

# All master integrals for three-jet production at NNLO

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# Towards an era of precision collider measurements

- Ever improving experimental precision at the LHC
- For many QCD processes Next-to-Leading Order approximation is insufficient, e.g. strong coupling from 3-jet/2-jet ratio:

$$\alpha_s(M_Z) = 0.1148 \pm 0.0014 \pm 0.0018 \pm \boxed{0.0050} \quad (\text{exp}) \quad (\text{PDF}) \quad (\text{theory})$$

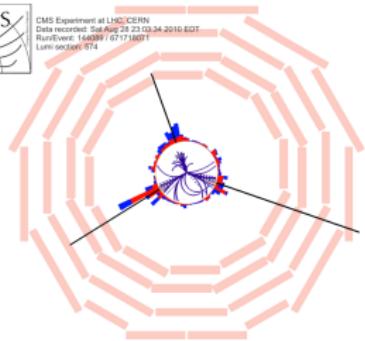
[CMS collaboration, Eur.Phys.J C73 (2013)]

**Large theoretical uncertainty!**

- NNLO predictions are required to fully exploit the LHC data

# Multi-jet processes at NNLO

CMS Experiment at LHC CERN  
Data recorded: Sat Apr 25 23:34:34 2010 EDT  
Run/Event: 146099 / 071718001  
Lumi section 874



- State of the art: two-to-two processes at NNLO
- Multi-jet processes are important for phenomenology:
  - ◊  $\alpha_s$  determination
  - ◊ tests of Standard Model
  - ◊ search for new physics
- Virtual three-jet corrections are bottleneck
- **This talk: two-loop five-particle amplitudes, analytic calculation**

process	known	desired
$pp \rightarrow 2 \text{ jets}$	$N^2\text{LO}_{\text{QCD}}$	
	$\text{NLO}_{\text{QCD}} + \text{NLO}_{\text{EW}}$	
$pp \rightarrow 3 \text{ jets}$	$\text{NLO}_{\text{QCD}}$	$N^2\text{LO}_{\text{QCD}}$

Table I.2: Precision wish list: jet final states.

# Amplitude calculation workflow



**Integrand:** Feynman diagrams, Generalized Unitarity

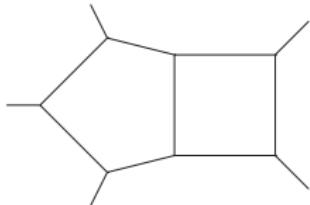
**Integration-by-Parts identities:** Feynman integrals are not all independent. IBP-reduction to a finite number of **master integrals**.  
⇒ Heavy computational problem

Finite fields significantly improve IBP reduction algorithms. [von Manteuffel, Schabinger '15][Peraro '16'19][Maierhoefer, Usovitsch '18][Smirnov, Chukharev '19]  
[talk by Hartanto]

**Master integrals:** Evaluate to pentagon functions

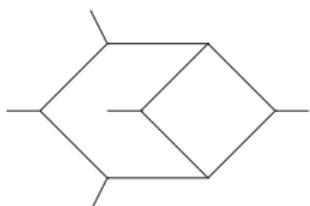
**Assembly of the amplitude.** Highly nontrivial task!

# Families of the master integrals for the massless two-loop scattering



[Gehrman, Henn, Lo Presti '15, '18]

[Papadopoulos, Tommasini, Wever '15]

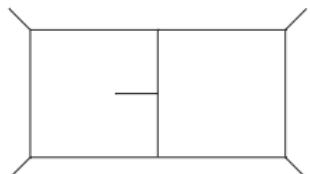


[D.C., Mitev, Henn '17]

[Boehm, Georgoudis, Larsen, Schoenemann, Zhang '18]

[Abreu, Dixon, Herrmann, Page, Zeng '18]

[D.C., Gehrman, Henn, Lo Presti, Mitev, Wasser '18]



[Abreu, Dixon, Herrmann, Page, Zeng '18]

[D.C., Gehrman, Henn, Wasser, Zhang, Zoia '18]

⇒ pentagon functions

# Pentagon functions

- Proposed in [D.C., Mitev, Henn '17]
- Confirmed in [Abreu, Dixon, Herrmann, Page, Zeng '18]  
[D.C., Gehrmann, Henn, Wasser, Zhang, Zoia '18]

