ANALYTICAL RESULTS FOR HADRONIC CONTRIBUTIONS TO THE MUON $g-2$

Wikipedia/Michiel Verbeek

Johan Bijnens

Lund University

bijnens@thep.lu.se
http://thep.lu.se/~bijnens

Vetenskapsrådet

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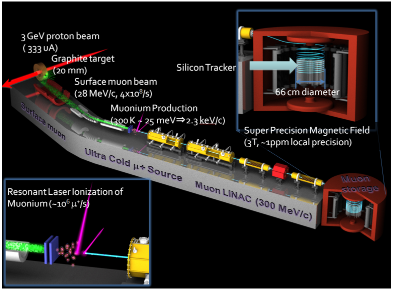
Why do we do this?

The muon $a_\mu = \dfrac{g_\mu - 2}{2}$ will be measured more precisely



J-PARC

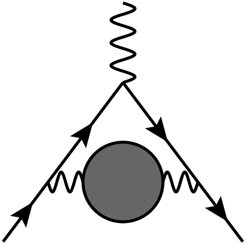
Fermilab

Why do we do this?

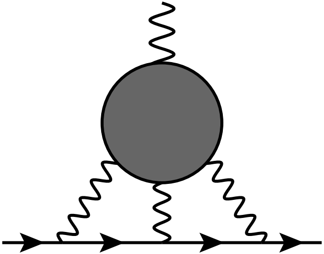
- Experiment dominated by BNL, FNAL error down by four
- Theory taken from PDG2018
- $a_\mu^{\text{SM}} = a_\mu^{\text{QED}} + a_\mu^{\text{EW}} + a_\mu^{\text{Had}}$
- $a_\mu^{\text{Had}} = a_\mu^{\text{LO-HVP}} + a_\mu^{\text{HO-HVP}} + a_\mu^{\text{HLbL}}$
- Impressive agreement with g_μ to 2×10^{-9}

Part	value	errors	units
a_μ^{EXP}:	116 592 091.x	(54)(33)	$\times 10^{-11}$
a_μ^{SM}:	116 591 823.x	(1)(34)(26)	$\times 10^{-11}$
Δa_μ:	268.x	(63)(43)	$\times 10^{-11}$
a_μ^{QED}:	116 584 719.0	(0.1)	$\times 10^{-11}$
a_μ^{EW} :	153.6	(1.0)	$\times 10^{-11}$
$a_\mu^{\text{LO-HVP}}$:	6 931.x	(33)(7)	$\times 10^{-11}$
$a_\mu^{\text{HO-HVP}}$:	-86.3	(0.9)	$\times 10^{-11}$
a_μ^{HLbL} :	105.x	(26)	$\times 10^{-11}$

Hadronic contributions



LO-HVP



HLbL

- The blobs are hadronic contributions
- I will present some results that are useful for LO-HVP and HLbL

My recent work related to the muon $g-2$

[1] J. Bijnens and J. Relefors, "Pion light-by-light contributions to the muon $g-2$," JHEP **1609** (2016) 113 [arXiv:1608.01454 [hep-ph]]

[2] J. Bijnens, N. Hermansson-Truedsson and A. Rodriguez-Sanchez, short-distance constraints on HLbL, work in progress.

[3] J. Bijnens and J. Relefors, "Connected, Disconnected and Strange Quark Contributions to HVP," JHEP **1611** (2016) 086 [1609.01573 [hep-lat]].

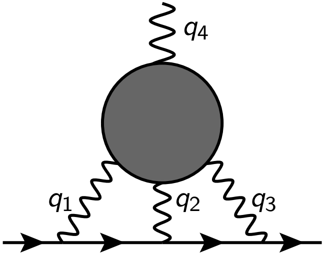
[4] J. Bijnens and J. Relefors, "Vector two-point functions in finite volume using partially quenched chiral perturbation theory at two loops," JHEP **1712** (2017) 114 [arXiv:1710.04479 [hep-lat]].

[5] J. Bijnens, J. Harrison, N. Hermansson-Truedsson, T. Janowski, A. Jüttner and A. Portelli, "Electromagnetic finite-size effects to the hadronic vacuum polarization," arXiv:1903.10591 [hep-lat].

