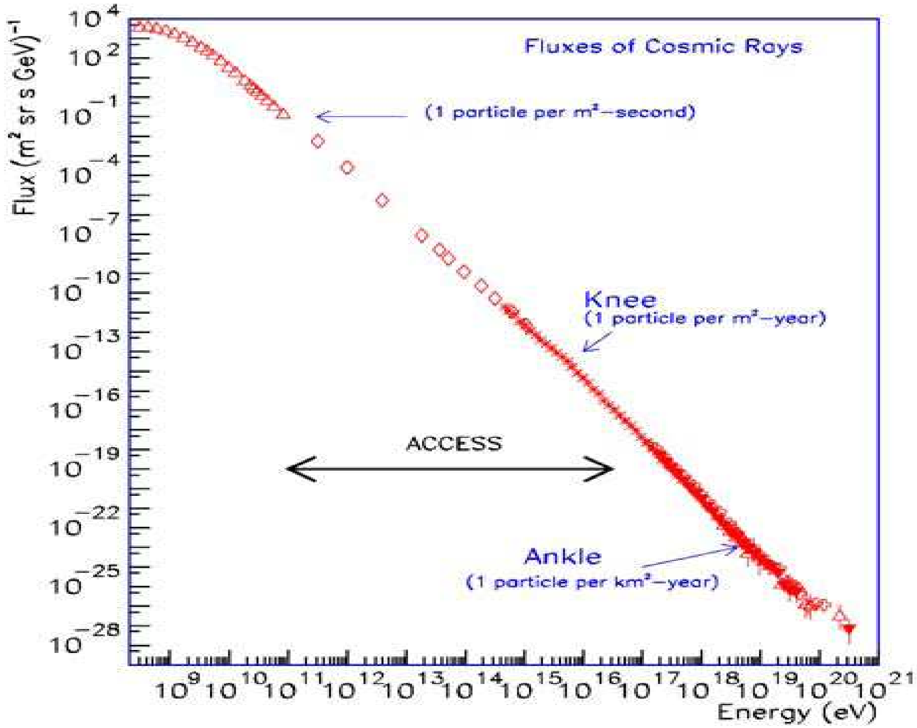
- **Special relativity is very simple, a few implications:**
  - **Newton: time is absolute, space is absolute**
  - **Einstein: time and space are relative, space-time is absolute**
  - **Different observers see a different reality**
  - **Of course: The common electromagnetic field are photons (How to transform a photon and what is the invariant ?)**

A last word ...

- Special relativity is very simple, a few implications:
  - Newton: time is absolute, space is absolute
  - Einstein: time and space are relative, space-time is absolute
  - Different observers see a different reality
  - Of course: The common electromagnetic field are photons
    (How to transform a photon and what is the invariant ?)
- Relevant Implications: next lecture "Kinematics and Multi-Particles"

If you do not yet have enough or are bored, look up some of the popular paradoxes (entertaining, but irrelevant for accelerators)

That's all for now ...



Cosmic ray flux ...

Personal comments:

Special relativity is very simple - but not intuitive, may violate common sense ...

We have to rely on the deductive procedure (and accept the results)
In any kind of theory the main difficulty is to formulate a problem mathematically.
A rudimentary knowledge of high school mathematics suffices to solve the resulting equations in this theory.
$\longrightarrow$ Derivations and proofs are avoided when they do not serve a didactic purpose (see e.g. [2, 4, 5])...

But no compromise on correctness, not oversimplified !

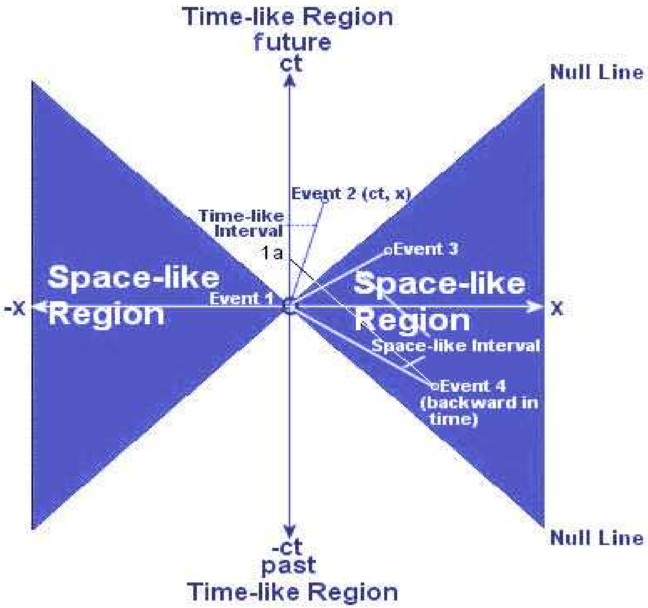
Small history

- 1678 (Römer, Huygens): Speed of light c is finite ($c \approx 3 \cdot 10^8$ m/s)
- 1630-1687 (Galilei,Newton): Principles of Relativity
- 1863 (Maxwell): Electromagnetic theory, light are waves moving through static ether with speed c
- 1887 (Michelson, Morley): Speed c independent of direction, $\rightarrow$ no ether
- 1892 (Lorentz, FitzGerald, Poincaré): Lorentz transformations, Lorentz contraction
- 1897 (Larmor): Time dilation
- 1905 (Einstein): Principles of Special Relativity
- 1907 (Einstein, Minkowski): Concepts of Spacetime

