Integrand Reduction for Two-Loop Five-Point Amplitudes

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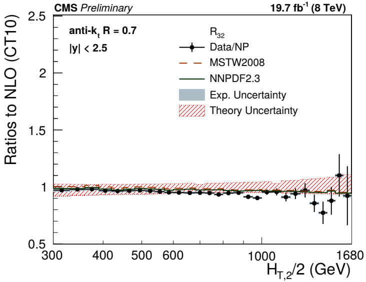
NNLO frontier: 2 to 3 scattering

- $pp \to jjj$: $R_{3/2}$, $m_{jjj} \Rightarrow \alpha_s$ determination at multi-TeV range

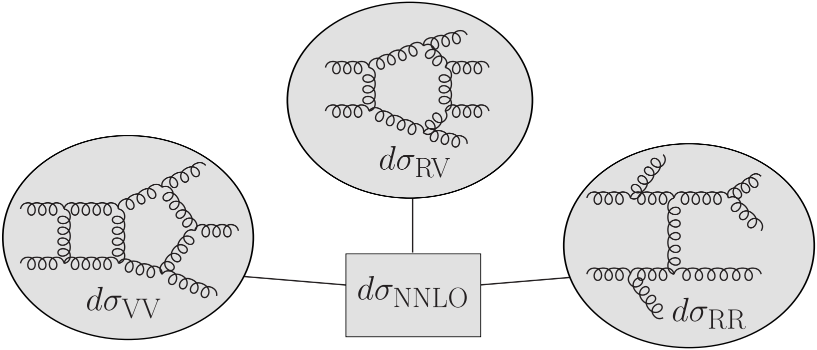
$R_{3/2} \sim \alpha_s \to$ cancellation of uncertainties



$$\alpha_s(M_Z) = 01150 \pm 0.0010(\text{exp}) \pm 0.0013(\text{PDF}) \pm 0.0015(\text{NP}) {}^{+0.0050}_{-0.0000}(\text{scale})$$

[CMS-PAS-SMP-16-008]

- $pp \to \gamma\gamma j$: background to Higgs p_T, signal/background interference effects
- $pp \to Hjj$: Higgs p_T, background to VBF (probes Higgs coupling)
- $pp \to Vjj$: Vector boson p_T, W^+/W^- ratios, multiplicity scaling
- $pp \to VVj$: background for new physics



q_T subtraction N-jettiness subtraction projection to Born

antenna subtraction CoLoRFulNNLO STRIPPER

Nested Soft-Collinear Subtraction geometric subtraction

+ ...

Why numerical approach?

Multi-scale process ⇒ algebraic and analytic complexities
talk by Dmitry

bottleneck: large intermediate expressions but simple final results

Numerical calculation ⇒ many 1-loop calculation with high multiplicity
(Njet, BlackHat, GoSam, ...)

What kind of numerical evaluation?

- floating-point evaluation ($x = 4.744955523489933 \times 10^6$)
  ✓ fast ✗ limited precision
- evaluation over rational field $\mathcal{Q}$ ($x = 706998373/149$)
  ✓ exact ✗ can be slow and expensive
- evaluation over finite fields $\mathcal{Z}_p$ ($x \bmod_{11} = 8$)
  $\mathcal{Z}_p$ ⇒ the field of integer numbers modulo a prime p
  ✓ exact+fast ✗ some information lost
  → need to reconstruct $\mathcal{Q}$ over several finite fields

Strategy ⇒ reconstruct analytic expressions from finite-field evaluations [Peraro 2016]

Analytics results: fast and stable for pheno applications

Computational Framework

Colour ordered amplitude:

$$\mathcal{A}^{(2)}(\{p\}) = \int [dk_1][dk_2] \frac{\mathcal{N}(\{k\},\{p\})}{\mathcal{D}_1 \cdots \mathcal{D}_n}$$

