A LOWER BOUND ON INELASTICITY IN PION-PION SCATTERING

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DEDICATED TO THE MEMORY OF STANLEY MANDELSTAM

EVERYBODY KNOWS THAT A RELATIVISTIC SCATTERING AMPLITUDE IS ZERO IF THERE IS NO INELASTICITY, BUT THE QUESTION, ASKED TO ONE OF US, DURING A SEMINAR IN LAUSANNE WAS "COULD INELASTICITY BE ARBITRARILY SMALL" THE ANSWER IS NO. FURTHERMORE, UNDER CERTAIN ASSUMPTIONS, WE PROVE THAT, AT SUFFICIENTLY HIGH ENERGIES

$$\boxed{\sigma_{\text{INELASTIC}} > \exp(-\sqrt{s}\ln s)}$$

s = square of c.m. energy

WHY DO WE THANK MANDELSTAM?

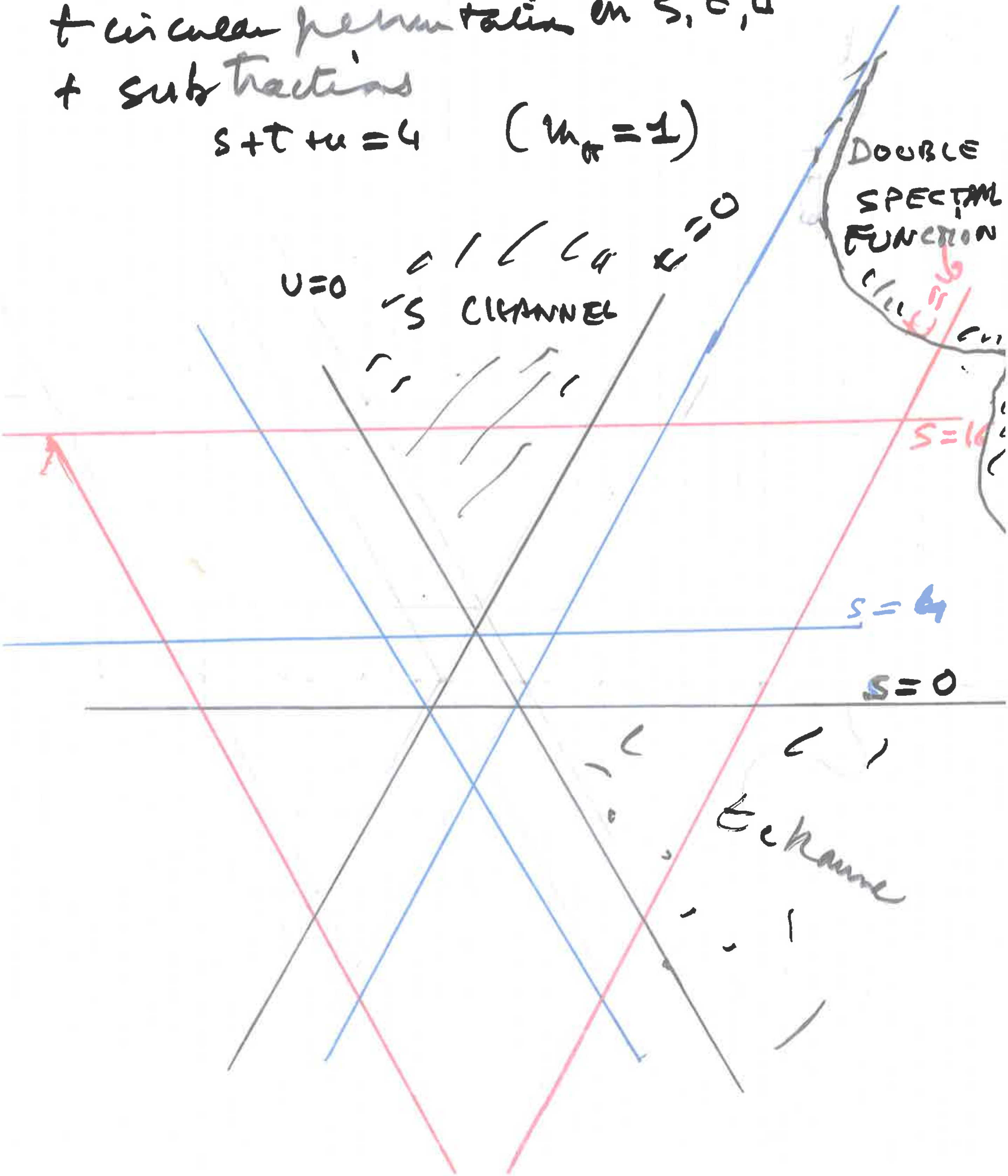
IN 1959 MANDELSTAM PROPOSED MANDELSTAM REPRESENTATION

$$F(s,t,u) = \frac{1}{\pi^2}\iint \frac{\rho(s',t')\,ds'\,dt'}{(s'-s)(t'-t)}$$

+ circular permutation on s, t, u

+ subtractions

$$s+t+u=4 \qquad (m_\pi = 1)$$



YOU MAY THINK: OBSOLETE (3)

BUT HISTORICAL ROLE