[1,2,3]: HLbl, [4,5]: HVP

I will concentrate on [2,5]

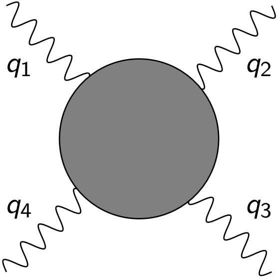
HLbL: the main object to calculate



- Muon line and photons: well known
- The blob: fill in with hadrons/QCD
- Trouble: low and high energy very mixed
- Double counting needs to be avoided: hadron exchanges versus quarks

General properties

$$\Pi^{\mu\nu\lambda\sigma}(q_1, q_2, q_3) \quad = $$



Actually we really need $\left.\dfrac{\delta\Pi^{\mu\nu\lambda\sigma}(q_1, q_2, q_3)}{\delta q_{4\rho}}\right|_{q_4=0}$

General properties

$\Pi^{\mu\nu\lambda\sigma}(q_1, q_2, q_3)$:

- In general 138 Lorentz structures (136 in 4 dimensions)
- Using $q_{1\mu}\Pi^{\mu\nu\lambda\sigma} = q_{2\nu}\Pi^{\mu\nu\lambda\sigma} = q_{3\lambda}\Pi^{\mu\nu\lambda\sigma} = q_{4\sigma}\Pi^{\mu\nu\lambda\sigma} = 0$
  43 (41) gauge invariant structures
- 41 helicity amplitudes
- Bose symmetry relates some of them
- Compare HVP: one function, one variable
- General calculation from experiment via dispersion relations: recent progress

  Colangelo, Hoferichter, Kubis, Procura, Stoffer,...
- Well defined separation between different contributions
- Theory initiative: paper under preparation
- One remaining problem: intermediate- and short-distances

Short-distance

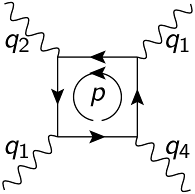
- Use (constituent) quark loop
- Used for full estimates since the beginning (1970s)
- Used for short-distance estimates
  JB, Pallante, Prades, 1996
- Is it a first term in a systematic OPE?
- OPE has been used as constraints on specific contributions
  - $\pi^0\gamma^*\gamma^*$ asymptotic behaviour
  - Constraints on many other hadronic formfactors
  - $q_1^2 \approx q_2^2 \gg q_3^2$ Melnikhov, Vainshtein 2003

Short-distance: first attempt

$$\Pi^{\mu\nu\lambda\sigma} = -i \int d^4x d^4y d^4z e^{-i(q_1\cdot x + q_2 \cdot y + q_3 \cdot z)} \left\langle T\left(j^\mu(x) j^\nu(y) j^\lambda(z) j^\sigma(0)\right)\right\rangle$$

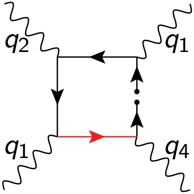
- Usual OPE: x, y, z all small
- First term in the expansion is the quark-loop
  no problem with $\partial/\partial q_4^\rho$ and $q_4 \to 0$

  p in loop $\Rightarrow$ no singular propagators:



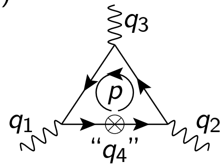
- Next term problems: no loop momentum;

  $q_4 \to 0$ propagator diverges:



Short-distance: correctly

- Similar problem in QCD sum rules for electromagnetic radii and magnetic moments
- Ioffe, Smilga, 1984
- For the q_4-leg use a constant background field and do the OPE in the presence of that constant background field
- Use radial gauge: $A_4^\lambda(w) = \frac{1}{2} w_\mu F^{\mu\lambda}$
  whole calculation is immediately with $q_4 = 0$.
- First term is exactly the usual quark loop
  (even including quark masses)

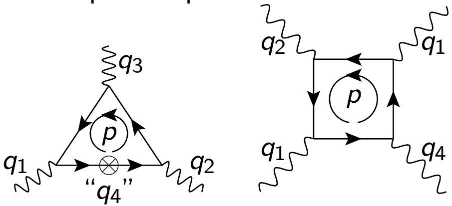


Short-distance: next term(s)