Iterated integrals along path  $\gamma$



$$\int\limits_{\gamma} d \log W_{i_1}(s) \dots d \log W_{i_n}(s)$$

$\{W_i(s)\}_{i=1}^{31}$  – functions of energies and scattering angles

Nice mathematical properties and fast numerical implementation for the planar sector [Gehrmann, Henn, Lo Presti '18]

# Iterated integrals in terms of familiar functions

- One-fold integrals: Logarithms, e.g.

$$\log(s_{12}), \log(-s_{23}), \dots$$

$$s_{ij} \equiv (p_i + p_j)^2$$

- Two-fold integrals: Dilogarithms, e.g.

$$\text{Li}_2\left(1 - \frac{s_{34}}{s_{12}}\right), \log(-s_{13}) \log(s_{34}), \dots$$

- Multi-fold integrals: Goncharov polylogarithms
  - ◊ Well-studied functions in math and HEP
  - ◊ Fast numerical routines

# Complicated integrals from simple differential equations

Change of master integral basis  $\vec{f}$  enormously simplifies DE [Henn '13]

$$d\vec{f}(s, \epsilon) = \epsilon d\tilde{A}(s) \vec{f}(s, \epsilon)$$

$$d\tilde{A}(s) = \sum_{i=1}^{31} a_i d \log W_i(s)$$

letters of the alphabet

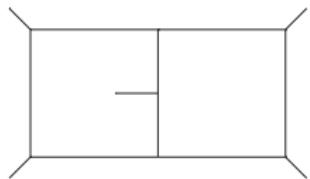
rational matrices

Elegant solution

$$\vec{f}(s, \epsilon) = \text{Pexp} \left( \epsilon \int_{\gamma} d\tilde{A}(s) \right) \vec{f}(s_0, \epsilon) \implies \text{Pentagon functions}$$

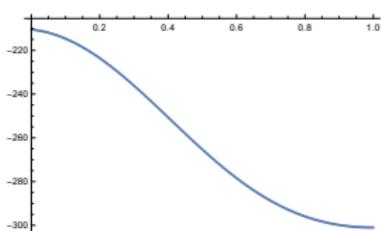
- ✓ Algorithmic construction of the canonical basis
- ✓ Absence of spurious singularities  $\implies$  boundary constants  $\vec{f}(s_0, \epsilon)$

# From analytic formulae to numeric values



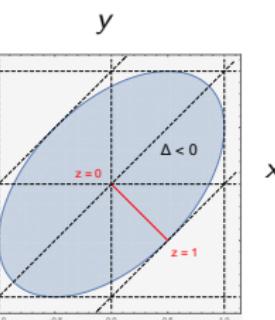
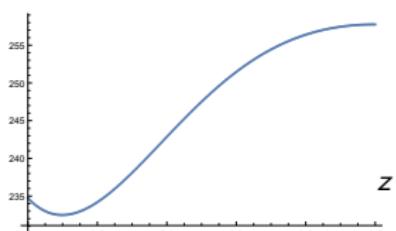
$\text{Re } f_{101}$

$z$



$\text{Im } f_{101}$

$z$



Example:

Two-dimensional  
slice of the kinematic  
space

Reference point:

$$x = y = 0$$

Integration path:

$$x = -y = \frac{z}{1 + z^2}$$

and  $0 < z < 1$

# Dramatic recent progress in calculation of the five-particle two-loop amplitudes

- All QCD amplitudes in the planar limit are known analytically [Abreu, Dormans, Febres Cordero, Ita, Page '18] Previous numerical [Badger, Brønnum-Hansen, Hartanto, Peraro '17][Abreu, Cordero, Ita, Page, Zeng '17] [Abreu, Cordero, Ita, Page, Sotnikov '18][Badger, Brønnum-Hansen, Gehrmann, Hartanto, Henn, Lo Presti, Peraro '18] and analytical results [Gehrmann, Henn, Lo Presti '15][Dunbar, Perkins '16] [Badger, Brønnum-Hansen, Hartanto, Peraro '18] in the planar approximation
- Full-color  $\mathcal{N} = 4$  super-Yang-Mills and  $\mathcal{N} = 8$  supergravity amplitudes (at symbol level) [D.C., Gehrmann, Henn, Wasser, Zhang, Zoia '18 '19][Abreu, Dixon, Herrmann, Page, Zeng '18 '19]
- Full-color five-gluon all-plus helicity amplitude [Badger, D.C., Gehrmann, Heinrich, Henn, Peraro, Wasser, Zhang, Zoia '19]  
⇒ Very first complete analytic two-loop five-particle amplitude!