EXEMPLES

1) GRIBOV'S THEOREM: ~(1960-61)
IMPOSSIBLE TO HAVE AN AMPLITUDE BEHAVING LIKE

$s\, f(t)$, for $s \to \infty$,

i.e. FIXED SHAPE OF DIFFRACTION PEAK

AND $\sigma_{TOTAL} \to$ Const $\neq 0$

WAY OUT PROPOSED BY GRIBOV:

$\sigma_{tot} \to 0$ UNFORTUNATELY WRONG
(IN FACT $\sigma_{tot} \to \infty$, PROBABLY)

2) THE FROISSART BOUND; $\sigma_{tot} < (\ln s)^2$
USED, INITIALLY, MANDELSTAM REP. (1961)
BUT IN 1966 I PROVED IT FROM FIRST PRINCIPLES, FROM ENLARGEMENT OF ANALYTICITY DOMAIN BY POSITIVITY.

TO PROVE INELASTICITY, WE DONT NEED THE FULL MANDELSTAM REP.
(NEVER PROVED NOR DISPROVED FOR $\pi\pi \to \pi\pi$)
WE NEED FIXED ENERGY ANALYTICITY IN AN ELLIPSE WITH FOCI AT $t=0$, $u=0$ AND RIGHT EXTREMITY AT $\boxed{t = 4+\varepsilon}$,
MINUS THE CUTS PREDICTED BY MANDELSTAM

IF YOU PREFER FOCI AT $\cos\theta_s = \pm 1$
extending at $\cos\theta_s = 1 + \frac{2}{s-4}(4+\varepsilon)$