- Do the usual QCD sum rule expansion in terms of vacuum condensates
- There are new condensates, induced by the constant magnetic field: $\langle \bar{q}\sigma_{\alpha\beta} q\rangle \equiv e_q F_{\alpha\beta} X_q$
- Lattice QCD Bali et al., arXiv:1206.4205
  $X_u = 40.7 \pm 1.3$ MeV,
  $X_d = 39.4 \pm 1.4$ MeV,
  $X_s = 53.0 \pm 7.2$ MeV
- Could have started at order $1/Q$, only starts at $1/Q^2$ via $m_q X_q$ corrections to the leading quark-loop result
- X_q and m_q are very small, only a very small correction
- Next order: very many condensates contribute, work in progress

Short-distance

- Formalism of Colangelo et al., 1702.07347
- $a_\mu = \frac{2\alpha^3}{3\pi^2}\int_0^\infty dQ_1 dQ_2 Q_1^3 Q_2^3 \int_{-1}^{1} d\tau \sqrt{1-\tau^2} \sum_{i=1,12} \hat{T}_i \overline{\Pi}_i$
- The $\overline{\Pi}_i$ are related to 6 $\hat{\Pi}_i$
- Bare quarkloop derived from both



- In agreement with quarkloop formulae from Hoferichter, Stoffer, private communication

Short-distance: next term(s)

For $N_c = 3$ with $e_q = 1$ and one quark we get $(q_i^2 = -Q_i^2)$ (preliminary)

$$\hat{\Pi}_1 = m_q X_q \frac{2(q_3^2 - q_1^2 - q_2^2)}{q_1^2 q_2^2 q_3^4}, \qquad \hat{\Pi}_4 = m_q X_q \frac{4}{q_1^2 q_2^2 q_3^2},$$

$$\hat{\Pi}_7 = 0, \qquad \hat{\Pi}_{17} = m_q X_q \frac{-4}{q_1^2 q_2^2 q_3^4}$$

$$\hat{\Pi}_{39} = 0, \qquad \hat{\Pi}_{54} = m_q X_q \frac{2(q_1^2 - q_2^2)}{q_1^4 q_2^4 q_3^2}$$

Short-distance: numerical results

- preliminary
- $Q_1, Q_2, Q_3 \geq Q_{\text{min}}$
- $m_u = m_d = m_s = 0$ for quark-loop
- $m_u = m_d = 5$ MeV and $m_s = 100$ MeV for $m_q X_q$

Q_{min}	quarkloop	$m_u X_u + m_d X_d$	$m_s X_s$
1 GeV	17×10^{-11}	-2.7×10^{-13}	-4.1×10^{-13}
2 GeV	4.3×10^{-11}	-1.7×10^{-14}	-2.6×10^{-14}

- Above 1 GeV still 15% of total value of HLbL
- Quarkloop goes roughly as $1/Q^2_{\text{min}}$
- $m_q X_q$ goes roughly as $1/Q^4_{\text{min}}$
- Naive suppression is $m_q X_q / Q^2_{\text{min}} \sim 2 \times 10^{-3}$
- Observed 10^{-3}

Short-distance: Conclusions

- We have shown that the quarkloop really is the first term of a proper OPE expansion for the HLbL
- We have calculated the next term which is suppressed by quark masses and a small X_q: negligible
- The next term contains both the usual vacuum and a large number of induced condensates but will not be suppressed by small quark masses
- Why do this: matching of the sum over hadronic contributions to the expected short distance domain
- Finding the onset of the asymptotic domain