Lorentz contraction

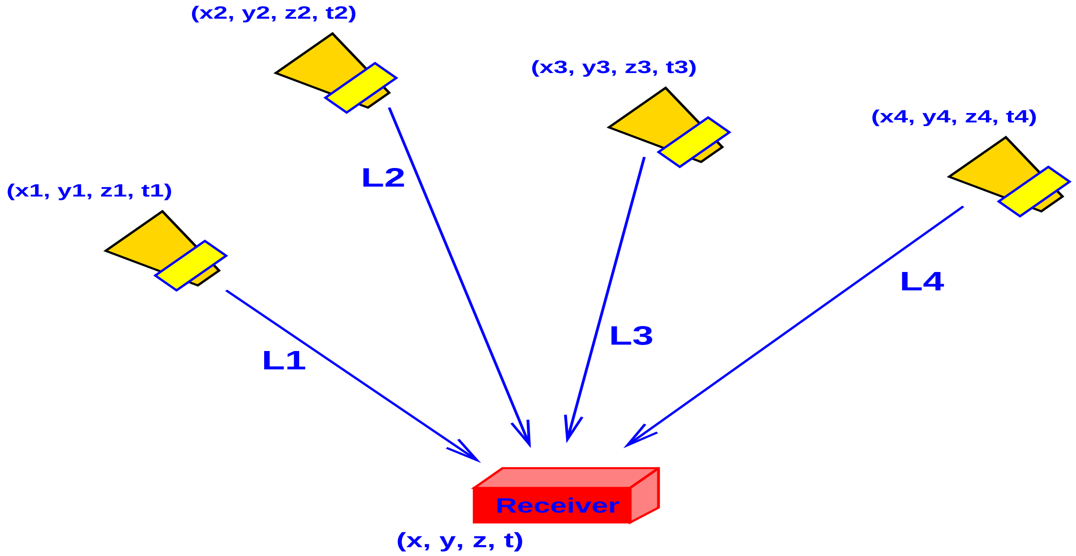
- **In moving frame an object has always the same length (it is invariant, our principle !)**
- **From stationary frame moving objects appear contracted by a factor γ (Lorentz contraction)**
- **Why do we care ?**
- **Turn the argument around: assume length of a proton bunch appears always at 0.1 m in laboratory frame (e.g. in the RF bucket), what is the length in its own (moving) frame ?**
  - **At 5 GeV ($\gamma \approx$ 5.3) $\quad \rightarrow \quad$ L' = 0.53 m**
  - **At 450 GeV ($\gamma \approx$ 480) $\quad \rightarrow \quad$ L' = 48.0 m**

Time-like and Space-like events



- **Event 1 can communicate with event 2**
- **Event 1 cannot communicate with event 3, would require to travel faster than the speed of light**

GPS principle ...



$$L_1 = \quad c(t - t_1) = \quad \sqrt{((x - x_1)^2 \; + \; (y - y_1)^2 \; + \; (z - z_1)^2}$$

$$L_2 = \quad c(t - t_2) = \quad \sqrt{((x - x_2)^2 \; + \; (y - y_2)^2 \; + \; (z - z_2)^2}$$

$$L_3 = \quad c(t - t_3) = \quad \sqrt{((x - x_3)^2 \; + \; (y - y_3)^2 \; + \; (z - z_3)^2}$$

$$L_4 = \quad c(t - t_4) = \quad \sqrt{((x - x_4)^2 \; + \; (y - y_4)^2 \; + \; (z - z_4)^2}$$

$t_1, t_2, t_3, t_4,$ **need relativistic correction !**

4 equations and 4 variables $\Rightarrow$ (x, y, z, t) **of the receiver !**

Gravitational time dilation

$$\frac{d\tau}{dt} = \sqrt{1 - \frac{2GM}{Rc^2}}$$
$$\frac{d\tau}{dt} \approx 1 - \frac{GM}{Rc^2}$$
$$\Delta\tau = \frac{GM}{c^2} \cdot \left(\frac{1}{R_{earth}} - \frac{1}{R_{gps}} \right)$$

With:

$R_{earth} = 6357000$ **m,** $R_{gps} = 26541000$ **m**
$G = 6.674 \cdot 10^{-11} \ \frac{N \cdot m^2}{kg^2}$ $M = 5.974 \cdot 10^{24}$ **kg**

We have:

$$\Delta\tau \approx 5.3 \cdot 10^{-10}$$

$$\Delta\tau \; = \; \frac{GM}{c^2} \cdot \left(\frac{1}{R_{earth}} - \frac{1}{R_{gps}} \right)$$

With:

$R_{earth} \; = \; R \; = \; 6357000$ **m**

$G = 6.674 \cdot 10^{-11} \; \dfrac{N \cdot m^2}{kg^2}$ $\quad M = 5.974 \cdot 10^{24}$ **kg**

At a height $h \; \lll \; R$ **above surface:**

$$\Delta\tau \; = \; \frac{GM}{c^2} \cdot \left(\frac{1}{R} - \frac{1}{R+h} \right) \; \approx \; \frac{GM}{c^2} \cdot \frac{h}{R}$$

We have:

$$\frac{\Delta f}{f} \; = \; \frac{g \cdot h}{c^2} \; \approx \; 1.1 \cdot 10^{-16} \;\; \text{per m}$$

Example: at $h \; = \; 0.33 \; m$**: relative shift is** $\approx \; 4 \cdot 10^{-17}$ **this was measured**

Do the math:

Orbital speed 14000 km/h $\approx$ 3.9 km/s

➡ $\beta \approx 1.3 \cdot 10^{-5}$, $\gamma \approx 1.000000000084$

Small, but accumulates 7 μs during one day compared to reference time on earth !

After one day: your position wrong by $\approx$ 2 km !!

(including general relativity error is 10 km per day, for the interested: backup slide, coffee break or after dinner discussions)

➡ **Countermeasures:**

(1) Minimum 4 satellites (avoid reference time on earth)

(2) Detune data transmission frequency from 1.023 MHz to 1.022999999543 MHz prior to launch