ANALYSIS OF SINGULARITIES AND THE 4D REPRESENTATION OF PHYSICAL OBSERVABLES WITHIN THE LTD FORMALISM

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Characterization of singularities with LTD
  - Location of IR singularities
  - Threshold singularities @ 1-loop

FDU approach @ NLO
  - Real-dual mappings
  - Applications (toy model & boson decays)

Conclusions and perspectives

Basic references for FDU/LTD:
**Introduction to Loop-tree duality**

**Dual representation of one-loop integrals**

**Loop Feynman integral**

\[
L^{(1)}(p_1, \ldots, p_N) = \int_{\ell} \prod_{i=1}^{N} G_F(q_i) = \int_{\ell} \prod_{i=1}^{N} \frac{1}{q_i^2 - m_i^2 + i0}
\]

**Dual integral**

\[
L^{(1)}(p_1, \ldots, p_N) = -\sum_{i=1}^{N} \int_{\ell} \tilde{\delta}(q_i) \prod_{j=1, j \neq i}^{N} G_D(q_i, q_j)
\]

**Sum of phase-space integrals!**

\[
G_D(q_i, q_j) = \frac{1}{q_j^2 - m_j^2 - i0\eta(q_j - q_i)}
\]

\[
\tilde{\delta}(q_i) = i2\pi \theta(q_i, 0) \delta(q_i^2 - m_i^2)
\]

See talks by William and Felix in this session!

Catani et al, JHEP09(2008)065; Rodrigo et al, JHEP02(2016)044
Characterization of singularities with LTD

**Location of IR singularities in the dual-space**

- Analize the dual integration region. It is obtained as the positive energy solution of the on-shell condition:

\[ G_F^{-1}(q_i) = q_i^2 - m_i^2 + i0 = 0 \]

\[ q_{i,0}^{(\pm)} = \pm \sqrt{q_i^2 + m_i^2 - i0} \]

- **Forward** (backward) on-shell hyperboloids associated with positive (negative) energy solutions.
- Degenerate to light-cones for massless propagators.
- Dual integrands become singular at intersections (two or more on-shell propagators)

Massive case: hyperboloids

Massless case: light-cones

Characterization of singularities with LTD

Location of IR singularities in the dual-space

- The application of LTD converts loop-integrals into PS ones: integration over forward light-cones.

- Only forward-backward interferences originate threshold or IR poles (other propagators become singular in the integration domain)

- Forward-forward singularities cancel among dual contributions

- Threshold and IR singularities associated with finite regions (i.e. contained in a compact region)

- No threshold or IR singularity at large loop momentum

- This structure suggests how to perform real-virtual combination! Also, how to overcome threshold singularities (integrable but numerically unstable)

Characterization of singularities with LTD

- The application of LTD converts loop-integrals into PS ones: *integration over forward light-cones.*

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Characterization of singularities with LTD

7 Description of threshold singularities @ 1-loop

- In general, the location of the singularities is given by the solutions of

$$\lambda_{ij}^{\pm\pm} = \pm q_{i,0}^{(+)} \pm q_{j,0}^{(+)} + k_{ji,0} = 0$$

with $q_i$ on-shell and $k_{ji} = q_j - q_i$.

- We consider the following test functions

$$S_{ij}^{(1)} = (2\pi\nu)^{-1} G_D(q_i; q_j) \tilde{\delta}(q_i) + (i \leftrightarrow j)$$

Up to 2 on-shell states
(standard thresholds)

$$S_{ijk}^{(1)} = (2\pi\nu)^{-1} G_D(q_i; q_k) G_D(q_i; q_j) \tilde{\delta}(q_i) + \text{perm.}$$

Up to 3 on-shell states
(anomalous thresholds)

- IMPORTANT: The singular structure of scattering amplitudes is dictated by their propagators. So, the proposed test functions are general enough to do a proper analysis of threshold singularities.

The singular structure depends on the separation among momenta:

- **Time-like separation (causal connection):**

  \[ k_{ji}^2 - (m_j + m_i)^2 \geq 0 \]

  Physical threshold singularities are originated.

  \[
  \lim_{\lambda_{ij}^{++} \to 0} S_{ij}^{(1)} = \frac{\theta(-k_{ji,0})\theta(k_{ji}^2 - (m_i + m_j)^2)}{x_{ij}(-\lambda_{ij}^{++} - i0k_{ji,0})} + O((\lambda_{ij}^{++})^0)
  \]

  \[
  x_{ij} = 4 q_{i,0}^{(+)} q_{j,0}^{(+)}
  \]

  The prescription is crucial to determine the imaginary part: it is always +i0 and corresponds to the usual Feynman prescription! For this configuration, LTD and FTT give equivalent descriptions!
Characterization of singularities with LTD

The singular structure depends on the separation among momenta:

- **Space-like separation:**
  
  \[ k_{ji}^2 - (m_j - m_i)^2 \leq 0 \]

  The dual-prescription changes sign within the different contributions, which allows a perfect cancellation of any singular behaviour.

  \[
  q_{j,0}^{(+)} G_D(q_i; q_j) \big|_{\lambda_{ij}^+ \to 0} = -q_{i,0}^{(+)} G_D(q_j; q_i) \big|_{\lambda_{ij}^+ \to 0}
  \]

  \[
  \lim_{\lambda_{ij}^+ \to 0} S_{ij}^{(1)} = O\left( (\lambda_{ij}^+)^0 \right)
  \]

  Cancellation codified by multiple-cuts in FTT!!