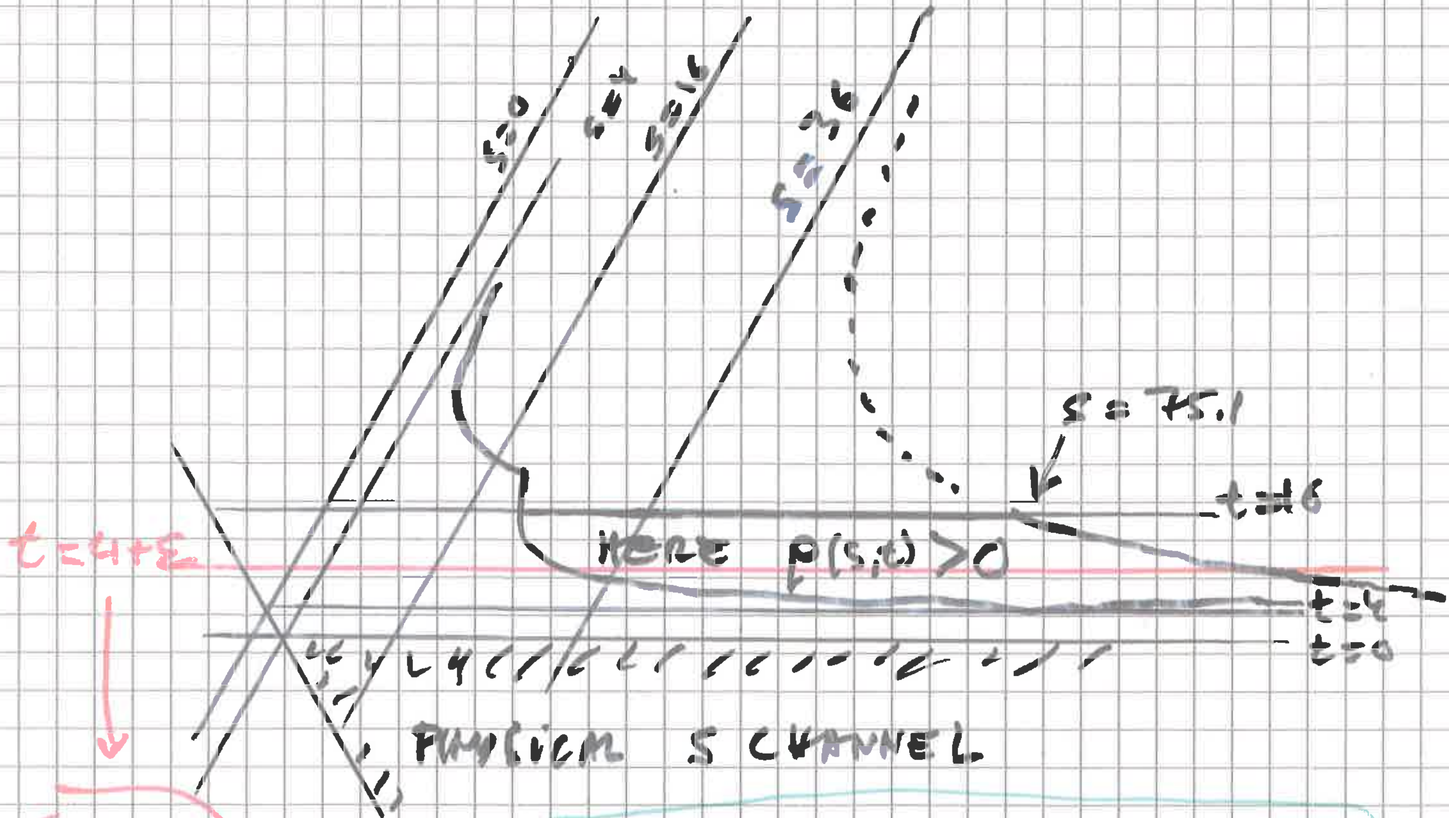
FROM FIELD THEORY, YOU CAN GO TO $\boxed{t=4}$

REMARK: THIS ENOUGH TO PROVE GRIBOV'S TH

OUR STRATEGY IS THE SAME AS THAT OF ALEX DRAGT (1967)

USE THE FACT THAT FOR LARGE ANGULAR MOMENTA THE PARTIAL WAVE AMPLITUDES ARE DOMINATED BY THE NEAREST SINGULARITIES

BUT, IN DRAGT APPROACH, SOMETHING IS MISSING, THE FACT THAT THERE IS A REGION WHERE THE DOUBLE SPECTRAL FUNCTION IS <u>POSITIVE</u> AND CALCULABLE (MAHOUX + MARTIN 1964), OBTAINED BY USING MANDELSTAM ANALYTIC CONTINUATION OF ELASTIC UNITARITY IN THE t CHANNEL



$\varepsilon \geq 4$

IN A FIRST STEP, WE USE ISOSPIN 0 PIONS!

WE OBTAINED THIS REGION BY USING MANDELSTAM'S CONTINUATION OF UNITARITY:

$$\rho(s,t) = \frac{2}{\pi}\sqrt{\frac{t-4}{t}} \int \frac{dz_1\, dz_2}{\sqrt{H(z,z_1,z_2)}} A_s(s_1,t) A_s^*(s_2,t)$$

$4 < t < 16$

$$z = 1+\frac{2s}{t-4}, \quad z_1 = 1+\frac{2s_1}{t-4}, \quad z_2 = 1+\frac{2s_2}{t-4}, \quad z_0 = 1+\frac{8}{t-4}$$

$$H = z^2+z_1^2+z_2^2-1-2zz_1z_2$$

$$= (z-z_+)(z-z_-), \quad z_\pm = z_1z_2 \pm\sqrt{z_1^2-1}\sqrt{z_2^2-1}$$

The domain of integration is such that $z_+ > 0$, z_1 and $z_2 > z_0$

if s is sufficiently small, $s_1\, t$ and $s_2\, t$ are in a region with $\rho = 0$. Then partial wave expansion of the A_s's converges, with POSITIVE COEFFICIENTS FROM UNITARITY, HENCE $\rho > 0$

this gives a region limited by

$t = 4$ and $t = 16$, $t = 4+\frac{64}{s-16}$ and $t = 4+\frac{32}{\sqrt{s}-6}$

NOT ONLY WE PROVE POSITIVITY BUT WE HAVE A LOWER BOUND ON ρ

BECAUSE $A_s(s_1,t) > A_s(s_1,0) \sim \sigma_{tot}(s_1)$ times something

$A_s(s_2,t) > A_s(s_2,0) \sim \sigma_{tot}(s_2)$ times ....