Integrand Reduction: construct irreducible numerators $\Delta_i(\{k\},\{p\})$
[Ossola,Papadopoulos,Pittau,Mastrolia,Badger,Frellesvig,Zhang,Peraro,Mirabella, ...]

$$\frac{\mathcal{N}(\{k\},\{p\})}{\mathcal{D}_1 \cdots \mathcal{D}_n} = \sum_{i \in T} \frac{\Delta_i(\{k\},\{p\})}{(\text{propagators})_i}$$

- $\mathcal{N}(\{k\},\{p\})$: process dependent numerator function

  ⇒ generalized unitarity cuts → product of tree amplitudes
  [Bern,Dixon,Dunbar,Kosower,1994;Britto,Cachazo,Feng,2004;Ellis,Giele,Kunszt,Melnikov,2007-2008; ...]

  ⇒ Feynman diagram input (`QGRAF` [Nogueira])

- Integrand parameterisation

$$\Delta_i(\{k\},\{p\}) = \sum_j c_{i,j}(\{p\})\, \mathbf{m}_j(\{k\})$$

- Fit coefficients $c_{i,j}(\{p\})$ on multiple cuts $\{\mathcal{D}_i = 0\}_{i \in T}$
  ⇒ solve linear system of equation

$$\mathcal{A}^{(2)}(\{p\}) = \int [dk_1][dk_2] \sum_{i \in T} \frac{\sum_j c_{i,j}(\{p\})\, \mathbf{m}_j(\{k\})}{(\text{propagators})_i} = \sum_i \tilde{c}_i G_i \underset{\text{IBP}}{=} \sum_i d_i \mathrm{MI}_i$$

- **IBP identities:** relations between integrals G_i → reduce to independent set of integrals → master integrals (MI_i)

$$\int [dk] \frac{\partial}{\partial k_\mu} \frac{v_\mu(k,p)}{(\text{propagators})} = 0$$

[Chetyrkin,Tkachov]

- allows for extraction of the coefficients of master integrals (d_i)

$$d_i = \underbrace{\begin{pmatrix} & \ldots & \end{pmatrix}}_{\text{all integrand coefficients}} \underbrace{\begin{pmatrix} \ddots & \\ & \ddots \end{pmatrix}}_{\text{IBP reduction}}$$

- All steps evaluated numerically over finite fields within `FiniteFlow` framework [Peraro,2019]
- Use momentum twistor variables [Hodges]: rational PS parametrisation

$$\mathcal{A}^{(2)}(\{p\}) = \int [dk_1][dk_2] \sum_{i\in T} \frac{\sum_j c_{i,j}(\{p\})\, \mathbf{m}_j(\{k\})}{(\text{propagators})_i} = \sum_i \tilde{c}_i G_i \underset{\text{IBP}}{=} \sum_i d_i \text{MI}_i$$

- **IBP identities:** relations between integrals G_i $\rightarrow$ reduce to independent set of integrals $\rightarrow$ master integrals (MI_i)

$$\int [dk] \frac{\partial}{\partial k_\mu} \frac{v_\mu(k,p)}{(\text{propagators})} = 0$$

[Chetyrkin,Tkachov]

- allows for extraction of the coefficients of master integrals (d_i)
  or special functions (+ Laurent expansion in ϵ + pole subtraction)

$$d_i = \underbrace{\begin{pmatrix} & \cdots & \end{pmatrix}}_{\text{all integrand coefficients}} \underbrace{\begin{pmatrix} \ddots & \\ & \ddots \end{pmatrix}}_{\text{IBP reduction}} \underbrace{\begin{pmatrix} \ddots & \\ & \ddots \end{pmatrix}}_{\text{map MIs to special functions}} - \underbrace{\begin{pmatrix} & \cdots & \end{pmatrix}}_{\epsilon\text{-pole in special functions basis}}$$

- All steps evaluated numerically over finite fields within `FiniteFlow` framework [Peraro,2019] reconstruct analytic form of **finite remainder**
- Use momentum twistor variables [Hodges]: rational PS parametrisation

