LUND UNIVERSITY
Faculty of Science

Anomaly-free model building: algebraic geometry and the Froggatt-Nielsen mechanism

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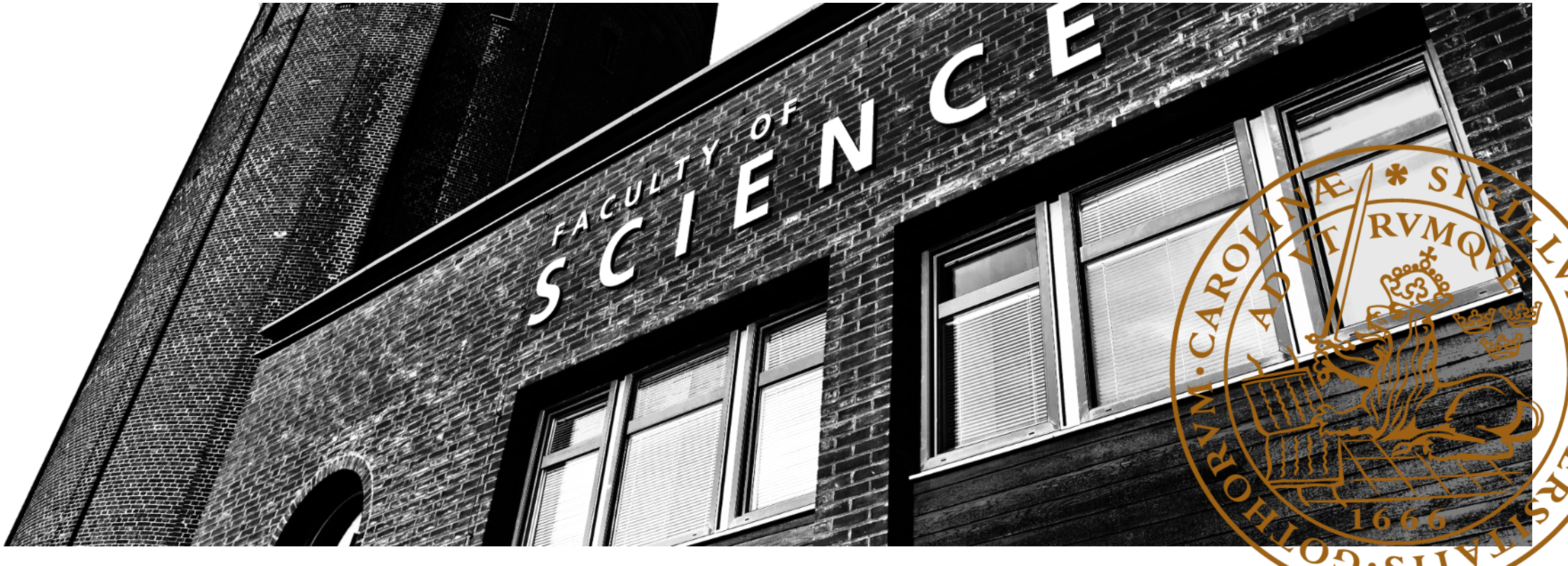
EPS-HEP 2019



Some trends in model building

- **Algebraic geometry:**
  Hilbert series methods – count invariant terms, calculate basis invariants
  Gröbner basis – find anomaly-free representations
- **Z' bosons:**
  Appears in many BSM models, light dark photons/heavy Z'
- **Flavour problem:**
  What is the origin for the different fermion masses and mixings?