HVP: Summary of results

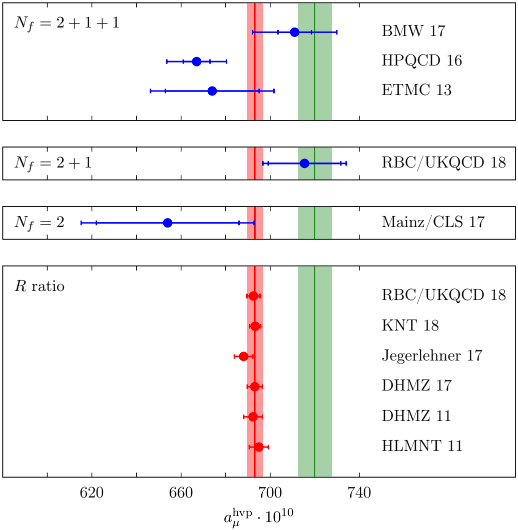


Figure from 1807.09370

Some discrepancies, lattice accuracies improve

Status lattice results

- Dispersive has reached below 0.5%
- Lattice accuracies a few %
- Improvement continuous
- Finite volume corrections known to two-loop order in ChPT [4]
- Next: isospin breaking
  - $m_u - m_d$: "easy"
  - Electromagnetism: possibly large finite volume corrections since $1/L^n$ rather than $\exp(-m_\pi L)$
- J. Bijnens, J. Harrison, N. Hermansson-Truedsson, T. Janowski, A. Jüttner and A. Portelli, "Electromagnetic finite-size effects to the hadronic vacuum polarization," arXiv:1903.10591 [hep-lat].

  Presented at $g-2$ meetings in KEK february 2018, Mainz June 2018

  They are expected to be small

Conventions

- Main object: $\boxed{\Pi^{\mu\nu}_{ab}(q) = i \int d^4x e^{iq\cdot x} \langle 0 | T(j^\mu_a(x) j^{\nu\dagger}_b(0)|0\rangle}$
- $j^\mu_{EM} = (2/3) j^\mu_U - (1/3) j^\mu_D - (1/3) j^\mu_S \qquad j^\mu_Q = \overline{q}\gamma^\mu q$
- Continuum/infinite volume:
  $\Pi^{\mu\nu}_{EM}(q) = \left(q^\mu q^\nu - q^2 g^{\mu\nu}\right) \Pi_{EM}(q^2)$
- Known positive weight functions v, w and $Q^2 = -q^2$:
  - $a_\mu = \int_{\text{threshold}}^\infty dq^2 w(q^2) \frac{1}{\pi} \mathrm{Im}\Pi_{EM}(q^2)$
  - $a_\mu = \int_0^\infty dQ^2 v(Q^2) \left(-\Pi(Q^2) + \Pi(0)\right)$
- Dispersion relation:

$$\Pi(q^2) = \Pi(0) + \frac{q^2}{\pi} \int_{\text{threshold}}^\infty ds \frac{1}{s(s-q^2)} \frac{1}{\pi} \mathrm{Im}\Pi(s)$$

First: Scalar QED

- Pion loops: finite volume effects suppressed by $e^{-m_\pi L}$ (if off-shell)
- Photon loops have suppression only by powers of $1/L$
- Dynamical photons: large finite volume effects possible
- Scalar QED in usual $\overline{MS}$ ($\mu^2_{ChPT} = \mu^2_{\overline{MS}} e$, $e = 2.71\ldots$)
- $\mathcal{L} = (\partial_\mu \Phi^* + ieA_\mu \Phi^*)(\partial_\mu \Phi - ieA_\mu \Phi) - m_0^2 \Phi^* \Phi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$ ($-\lambda (\phi^* \phi)^2$ not needed)

Finite volume

- Finite volume for photons started in earnest with
  N. Hayakawa, S. Uno, Prog. Theor. Phys. **120**(2008)413 [0804.2044]
- Get the explicit $1/L$ behaviour from one-loop integrals:
  Z. Davoudi, M. Savage, Phys. Rev. D90(2014)054503 [1402,6471]
  S. Borsyani et al, Science 347(2015)1452 [1406.4088]
- Problem for photon: $\int d^d k \frac{1}{k^2} \rightarrow \int dk^0 \sum_{\vec{k}} \frac{1}{(k^0)^2 - \vec{k}^2}$
  Singularity no longer has the k^{d-1} to soften it
- Solution QED_L: drop all modes with $\vec{k} = 0$ Hayakawa-Uno
- We extend the arguments of Davoudi-Savage to two-loop order, use QED_L and a lattice with infinite time extension
- We only calculate corrections suppressed by $1/L$ and powers, not exponentially suppressed contributions
- Below threshold so mesons (pions) are off-shell

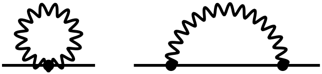
Integrals at finite volume

- $S = \frac{1}{i^2} \int \frac{d^d l}{(2\pi)^d} \frac{d^d k}{(2\pi)^d} \frac{1}{k^2(l^2-m^2)((k+l-p)^2-m^2))}$
- do l^0, k^0 integrals via contour integration
- $\vec{k} = \frac{2\pi}{L}\vec{n}$ and expand in $1/L$
- Write the $\vec{k}$ part as