# First Result for a Full Two-Loop Five-Gluon Amplitude

Based on arXiv:2303.03733

**Towards an Era of Precision Measurements at the LHC**

A new era of precision measurements at the LHC is dawning. CERN's Large Hadron Collider (LHC) will start Run 3 in 2021. To fully exploit the machine's potential, accurate theoretical predictions are required. These can be obtained through the computation of higher orders in perturbation theory.

**For Many QCD Processes, Next-To-Leading Order is Insufficient**

E.g. strong coupling from 3-jet/2-jet ratio [1]:  
 $\alpha_s / M_Z = 0.1148 \pm 0.0014 \pm 0.0018 \pm 0.0050$   
 $(\text{stat}) \quad (\text{syst}) \quad (\text{lum})$

Large theoretical uncertainty!

**Five-Particle Processes at the LHC**

- Precision measurements of the strong coupling
- In-depth tests of the Standard Model
- Improved background to new physics searches

**Bottleneck:** two-loop five-particle amplitudes



**New Result: Full-Color All-Plus Helicity Amplitude**

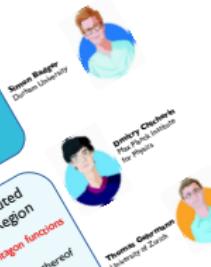
We find a remarkably simple expression for the infrared subtracted hard function:

$$\text{Permutations of the external legs} \quad \text{Color-SU}(N) \quad \text{Gluon spin dimension}$$

$$K_{\text{trace}}^{(2)} = \sum_{\text{partitions}} \text{Tr}(12)(\text{Tr}(345) - \text{Tr}(543)) \prod_{i=1}^n \frac{[(24)(14)(23)]}{[(12)(23)(34)(45)]} \prod_{j=1}^m \frac{[(24)(14)(23)]}{[(12)(34)(45)]}$$

Partitions appear:  
 $x_1 x_2 \rightarrow x_1 + x_2$   
 $x_1 x_2 x_3 \rightarrow x_1 + x_2 + x_3$   
 $\dots$

Conformally invariant



**All Feynman Integrals Computed Analytically in the Physical Region**

Base for all QCD amplitudes → **pentagon functions**  
 $f(p_1, p_2, p_3, p_4, p_5)$   
 (pentagon, dlogarithms, and generalizations thereof)  
 Logarithms, dilogarithms,  
 Goncharov polylogarithms,  
 and analytic properties

- > Elegant and accurate numerical evaluation

**A Refined Amplitude Assembly**

Integration-by-parts reduction → Finite fields

Feynman integrand → Infrared subtraction → Finite field function

Master integrals → Leading singularities

The code can be parallelized, and runs in a few hours using multi-threading on a modern computing node.

**Cutting-Edge Methods**

- > Generalized unitarity for Feynman integrands [4]
- > Differential equations for Feynman integrals in the canonical form [2-4]
- > Algorithmic construction of canonical basis using D-dimensional leading singularities [6]
- > Computational algebraic geometry [7]
- > **Finite fields** and functional reconstruction [8]



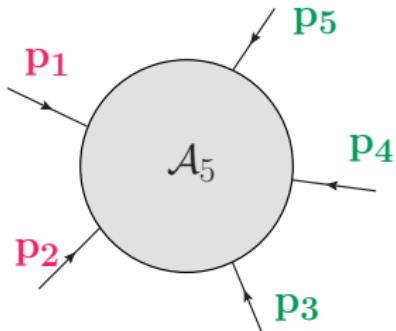
# Want to learn more? Have a look at my poster!