Two-loop five-point amplitudes

Five-point master integrals talk by Dmitry

[Papadopoulos,Tommasini,Wever 2015], [Gehrmann,Henn,Lo Presti 2015,2018], [Abreu,Page,Zeng 2018], [Chicherin,Gehrmann,Henn,Lo Presti,Mitev,Wasser 2018], [Abreu,Dixon,Herrmann,Page,Zeng 2018,2019] [Chicherin,Gehrmann,Henn,Wasser,Zhang,Zoia 2018,2019]

Numerical evaluation of two-loop five-point amplitudes in QCD:

- planar five-gluon [Badger,Brønnum-Hansen,HBH,Peraro 2017] [Abreu,Ita,Febres Cordero,Page,Zeng 2017]
- planar five-parton [Badger,Brønnum-Hansen,HBH,Peraro 2018] [Abreu,Ita,Febres Cordero,Page,Sotnikov 2018]
- planar $W+4$ parton [Badger,Brønnum-Hansen,HBH,Peraro arXiv:1906.11862]

Analytic form of two-loop five-point amplitudes in QCD

- planar five-gluon all-plus [Gehrmann,Henn,Lo Presti 2015]
- planar five-gluon single-minus [Badger,Brønnum-Hansen,HBH,Peraro arXiv:1811.11699]
- planar five-gluon MHV [Abreu,Dormans,Ita,Febres Cordero,Page 2018]
- planar five-parton MHV [Abreu,Dormans,Ita,Febres Cordero,Page,Sotnikov 2019]
- non-planar five-gluon all-plus [Badger,Chicherin,Gehrmann,Heinrich,Henn,Peraro,Wasser,Zhang,Zoia 2019]

Planar two-loop five-gluon amplitude: single minus

[Badger,Brønnum-Hansen,HBH,Peraro arXiv:1811.11699]

Unitarity cuts (6D tree amplitudes via Berends-Giele recursion) | integrand reduction

IBP reduction | finite-field sampling | analytic reconstruction

Master integrals in terms of **pentagon functions** f: $\mathrm{MI}_x = \sum_{y,z} c_{xyz} m_{xyz}(f)$

[Gehrmann,Henn,Lo Presti 2018]

Subtract IR poles, reconstruct finite remainder $\mathcal{F}$

$$\mathcal{A}^{(2)}_{-++++} = I^{(1)}\mathcal{A}^{(1)}_{-++++} + \mathcal{F}_{-++++}, \qquad \mathcal{A}^{(0)}_{-++++} = 0$$

Obtain compact expressions for $\mathcal{F}$ (decomposition in $(d_s-2)^i$, $d_s = g^\mu_\mu$)

$$F^{(2),[i]}\left(1^-,2^+,3^+,4^+,5^+\right) = \frac{[25]^2}{[12]\langle 23\rangle\langle 34\rangle\langle 45\rangle[51]}\left(F^{(2),[i]}_{\mathrm{sym}}(1,2,3,4,5) + F^{(2),[i]}_{\mathrm{sym}}(1,5,4,3,2)\right)$$

$$\begin{aligned} F^{(2),[1]}_{\mathrm{sym}}(1,2,3,4,5) = \; & c^{(2)}_{51}F^{(2)}_{\mathrm{box}}(s_{23},s_{34},s_{15}) + c^{(1)}_{51}F^{(1)}_{\mathrm{box}}(s_{23},s_{34},s_{15}) + c^{(0)}_{51}F^{(0)}_{\mathrm{box}}(s_{23},s_{34},s_{15}) \\ & + c^{(2)}_{34}F^{(2)}_{\mathrm{box}}(s_{12},s_{15},s_{34}) + c^{(1)}_{34}F^{(1)}_{\mathrm{box}}(s_{12},s_{15},s_{34}) + c^{(0)}_{34}F^{(0)}_{\mathrm{box}}(s_{12},s_{15},s_{34}) \\ & + c_{45}F^{(0)}_{\mathrm{box}}(s_{12},s_{23},s_{45}) + c_{34;51}\hat{L}_1(s_{34},s_{15}) + c_{51;23}\hat{L}_1(s_{15},s_{23}) + c_{\mathrm{rat}} \end{aligned}$$