Gauged Froggatt-Nielsen mechanism

Extend the SM gauge symmetries: $SU(3)_C \times SU(2)_L \times U(1)_Y$ to $SU(3)_C \times SU(2)_L \times U(1)_Y \times U(1)_F$

Left-handed fields: $\{Q_L^i, \quad (U_R^i)^c, \quad (D_R^i)^c, \quad L_L^i, \quad (E_R^i)^c, \quad (N_R^i)^c\}$ and Higgs Φ_n

F –charges: $\{Q_i, \quad u_i, \quad d_i, \quad L_i, \quad e_i, \quad \nu_i\}$ H_n

$U(1)_F$ is spontaneously broken when the scalar flavon S (F-charge -1) attains a vev $\langle S \rangle$

A high energy scale Λ_{FN} with new physics

Idea is that Yukawa couplings become suppressed by factors of $\epsilon = \frac{\langle S \rangle}{\Lambda_{FN}}$

C. Froggatt and H. Nielsen,
Nucl. Phys. B 147 (1979) 277 - 298

Yukawa couplings:

$$-\mathcal{L}_Y = \overline{Q}_L \widetilde{\Phi} Y^U U_R + \overline{Q}_L \Phi Y^D D_R + \overline{L}_L \Phi Y^L E_R + \overline{L}_L \widetilde{\Phi} Y^N N_R + \mathrm{H.c}$$

Froggatt-Nielsen:

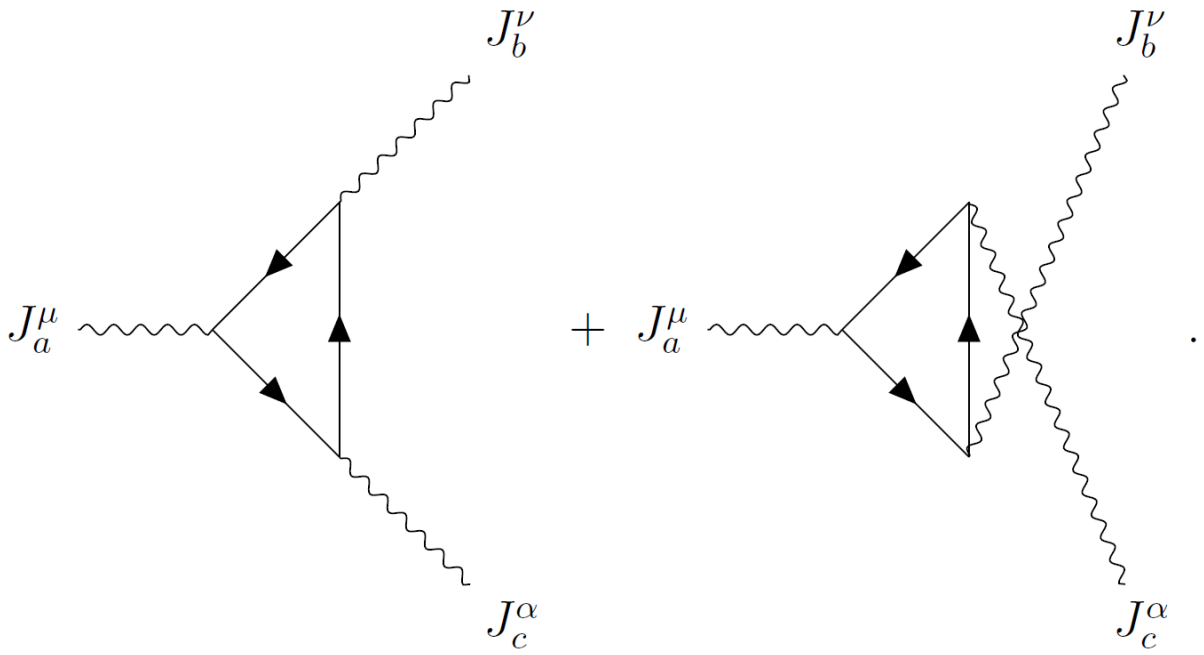
$$M_{ij}^F = \langle \Phi \rangle_0 Y_{ij}^F = \langle \Phi \rangle_0 g_{ij}^F \left(\frac{\langle S \rangle}{\Lambda_{FN}} \right)^{|n_{ij}^F|}$$

F=U,D,L,N
g_{ij}^F order one

$U(1)_F$ invariance:

$$n_{ij}^U = Q_i + u_j + H, \qquad n_{ij}^N = L_i + \nu_j + H$$
$$n_{ij}^D = Q_i + d_j - H, \qquad n_{ij}^L = L_i + e_j - H$$