# Summary

- Pentagon functions describe all massless five-particle two-loop amplitudes
- All master integrals for 3-jet production at NNLO are known analytically
- First analytic results for full-color five-particle two-loop amplitudes
- Many more five-particle two-loop amplitudes become available

# Backup slides

# Kinematics of the five-particle scattering



Physical scattering region  $\mathbf{12} \rightarrow \mathbf{345}$

$$s_{12}, s_{34}, s_{45}, s_{35} > 0$$

$$s_{13}, s_{14}, s_{15}, s_{23}, s_{24}, s_{25} < 0$$

Massless particles:  $p_i^2 = 0$   $(\epsilon_5)^2 < 0$

Mandelstam invariants:  $s_{ij} = (p_i + p_j)^2$

Five independent:  $s_{12}, s_{23}, s_{34}, s_{45}, s_{15}$

One pseudo-scalar:  $\epsilon_5 \equiv i \epsilon_{\mu_1 \mu_2 \mu_3 \mu_4} p_1^{\mu_1} p_2^{\mu_2} p_3^{\mu_3} p_4^{\mu_4}$

- Chen iterated integral along path  $\gamma$

$$I(\gamma) \equiv \int_{\gamma} \omega_n \circ \dots \circ \omega_2 \circ \omega_1 = \int_0^1 dt_1 f_1(t_1) \int_0^{t_1} dt_2 f_2(t_2) \int_0^{t_2} dt_3 f_3(t_3) \dots \int_0^{t_{n-1}} dt_n f_n(t_n)$$

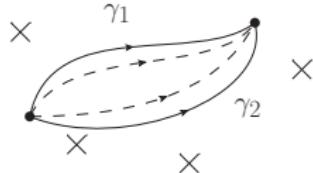
$$\gamma^*(\omega_i) = f_i(t_i)dt_i$$

$\omega_i$  – differential 1-form



- d-log forms  $\omega_i = d \log \alpha_i$
- Alphabet  $\mathbb{A} = \{\alpha_i\}$
- transcendental weight = number of integrations
- homotopy invariance of the iterated integral  $\gamma_1 \sim \gamma_2 \Rightarrow I(\gamma_1) = I(\gamma_2)$  if

$$\omega_i \wedge \omega_{i+1} = 0$$



## Example. Dilogarithm. One variable

$$\text{Li}_2(x) = - \int_{\gamma} d \log(1-x) \circ d \log(x) = - \int_0^1 \frac{dt}{t} \int_0^t \frac{x dt'}{1-xt'}$$

## Example. Multiple polylogarithms. Several variables

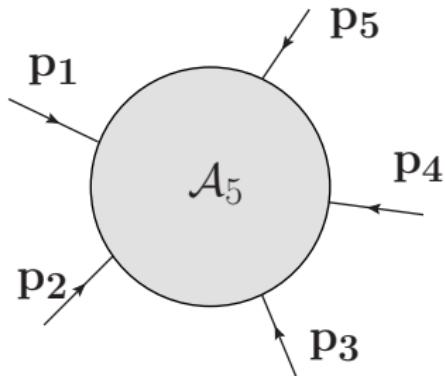
$$\text{Li}_{1,1}(x, y) \equiv \sum_{0 < k_1 < k_2} \frac{x^{k_1} y^{k_2}}{k_1 k_2}$$

$$\text{Li}_{1,1}(x, y) = \int_{\gamma} d \log(1-xy) \circ d \log \left( \frac{y(1-x)}{1-y} \right) + d \log(1-x) \circ d \log(1-y)$$

# Pentagon alphabet

31-letter alphabet  $\mathbb{A} = \left\{ W_j(s) \right\}_{j=1}^{31}$

$W_1$	$2 p_1 \cdot p_2$	$+(4)$
$W_6$	$2 p_4 \cdot (p_3 + p_5)$	$+(4)$
$W_{11}$	$2 p_3 \cdot (p_4 + p_5)$	$+(4)$
$W_{16}$	$2 p_1 \cdot p_3$	$+(4)$
$W_{21}$	$2 p_3 \cdot (p_1 + p_4)$	$+(4)$
$W_{26}$	$\begin{matrix} \langle 12 \rangle [24] \langle 45 \rangle [51] \\ [12] \langle 24 \rangle [45] \langle 51 \rangle \end{matrix}$	$+(4)$
$W_{31}$	$\epsilon_5$	



- The alphabet splits into orbits of  $\mathbb{Z}_5$
- Invariance under  $S_5$
- 26 parity-even and 5 parity-odd letters
- Zero locus of letters: branch points of the master integrals