- c: rational coefficients in s_{ij} and tr_+
- $F^{(i)}_{\mathrm{box}}$, $\hat{L}_i$: analytic function basis, free of spurious singularities [Bern,Dixon,Kosower]

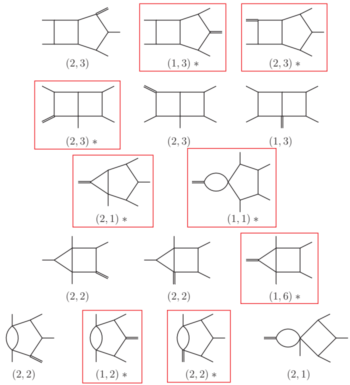
2-loop $W+4$ parton amplitudes

[Badger,Brønnum-Hansen,HBH,Peraro arXiv:1906.11862]

Leading colour $q\bar{Q}Q\bar{q}'\bar{\nu}\ell$ and $qgg\bar{q}'\bar{\nu}\ell$ amplitudes

$$\mathcal{A}^{(2)}(1_q,2_{\bar{Q}},3_Q,4_{\bar{q}'},5_{\bar{\nu}},6_\ell) \sim g_s^6 g_W^2\ N_c^2\ \delta_{i_1}^{\ \bar{i}_2}\delta_{i_3}^{\ \bar{i}_4}\ A^{(2)}(1_q,2_{\bar{Q}},3_Q,4_{\bar{q}'},5_{\bar{\nu}},6_\ell)$$

$$\mathcal{A}^{(2)}(1_q,2_g,3_g,4_{\bar{q}'},5_{\bar{\nu}},6_\ell) \sim g_s^6 g_W^2 \left[N_c^2\,(\,T^{a_2}T^{a_3})_{i_1}^{\ \bar{i}_4}\ A^{(2)}(1_q,2_g,3_g,4_{\bar{q}'},5_{\bar{\nu}},6_\ell) + (2\leftrightarrow 3)\right]$$



- Feynman diagrams + integrand reduction + IBP relations
- Coefficients of master integrals (MIs) are computed numerically over finite fields
- Use mom. twistor variables: $x_1, \cdots, x_8$
- only some of MIs known analytically
  [Papadopoulos,Tomassini,Wever 2015]
  [Gehrmann,von Manteuffel,Tancredi 2015]
  [Henn,Melnikov,Smirnov 2014]
  [Gehrman,Remiddi 2000]
- unknown MIs are evaluated numerically using `pySecDec`/`Fiesta`
  $\Rightarrow$ change MI basis to improve accuracy

2-loop $W+4$ parton amplitudes

Numerical benchmark: Euclidean phase-space point

$$x_1 = -1, \quad x_2 = \frac{79}{270}, \quad x_3 = \frac{64}{61}, \quad x_4 = -\frac{37}{78}, \quad x_5 = \frac{83}{102}, \quad x_6 = \frac{4723}{9207}, \quad x_7 = -\frac{12086}{7451}, \quad x_8 = \frac{3226}{2287}.$$

$qgg\bar{q}'\bar{\nu}\ell$	ϵ^{-4}	ϵ^{-3}	ϵ^{-2}	ϵ^{-1}	ϵ^{0}
$\widehat{A}^{(2)}_{-++++-}$	4.50000	-3.63577(3)	-277.2182(7)	-344.56(1)	2051.1(2)
$P^{(2)}_{-++++-}$	4.5	-3.63576	-277.2186	-344.569(6)	—
$\widehat{A}^{(2)}_{-+-++-}$	4.50000	-3.63581(9)	-13.6826(2)	6.143(5)	66.21(7)
$P^{(2)}_{-+-++-}$	4.5	-3.63576	-13.6824	6.145(1)	—
$\widehat{A}^{(2)}_{--+++-}$	4.50000	-3.63579(5)	-18.79219(7)	-6.633(6)	79.02(4)
$P^{(2)}_{--+++-}$	4.5	-3.63576	-18.79212	-6.6303(5)	—