Cancellation of triangle anomalies



$$\mathcal{A}_{1FF} = 2\sum_{j=1}^{3}\left(Q_j^2 - 2u_j^2 + d_j^2 - L_j^2 + e_j^2\right) = 0$$

$$\mathcal{A}_{11F} = \frac{2}{3}\sum_{j=1}^{3}\left(Q_j + 8u_j + 2d_j + 3L_j + 6e_j\right) = 0$$

$$\mathcal{A}_{33F} = \frac{1}{2}\sum_{j=1}^{3}\left(2Q_j + u_j + d_j\right) = 0$$

$$\mathcal{A}_{22F} = \frac{1}{2}\sum_{j=1}^{3}\left(3Q_j + L_j\right) = 0$$

$$\mathcal{A}_{FFF} = \sum_{j=1}^{3}\left(6Q_j^3 + 3u_j^3 + 3d_j^3 + 2L_j^3 + e_j^3 + \nu_j^3\right) = 0$$

$$\mathcal{A}_{ggF} = \sum_{j=1}^{3}\left(6Q_j + 3u_j + 3d_j + 2L_j + e_j + \nu_j\right) = 0$$

Model example I

https://arxiv.org/abs/1902.08529

- 2HDM, $\tan\beta = 1$
- 3 SM singlets
- Neutrino masses through Weinberg operator

$$Y_2^U \sim \begin{pmatrix} \epsilon^7 & \epsilon^5 & \epsilon^3 \\ \epsilon^6 & \epsilon^4 & \epsilon^2 \\ \epsilon^4 & \epsilon^2 & \epsilon^0 \end{pmatrix}, \quad Y_1^D \sim \begin{pmatrix} \epsilon^7 & \epsilon^6 & \epsilon^6 \\ \epsilon^6 & \epsilon^5 & \epsilon^5 \\ \epsilon^4 & \epsilon^3 & \epsilon^3 \end{pmatrix}, \quad Y_1^L \sim \begin{pmatrix} \epsilon^8 & * & * \\ * & \epsilon^4 & \epsilon^3 \\ * & \epsilon^4 & \epsilon^3 \end{pmatrix} \quad \kappa_{11} \sim \begin{pmatrix} * & * & * \\ * & \epsilon^0 & \epsilon^0 \\ * & \epsilon^0 & \epsilon^0 \end{pmatrix}$$

$$V_{CKM} \sim \begin{pmatrix} 1 & \epsilon & \epsilon^3 \\ \epsilon & 1 & \epsilon^2 \\ \epsilon^3 & \epsilon^2 & 1 \end{pmatrix}$$

Model example I

- 2HDM, tan$\beta = 1$
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$$Y_2^U \sim \begin{pmatrix} \epsilon^7 & \epsilon^5 & \epsilon^3 \\ \epsilon^6 & \epsilon^4 & \epsilon^2 \\ \epsilon^4 & \epsilon^2 & \epsilon^0 \end{pmatrix}, \quad Y_1^D \sim \begin{pmatrix} \epsilon^7 & \epsilon^6 & \epsilon^6 \\ \epsilon^6 & \epsilon^5 & \epsilon^5 \\ \epsilon^4 & \epsilon^3 & \epsilon^3 \end{pmatrix}, \quad Y_1^L \sim \begin{pmatrix} \epsilon^8 & * & * \\ * & \epsilon^4 & \epsilon^3 \\ * & \epsilon^4 & \epsilon^3 \end{pmatrix} \quad \kappa_{11} \sim \begin{pmatrix} * & * & * \\ * & \epsilon^0 & \epsilon^0 \\ * & \epsilon^0 & \epsilon^0 \end{pmatrix}$$

$$V_{CKM} \sim \begin{pmatrix} 1 & \epsilon & \epsilon^3 \\ \epsilon & 1 & \epsilon^2 \\ \epsilon^3 & \epsilon^2 & 1 \end{pmatrix}$$