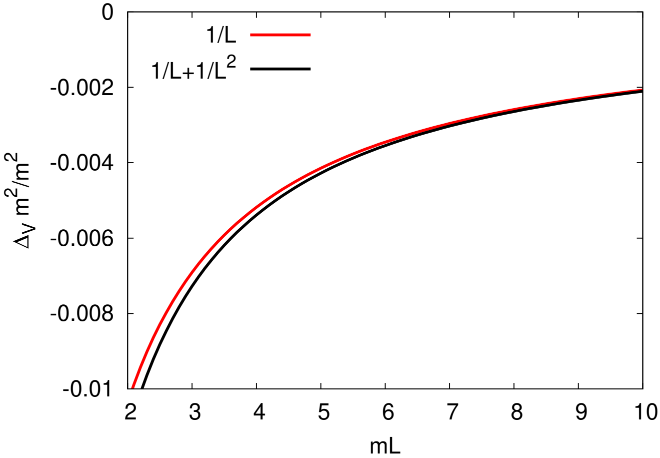
$$\frac{1}{L^{d-1}}\sum_{\vec{n}\neq\vec{0}} = \int \frac{d^{d-1}k}{(2\pi)^{d-1}} + \left[\frac{1}{L^{d-1}}\sum_{\vec{n}\neq\vec{0}} - \int \frac{d^{d-1}k}{(2\pi)^{d-1}}\right]$$

- In the first term resum the series in $1/L$: infinite volume contribution
- Call the quantity in brackets $\left(1/L^{d-1}\right)\Delta'_{\vec{n}}$
- Define $c_m = \Delta_{\vec{n}}\frac{1}{|\vec{n}|^m}$
- These are known numerically

Example: mass

- 
- $\frac{\Delta_V m^2}{m^2} = e^2 \left(\frac{4c_2}{16\pi^2 mL} + \frac{2c_1}{16\pi^2 m^2 L^2} + \mathcal{O}\left(\frac{1}{L^4}, e^{-mL}\right)\right)$
- Agrees with earlier results
- c_2 shows up since BOTH propagators can be 'on-shell'
- For $\Pi_{\mu\nu}$ below threshold only photon line goes 'on-shell'
  $\Longrightarrow$ corrections start only at $1/L^2$
- Only true if infinite volume mass is used in the expressions at LO

Numerical results mass



Infinite volume is $\dfrac{\delta m^2}{m^2} = 0.00852$

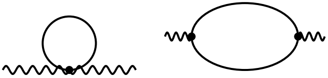
So very large finite volume corrections to electromagnetic part

Twopoint function

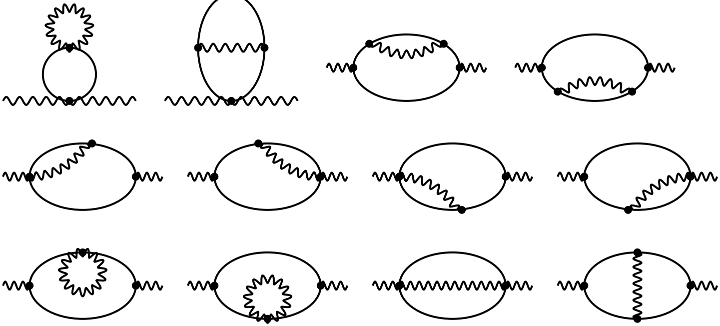
- Do the k^0, l^0 integral
- expand for large L as explained earlier
- Rest can be expressed in terms of
  $$Z_{ij}(m^2,p^2) = \int \frac{d^{d-1}l}{(2\pi)^{d-1}} \frac{1}{(\vec{l}^2+m^2)^{i/2}(4\vec{l}^2+4m^2-p^2)^j}$$
- these correspond to one-loop integrals with masses
- $\Omega_{ij} \equiv Z_{ij}(m^2,p^2)m^{i+2j-d+1}$
- Calculate in center of mass frame: $p=(p^0,\vec{0})$
- QED$_L$: no disconnected contribution
- $t_{\mu\nu}$ spatial part of $g_{\mu\nu}$
- $\widetilde{\Pi}((p^0)^2) \equiv \frac{-1}{3p^2} t_{\mu\nu}\left(\Pi^{\mu\nu}(p) - \Pi^{\mu\nu}(p=0)\right)$
- Infinite volume: $\widetilde{\Pi}(p^2) = \Pi(p^2)$