$q\bar{Q}Q\bar{q}'\bar{\nu}\ell$	ϵ^{-4}	ϵ^{-3}	ϵ^{-2}	ϵ^{-1}	ϵ^{0}
$\widehat{A}^{(2)}_{-+-++-}$	2.00000	-7.16949(9)	-9.9055(2)	39.922(6)	154.79(7)
$P^{(2)}_{--+++-}$	2	-7.16944	-9.9054	39.9245(8)	—
$\widehat{A}^{(2)}_{--+++-}$	2.00000	-7.16948(8)	-12.9371(1)	41.432(8)	189.53(6)
$P^{(2)}_{--+++-}$	2	-7.16944	-12.9370	41.4353(6)	—

Comparison against universal pole structures in t'Hooft-Veltman scheme

Conclusions

✓ Numerical framework to compute two-loop five-point amplitudes
$\Rightarrow$ integrand reduction + IBP relations
$\Rightarrow$ numerical evaluation over finite field

✓ Analytical reconstruction from finite field sampling
$\Rightarrow$ avoid large intermediate expressions encountered in traditional approach

✗ More processes to explore

✗ From amplitudes to NNLO cross sections?

Conclusions

- ✓ Numerical framework to compute two-loop five-point amplitudes
  - $\Rightarrow$ integrand reduction + IBP relations
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  - $\Rightarrow$ avoid large intermediate expressions encountered in traditional approach
- ✗ More processes to explore
- ✗ From amplitudes to NNLO cross sections?

THANK YOU!!!

Back-up Slides

Two-loop five-gluon: single minus

$$F^{(-1)}_{\text{box}}(s,t,m^2) = \mathrm{Li}_2\left(1-\frac{s}{m^2}\right) + \mathrm{Li}_2\left(1-\frac{t}{m^2}\right) + \log\left(\frac{s}{m^2}\right) + \log\left(\frac{t}{m^2}\right) - \frac{\pi^2}{6},$$

$$F^{(0)}_{\text{box}}(s,t,m^2) = \frac{1}{u(s,t,m^2)} F^{(-1)}_{\text{box}}(s,t,m^2),$$

$$F^{(1)}_{\text{box}}(s,t,m^2) = \frac{1}{u(s,t,m^2)}\left[F^{(0)}_{\text{box}}(s,t,m^2) + \hat{L}_1(s,m^2) + \hat{L}_1(m^2,t)\right],$$

$$F^{(2)}_{\text{box}}(s,t,m^2) = \frac{1}{u(s,t,m^2)}\Bigg[F^{(1)}_{\text{box}}(s,t,m^2) + \frac{s-m^2}{2t}\hat{L}_2(s,m^2) + \frac{m^2-t}{2s}\hat{L}_2(m^2,t) - \left(\frac{1}{s}+\frac{1}{t}\right)\frac{1}{4m^2}\Bigg],$$

$$u(s,t,m^2) = m^2 - s - t$$

$$L_k(s,t) = \frac{\log(t/s)}{(s-t)^k}$$

$$\hat{L}_0(s,t) = L_0(s,t), \qquad \hat{L}_1(s,t) = L_1(s,t),$$

$$\hat{L}_2(s,t) = L_2(s,t) + \frac{1}{2(s-t)}\left(\frac{1}{s}+\frac{1}{t}\right), \qquad \hat{L}_3(s,t) = L_3(s,t) + \frac{1}{2(s-t)^2}\left(\frac{1}{s}+\frac{1}{t}\right).$$