No $Z - Z_F$ mixing through fermion loops:

$$\sum_{j=1}^{3} (2Q_j - 4u_j + 2d_j - 2L_j + 2e_j) = 0$$

Find rational solutions to non-linear system of 20 equations:

$$
\begin{cases}
\sum_{j=1}^{3}\left(Q_j^2 - 2u_j^2 + d_j^2 - L_j^2 + e_j^2\right) = 0 \\
\sum_{j=1}^{3}\left(Q_j + 8u_j + 2d_j + 3L_j + 6e_j\right) = 0 \\
\sum_{j=1}^{3}\left(2Q_j + u_j + d_j\right) = 0 \\
\sum_{j=1}^{3}\left(3Q_j + L_j\right) = 0 \\
\sum_{j=1}^{3}\left(6Q_j^3 + 3u_j^3 + 3d_j^3 + 2L_j^3 + e_j^3 + \nu_j^3\right) = 0 \\
\sum_{j=1}^{3}\left(2L_j + e_j + \nu_j\right) = 0 \\
\sum_{j=1}^{3}\left(2Q_j - 4u_j + 2d_j - 2L_j + 2e_j\right) = 0 \\
Q_3 + u_3 + H_2 = 0,\ Q_2 + u_2 + H_2 = 4,\ Q_1 + u_1 + H_2 = 7 \\
Q_3 + d_3 - H_1 = 3,\ Q_2 + d_2 - H_1 = 5,\ Q_1 + d_1 - H_1 = 7 \\
L_3 + e_3 - H_1 = 3,\ L_2 + e_2 - H_1 = 4,\ L_1 + e_1 - H_1 = 8 \\
Q_1 - Q_2 = 1,\ Q_2 - Q_3 = 2 \\
L_2 - L_3 = 0,\ L_2 + H_1 = 0
\end{cases}
$$

Gröbner bases and Mordell-Weil generators

Gröbner basis: the most reduced set of equations with the same solutions.
As many terms and variables as possible have been eliminated.
(Like Gauss-Jordan elimination in linear algebra)

Mordell-Weil's theorem:
The set of rational points on an elliptic curve (i.e. smooth curve of genus 1) is finitely generated

Mordell-Weil generators: The generators of the group of rational points

$$
\begin{aligned}
Q_1 - 8/27 &= 0, & Q_2 + 19/27 &= 0, & Q_3 + 73/27 &= 0,\\
u_1 - 26/27 &= 0, & u_2 + 28/27 &= 0, & u_3 + 82/27 &= 0,\\
d_1 - 34/9 &= 0, & d_2 - 25/9 &= 0, & d_3 - 25/9 &= 0,\\
L_1 - 94/27 &= 0, & L_2 - 79/27 &= 0, & L_3 - 79/27 &= 0,\\
e_1 - 43/27 &= 0, & e_2 + 50/27 &= 0, & e_3 + 77/27 &= 0,\\
H_1 + 79/27 &= 0, & H_2 - 155/27 &= 0,
\end{aligned}
$$

Gröbner basis:

$$
\begin{aligned}
&\nu_1 + \nu_2 + \nu_3 + 140/9 = 0,\\
&\nu_2^2 \cdot \nu_3 + 140/9 \cdot \nu_2^2 + \nu_2 \cdot \nu_3^2 + 280/9 \cdot \nu_2 \cdot \nu_3 + 19600/81 \cdot \nu_2 +\\
&+ 140/9 \cdot \nu_3^2 + 19600/81 \cdot \nu_3 + 95036/81 = 0.
\end{aligned}
$$

$$
\begin{aligned}
Q_1 - 8/27 &= 0, & Q_2 + 19/27 &= 0, & Q_3 + 73/27 &= 0,\\
u_1 - 26/27 &= 0, & u_2 + 28/27 &= 0, & u_3 + 82/27 &= 0,\\
d_1 - 34/9 &= 0, & d_2 - 25/9 &= 0, & d_3 - 25/9 &= 0,\\
L_1 - 94/27 &= 0, & L_2 - 79/27 &= 0, & L_3 - 79/27 &= 0,\\
e_1 - 43/27 &= 0, & e_2 + 50/27 &= 0, & e_3 + 77/27 &= 0,\\
H_1 + 79/27 &= 0, & H_2 - 155/27 &= 0,
\end{aligned}
$$