⑥

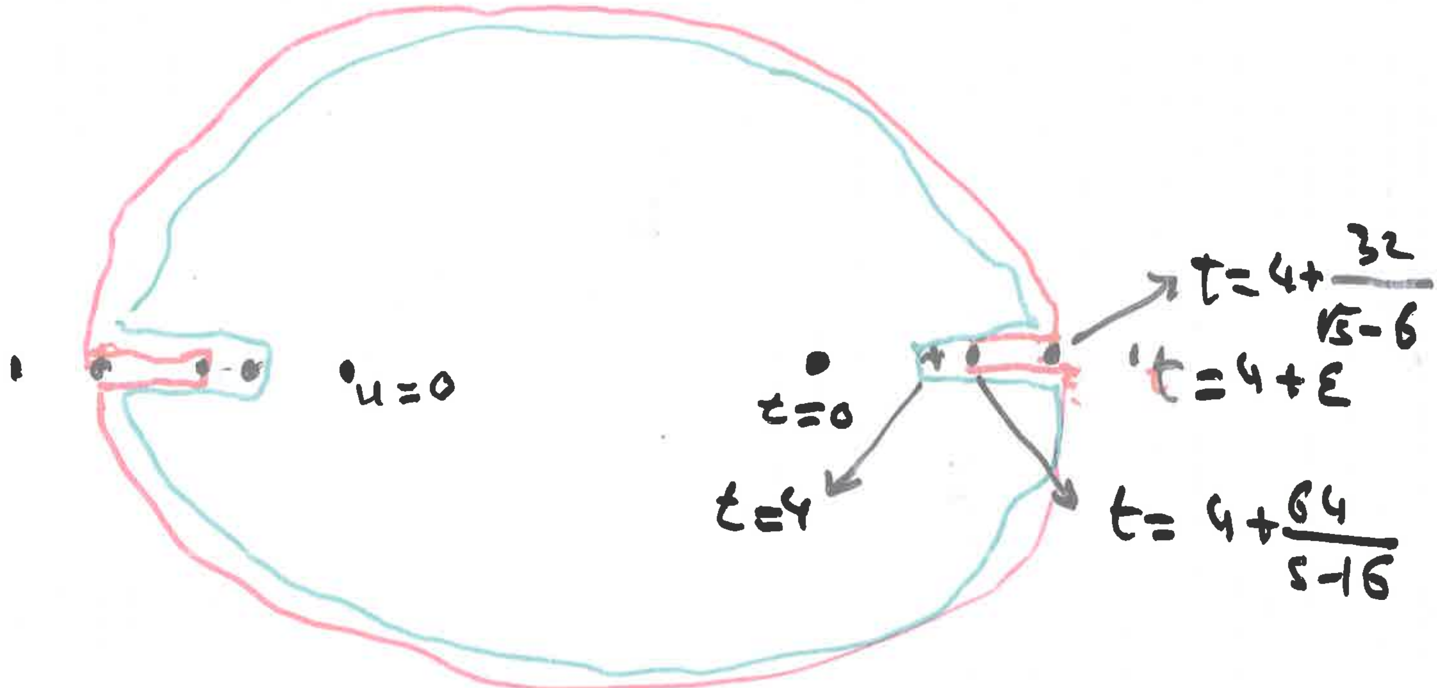
THIS LOWER WILL BE USED TO CALCULATE $\operatorname{Im} f_\ell$

ESTIMATES OF $|f_\ell|$ and $\operatorname{Im} f_\ell$

WE WRITE f_ℓ and $\operatorname{Im} f_\ell$ as contour integrals along the ellipse and the cuts

$$f_\ell = \frac{2}{s-4}\int Q_\ell\left(1+\frac{t}{2k^2}\right) F(s,t)\,dt$$

$$\operatorname{Im} f_\ell = \frac{2}{s-4}\int Q_\ell\left(1+\frac{t}{2k^2}\right) A_s(s,t)\,dt$$