# Differential equations

$\partial_s g_i(s, \epsilon) = \text{linear combination of } \mathcal{I}_{a_1, \dots, a_{11}} \stackrel{\text{IBP}}{=} \text{linear combination of } g_k$

[Kotikov '91][Bern, Dixon, Kosower '93][Gehrman, Remiddi '99]

$$d\vec{g}(s, \epsilon) = [dA(s, \epsilon)] \vec{g}(s, \epsilon)$$

$$g(s, \epsilon) = \sum_{p=0}^{\infty} \frac{1}{\epsilon^{4-p}} \sum_k r_k(s) \sum_{q=0}^p h_{k,p}^{(q)}(s) \xrightarrow{\text{basis change}} f(s, \epsilon) = \sum_{q=0}^{\infty} \frac{1}{\epsilon^{4-q}} h^{(q)}(s)$$

rational/algebraic    q-fold iterated integral

Canonical form of the differential equation

[Henn '13]

$$d\vec{f}(s, \epsilon) = [\epsilon d\tilde{A}(s)] \vec{f}(s, \epsilon)$$

where the matrix contains letters of the pentagon alphabet

$$d\tilde{A}(s) = \sum_{i=1}^{31} a_i d \log W_i(s)$$

numerical matrices

# Structure of the two-loop five-particle amplitude

Color decomposition (single and double trace)

$$\begin{aligned} [\mathcal{A}^{(2)}]^{a_1 a_2 a_3 a_4 a_5} = & N_c^2 [\text{Tr}(T^{a_1} T^{a_2} T^{a_3} T^{a_4} T^{a_5}) - \text{Tr}(T^{a_1} T^{a_5} T^{a_4} T^{a_3} T^{a_2})] \mathcal{A}_{12345}^{(2)} \\ & + N_c \text{Tr}(T^{a_1} T^{a_2}) [\text{Tr}(T^{a_3} T^{a_4} T^{a_5}) - \text{Tr}(T^{a_5} T^{a_4} T^{a_3})] \mathcal{A}_{12;345}^{(2)} \\ & + N_c^0 \times \text{single trace} + \text{permutations} \end{aligned}$$

known ↘  
new ↗

Universal factorization of the infrared divergences

$$\mathcal{A} = \mathcal{Z} \cdot \mathcal{A}^f$$

finite ↙

The finite hard function

$$\mathcal{H} = \lim_{\epsilon \rightarrow 0} \mathcal{A}^f$$

- Truly new piece of information
- Relevant for cross sections

# Double-trace two-loop all-plus helicity hard function

$$\begin{aligned} \mathcal{H}_{\substack{\text{double} \\ \text{trace}}}^{(2)} = & \sum_{\mathcal{S}_5/\Sigma} \text{Tr}(12) [\text{Tr}(345) - \text{Tr}(543)] \sum_{\Sigma} \left\{ 6\kappa^2 \left[ \frac{\langle 24 \rangle [14][23]}{\langle 12 \rangle \langle 23 \rangle \langle 45 \rangle^2} + 9 \frac{\langle 24 \rangle [12][23]}{\langle 12 \rangle \langle 34 \rangle \langle 45 \rangle^2} \right. \right. \\ & \left. \left. + \kappa \frac{[15]^2}{\langle 23 \rangle \langle 34 \rangle \langle 42 \rangle} \left[ I_{234;15} + I_{243;15} - I_{324;15} - 4I_{345;12} - 4I_{354;12} - 4I_{435;12} \right] \right\} \right. \end{aligned}$$

Finite part of the one-mass box function:

$$I_{123;45} = \text{Li}_2 \left( 1 - \frac{s_{12}}{s_{45}} \right) + \text{Li}_2 \left( 1 - \frac{s_{23}}{s_{45}} \right) + \log^2 \left( \frac{s_{12}}{s_{23}} \right) + \frac{\pi^2}{6}$$

Spinor-helicity variables:  $\langle ij \rangle = \sqrt{s_{ij}} e^{i\varphi_{ij}}$  and  $[ij] = \sqrt{s_{ij}} e^{-i\varphi_{ij}}$

Gluon spin dimension:  $\kappa \equiv \frac{g^\mu{}_\mu - 2}{6}$