Gröbner basis:

$$
\begin{aligned}
&\nu_1 + \nu_2 + \nu_3 + 140/9 = 0,\\
&\nu_2^2 \cdot \nu_3 + 140/9 \cdot \nu_2^2 + \nu_2 \cdot \nu_3^2 + 280/9 \cdot \nu_2 \cdot \nu_3 + 19600/81 \cdot \nu_2 +\\
&+ 140/9 \cdot \nu_3^2 + 19600/81 \cdot \nu_3 + 95036/81 = 0.
\end{aligned}
$$

$$
E: \; y^2 + 2xy + \frac{95036}{81} y = x^3 + \frac{19519}{81} x^2 + \frac{12449716}{729} x
$$

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Q_1 - 8/27 &= 0, & Q_2 + 19/27 &= 0, & Q_3 + 73/27 &= 0,\\
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L_1 - 94/27 &= 0, & L_2 - 79/27 &= 0, & L_3 - 79/27 &= 0,\\
e_1 - 43/27 &= 0, & e_2 + 50/27 &= 0, & e_3 + 77/27 &= 0,\\
H_1 + 79/27 &= 0, & H_2 - 155/27 &= 0,
\end{aligned}
$$

Gröbner basis:

$$
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&\nu_1 + \nu_2 + \nu_3 + 140/9 = 0,\\
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&+ 140/9 \cdot \nu_3^2 + 19600/81 \cdot \nu_3 + 95036/81 = 0.
\end{aligned}
$$

$$E: \; y^2 + 2xy + \frac{95036}{81}y = x^3 + \frac{19519}{81}x^2 + \frac{12449716}{729}x$$

$$E(\mathbb{Q}) = E(\mathbb{Q})_{\text{tors}} \oplus \mathbb{Z}P_1$$

$$
\begin{aligned}
E(\mathbb{Q})_{\text{tors}} &= \{(0 : -95036/81 : 1), (0 : 0 : 1), (0 : 1 : 0)\}\\
P_1 &= (2041940/81 : 323674124/81 : 1)
\end{aligned}
$$

$$(\nu_2, \nu_3) = \left(\frac{30795}{193}, -\frac{18344}{115}\right) \qquad \nu_1 = -\frac{140}{9} - \nu_2 - \nu_3 = -\frac{3116597}{199755}$$

$$
\begin{aligned}
Q_1 - 8/27 &= 0, & Q_2 + 19/27 &= 0, & Q_3 + 73/27 &= 0,\\
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$$(\nu_2, \nu_3) = \left(\frac{30795}{193}, -\frac{18344}{115}\right)$$

$$\nu_1 = -\frac{140}{9} - \nu_2 - \nu_3 = -\frac{3116597}{199755}$$

$$
Y_2^U \sim \begin{pmatrix} \epsilon^7 & \epsilon^5 & \epsilon^3 \\ \epsilon^6 & \epsilon^4 & \epsilon^2 \\ \epsilon^4 & \epsilon^2 & \epsilon^0 \end{pmatrix}, \qquad
Y_1^D \sim \begin{pmatrix} \epsilon^7 & \epsilon^6 & \epsilon^6 \\ \epsilon^6 & \epsilon^5 & \epsilon^5 \\ \epsilon^4 & \epsilon^3 & \epsilon^3 \end{pmatrix} \qquad
V_{CKM} \sim \begin{pmatrix} 1 & \epsilon & \epsilon^3 \\ \epsilon & 1 & \epsilon^2 \\ \epsilon^3 & \epsilon^2 & 1 \end{pmatrix}
$$

$$
Y_1^L \sim \begin{pmatrix} \epsilon^8 & 0 & 0 \\ 0 & \epsilon^4 & \epsilon^3 \\ 0 & \epsilon^4 & \epsilon^3 \end{pmatrix} \qquad
\kappa_{11} \sim \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \qquad
U_{PMNS} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}.
$$