— = CONTOUR FOR $\operatorname{Im} f_\ell$

— = CONTOUR FOR f_ℓ

INSIDE THIS REGION F and A_s ARE BOUNDED BY $B(s)$, THAT WE LEAVE ARBITRARY, FOR THE TIME BEING

Now we proceed in 2 steps

1) SHOW THAT BEYOND A CERTAIN L THE CONTRIBUTION OF THE CUT DOMINATES ON THE CONTRIBUTION OF THE ELLIPSE FOR $\mathrm{Im} f_L$. WE MUTILATE THE CONTRIBUTION OF THE CUT AND GET

$$4k^2\, \mathrm{Im} f_L > Q_L\left(1+\frac{1}{s-4}\left(8+\frac{128+P(s)}{s-16}\right)\right) \times \int_{4+\frac{64}{s-16}}^{4+\frac{64+P}{s-16}} \rho(s,t)$$

$$-\frac{1}{2} B(s)\, L(s)\, Q_L\left(1+\frac{1}{s-4}\left(8+\frac{64}{\sqrt{s}-6}\right)\right)$$

$L(s) =$ LENGHT OF THE ELLIPSE

Since $s-16 \gg \sqrt{s}-6$

AND SINCE ~~TH~~ Q_ℓ's DECREASE EXPONENTIALLY WITH L

IT IS OBVIOUS THAT FOR SOME $L > L_0$ THE CUT CONTRIBUTION DOMINATES WHATEVER IS $B(s)$

2) THEN, WE COMPARE THE CUT CONTRIBUTION IN $\mathrm{Im} f_L$ with $|f_L|^2$. $|f_L|^2$ is less than

$$[B(s)]^2 |L(s)|^2 \left(Q_L\left(1+\frac{8}{s-4}\right)\right)^2$$

AGAIN,

IF $P(s)$ IS BOUNDED

$$\frac{\left(Q_L\left(1+\frac{8}{s-4}\right)\right)^2}{Q_L\left(1+\frac{1}{s-4}\left(8+\frac{128+P(s)}{s-16}\right)\right)}$$

DECREASES EXPONENTIALLY WITH L

SO, THERE AN L_1 SUCH THAT

if $L > L_1$ $\quad \mathrm{Im}\, f_L \gg |f_L|^2$

AND THE INELASTIC CROSS-SECTION IS STRICTLY POSITIVE

ASYMPTOTIC BEHAVIOUR FOR LARGE L

HERE, WE HAVE TO MAKE AN ASSUMPTION ON $B(s)$. IF WE BELIEVE IN MANDELSTAM REPRESENTATION WITH A FINITE NUMBER OF SUBTRACTIONS, WE HAVE

$$B(s) < s^N$$

EVEN IF MANDELSTAM IS NOT VALID REMEMBER THAT THE ELLIPSE WITH RIGHT EXTREMITY AT $t = 4+\epsilon$ IS VERY CLOSE TO THE ELLIPSE WHERE WE KNOW, FROM FIRST PRINCIPLES

THAT AT LEAST THE ABSORPTIVE PART IS BOUNDED FOR ALMOST EVERY ENERGY BY s^2. (THIS ELLIPSE IS THE ONE WITH EXTREMITY AT $t=4$) HOPEFULLY THIS MIGHT BE THE CASE FOR THE DISPERSIVE PART. WE FEEL THAT IT IS NOT UNREASONABLE TO POSTULATE

$$B(s) < s^N$$

IT IS RELATIVELY EASY TO FIND A LOWER BOUND ON ρ, USING THE UPPER BOUND ON H (eq ⊛)

$$H < |z-1|^2$$

RETURNING TO ⊛ ⊛

WE FIND THAT IN $\mathrm{Im}\, f_L$ the CUT CONTRIBUTION DOMINATES THE INTEGRAL ALONG THE ELLIPSE FOR

$$L > L_0(s) = \frac{N+5/2}{16}\, s \ln s ,$$

and, CALCULATING $\mathrm{Im}\, f_L$ FOR THIS VALUE OF L $\mathrm{Im}\, f_L \gg |f_L|^2$ WE FIND

$$\boxed{\sigma_{\text{INELASTIC}} > \frac{C}{s^{5/2}} \exp -\frac{\sqrt{s}}{4}\left(N+5/2\right)\ln s}$$

THE CASE OF REAL PIONS, $\pi^+ \pi^- \pi^0$ CAN BE TREATED BY GOOD ALGEBRISTS LIKE G. MAHOUX and S.M. ROY

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17 juin 2017