Diagrams

- Lowest order:



- NLO:



contributions from counterterms and 

Results

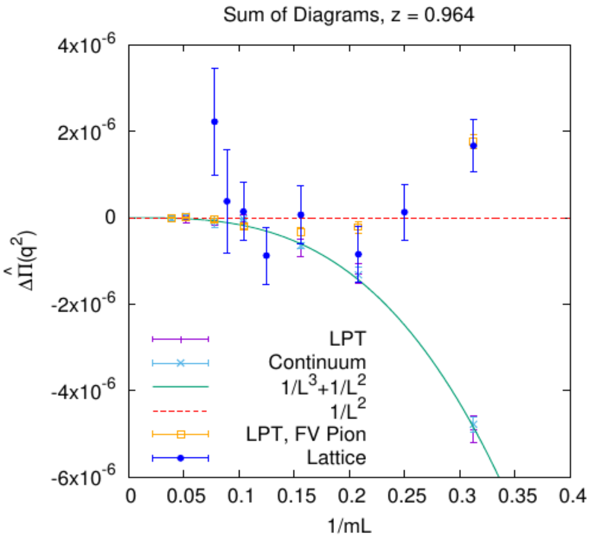
$$\widetilde{\Pi}(p^2) = \frac{c_1}{\pi m^2 L^2}\left(\frac{16}{3}\Omega_{-1,3} - \frac{1}{3}\Omega_{1,2} - \frac{32}{3}\Omega_{1,3} - \frac{2}{3}\Omega_{3,2} + \frac{16}{3}\Omega_{3,3} - \frac{1}{8}\Omega_{5,1} + \Omega_{5,2}\right)$$
$$+ \frac{c_0}{m^3 L^3}\Bigg(-\frac{128}{3}\Omega_{-2,4} + \frac{256}{3}\Omega_{0,4} - \frac{5}{3}\Omega_{2,2} + \frac{8}{3}\Omega_{2,3} - \frac{128}{3}\Omega_{2,4}$$
$$- \frac{3}{8}\Omega_{4,1} + \frac{7}{6}\Omega_{4,2} - \frac{8}{3}\Omega_{4,3}\Bigg) + \mathcal{O}\left(\frac{1}{L^4}, e^{-mL}\right)$$

Simplify using relations of the Ω_{ij} to

$$\widetilde{\Pi}(p^2) = + \frac{c_0}{m^3 L^3}\left(-\frac{16}{3}\Omega_{0,3} - \frac{5}{3}\Omega_{2,2} + \frac{40}{9}\Omega_{2,3} - \frac{3}{8}\Omega_{4,1} + \frac{7}{6}\Omega_{4,2} + \frac{8}{9}\Omega_{4,3}\right)$$
$$+ \mathcal{O}\left(\frac{1}{L^4}, e^{-mL}\right)$$

the $1/L^2$ cancels: expected: far away the photon sees no charge since it is a neutral current: only dipole effect

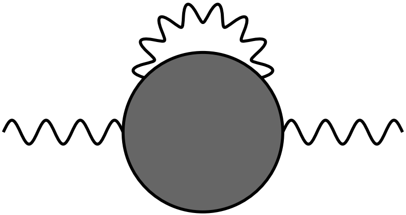
Scalar QED on a lattice



Correction for $mL \sim 5$ less than 1%

Generality

the $1/L^2$ cancels: expected: far away the photon sees no charge since it is a neutral current: only dipole effect



- The blob (hadrons) is a four-point function of electromagnetic currents and has no singularities in the regime needed
- The conclusion about $1/L^3$ is general

Conclusions HVP em finite volume

Showed you results for:

- Finite volume corrections to the electromagnetic contribution as estimated in scalar QED are small
- This is a universal feature: the object under study is neutral and contains no hadronic on-shell propagators
- At the level of precision needed for the next level: negligible