Note: $H_2 - H_1 = 26/3$ non-integer so we have type-II Z_2 symmetry as an effect from $U(1)_F$ breaking

Model example II

https://arxiv.org/abs/1902.08529

- Standard Model + three right-handed neutrinos
- **All** fermion masses are Dirac masses

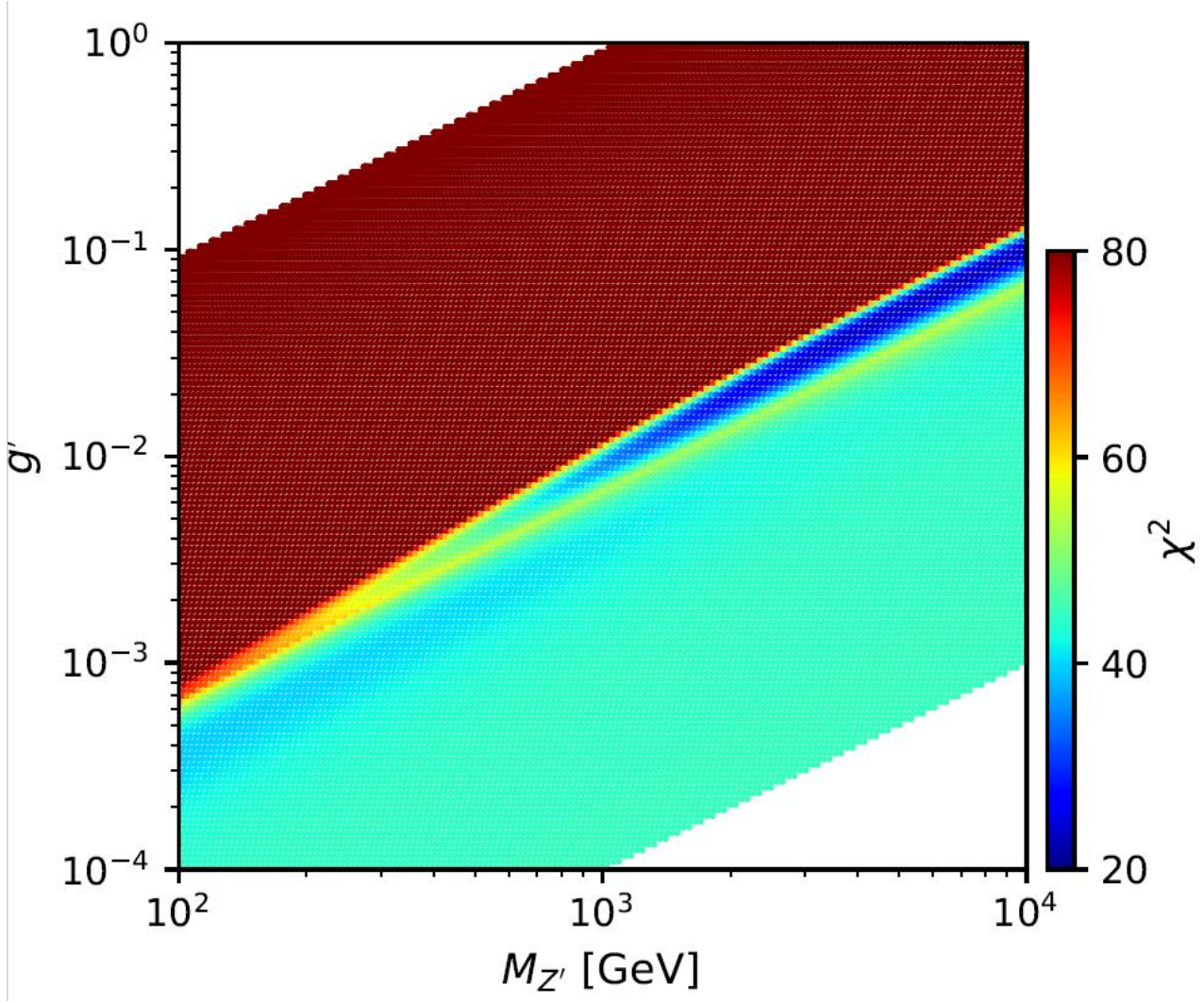
$$Y^U \sim \begin{pmatrix} \epsilon^7 & \epsilon^6 & \epsilon^4 \\ \epsilon^8 & \epsilon^5 & \epsilon^3 \\ \epsilon^{10} & \epsilon^3 & \epsilon \end{pmatrix}, \quad Y^D \sim \begin{pmatrix} \epsilon^7 & \epsilon^6 & \epsilon^6 \\ \epsilon^8 & \epsilon^5 & \epsilon^5 \\ \epsilon^{10} & \epsilon^3 & \epsilon^3 \end{pmatrix}, \quad Y^L \sim \begin{pmatrix} \epsilon^8 & 0 & 0 \\ 0 & \epsilon^4 & \epsilon^3 \\ 0 & \epsilon^4 & \epsilon^3 \end{pmatrix}, \quad Y^N \sim \begin{pmatrix} \epsilon^{37} & 0 & 0 \\ 0 & \epsilon^{18} & \epsilon^{18} \\ 0 & \epsilon^{18} & \epsilon^{18} \end{pmatrix}$$

Again the Higgs charge is free so we fix it by the $Z - Z_F$ mixing condition

$$Q_1 = 21006391/7704195, \quad Q_2 = 13302196/7704195, \quad Q_3 = -2106194/7704195,$$
$$u_1 = -50899429/7704195, \quad u_2 = 49255106/7704195, \quad u_3 = 33846716/7704195,$$
$$d_1 = -98972083/7704195, \quad d_2 = 1182452/7704195, \quad d_3 = 1182452/7704195,$$
$$L_1 = -9617581/2568065, \quad L_2 = -11292406/2568065, \quad L_3 = -11292406/2568065,$$
$$e_1 = 22149992/2568065, \quad e_2 = -6991963/2568065, \quad e_3 = -4423898/2568065,$$
$$\nu_1 = -15477743/513613, \quad \nu_2 = 13105937/513613, \quad \nu_3 = 13105937/513613$$
$$H = -8012109/2568065,$$

- Highly non-trivial charges
- Non need for ”anomalous U(1)s”
- All masses and mixings come from the same mechanism
- Plastic surgery could be with some additional fields

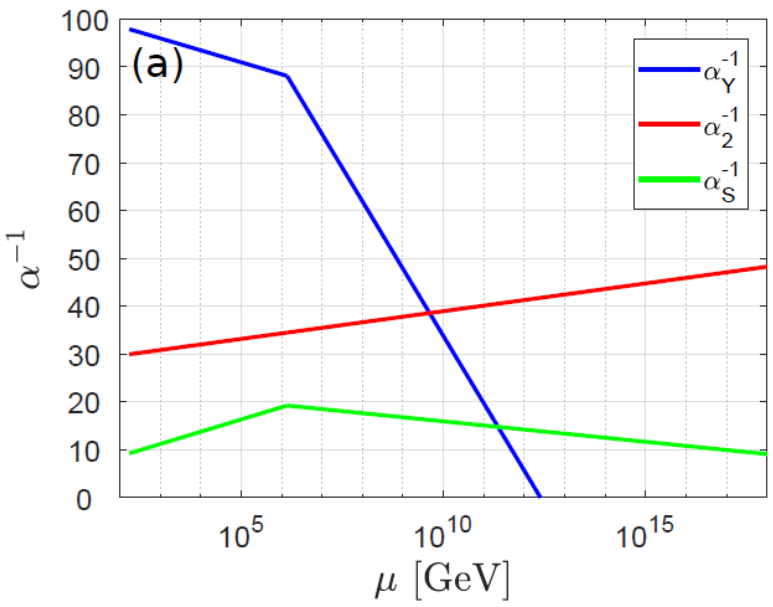
Global fit



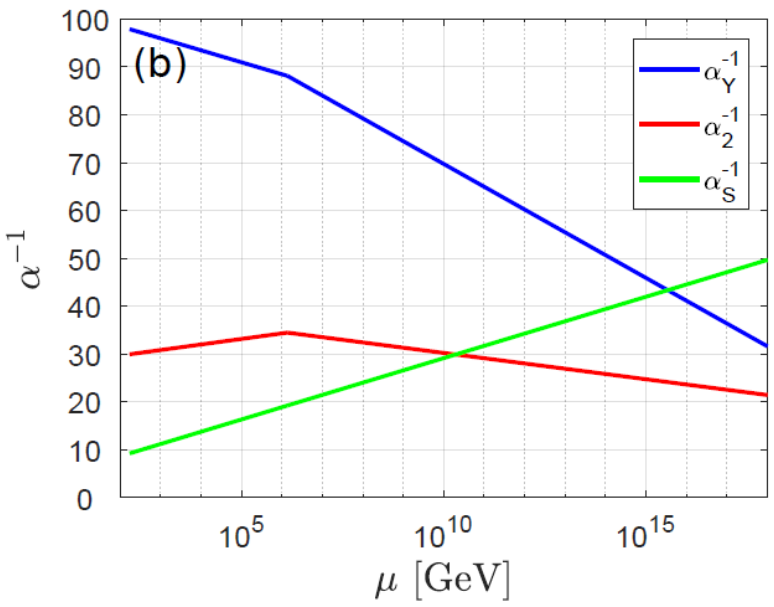
- Fermion masses and mixings
- Electroweak precision tests and oblique parameters
- Several rare decays

An interesting scale for new physics is $\Lambda_{FN} = 10^6$ GeV

UV completion

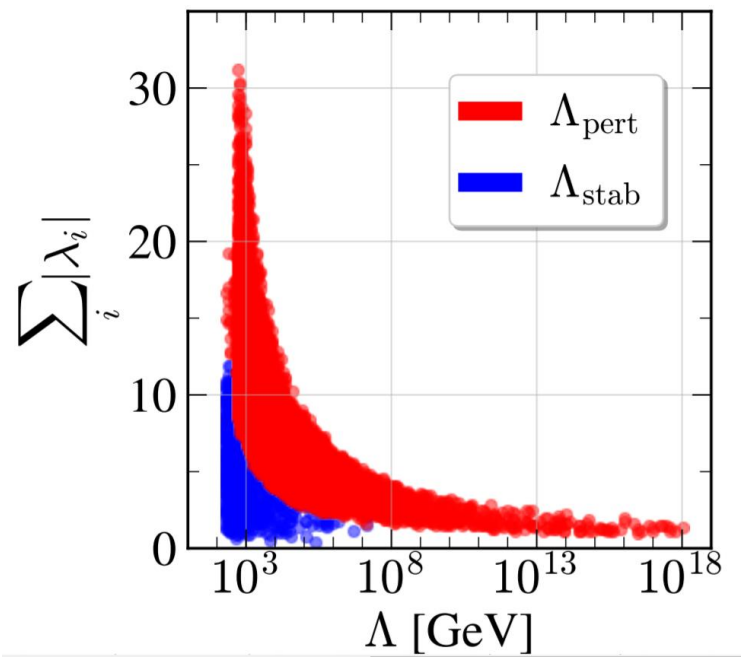


FNs original suggestion with vector-like fermions will in general produce Landau poles



Using scalars as suggested by Bijnens and Wetterich yields a consistent completion

J. Bijnens and C. Wetterich, Nucl. Phys. B 283 (1987) 237



Breakdown of perturbativity 2HDM with soft Z_2-breaking

J. Oredsson and J. Rathsman JHEP 2019:152


Thank you!
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AD VTRVMQVE
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