- **Light-like separation:**
  It originates IR and threshold singularities that remain in a compact region of the integration domain. There is a partial cancellation among dual contributions, but IR might remain!

Characterization of singularities with LTD

**Description of threshold singularities @ 1-loop**

- **Anomalous thresholds**: causal (i.e. time-like separated) singularities originated by multiple propagators going on-shell.

\[
\lim_{\lambda_{ij}^{++}, \lambda_{ik}^{++} \to 0} S_{ijk}^{(1)} = \frac{1}{x_{ijk}} \prod_{r=j,k} \theta(-k_{ri,0}) \theta(k_{ri}^2 - (m_i + m_r)^2) \frac{\theta(-\lambda_{ir}^{++} - i0k_{ri,0})}{(-\lambda_{ir}^{++} - i0k_{ri,0})} \\
+ \mathcal{O}\left( (\lambda_{ij}^{++})^{-1}, (\lambda_{ik}^{++})^{-1} \right)
\]

- Intersections of two hyperboloids lead to the standard IR and threshold singularities.

- Anomalous thresholds are originated from the intersection of two forward (backward) and one backward (forward) hyperboloids.

- There are not singularities for \( \lambda_{jk}^{-+} = \lambda_{ik}^{++} - \lambda_{ij}^{++} \to 0 \) !!!

Suppose one-loop scalar scattering amplitude given by the triangle (scalar toy-model!):

\[ |\mathcal{M}^{(0)}(p_1, p_2; p_3)\rangle = ig \]
\[ |\mathcal{M}^{(1)}(p_1, p_2; p_3)\rangle = -ig^3 L^{(1)}(p_1, p_2, -p_3) \]
\[ \Rightarrow \text{Re} \langle \mathcal{M}^{(0)}|\mathcal{M}^{(1)} \rangle \]

1->2 one-loop process  \[\rightarrow\]  1->3 with unresolved extra-parton

Add scalar tree-level contributions with one extra-particle; consider interference terms:

Generate 1->3 kinematics starting from 1->2 configuration plus the loop three-momentum \( \vec{l} \) !!!
Mapping of momenta: generate $1\rightarrow 3$ real emission kinematics ($3$ external on-shell momenta) starting from the variables available in the dual description of $1\rightarrow 2$ virtual contributions ($2$ external on-shell momenta and $1$ free three-momentum)

- Split the real phase space into two regions, i.e. $y'_{1r}<y'_{2r}$ and $y'_{2r}<y'_{1r}$, to separate the possible collinear singularities
- Implement an optimized mapping in each region, to allow a fully local cancellation of IR singularities with those present in the dual terms

REGION 1:

\[
\begin{align*}
p'^{\mu}_{1r} &= q'^{\mu}_{1}, & \quad y'_{1r} &= \frac{v_{1} \xi_{1,0}}{1 - (1 - v_{1}) \xi_{1,0}} & \quad y'_{12} &= 1 - \xi_{1,0} \\
p'^{\mu}_{1} &= p'^{\mu}_{1} - q'^{\mu}_{1} + \alpha_{1} p'^{\mu}_{2}, & \quad \alpha_{1} &= \frac{q_{3}^{2}}{2 q_{3} \cdot p_{2}} \\
p'^{\mu}_{2} &= (1 - \alpha_{1}) p'^{\mu}_{2}, & \quad y'_{2r} &= \frac{(1 - v_{1})(1 - \xi_{1,0}) \xi_{1,0}}{1 - (1 - v_{1}) \xi_{1,0}} \\
\end{align*}
\]

REGION 2:

\[
\begin{align*}
p'^{\mu}_{2} &= q'^{\mu}_{2}, & \quad y'_{1r} &= 1 - \xi_{2,0} & \quad y'_{12} &= \frac{v_{2} (1 - \xi_{2,0}) \xi_{2,0}}{1 - v_{2} \xi_{2,0}} \\
p'^{\mu}_{r} &= p'^{\mu}_{2} - q'^{\mu}_{2} + \alpha_{2} p'^{\mu}_{1}, & \quad \alpha_{2} &= \frac{q_{1}^{2}}{2 q_{1} \cdot p_{1}} \\
p'^{\mu}_{1} &= (1 - \alpha_{2}) p'^{\mu}_{1}, & \quad y'_{2r} &= \frac{(1 - v_{2}) \xi_{2,0}}{1 - v_{2} \xi_{2,0}} \\
\end{align*}
\]
We combine the dual contributions with the real terms (after applying the proper mapping) to get the total decay rate in the scalar toy-model.

- The result agrees perfectly with standard DREG.
- Massless limit is smoothly approached due to proper treatment of quasi-collinear configurations in the RV mapping.
FDU approach @ NLO

Example: vector boson decays

- Total decay rate for a vector particle into a pair of massive quarks:
  - Agreement with the standard DREG result
  - Smoothly achieves the massless limit
  - Efficient numerical implementation
  - Cancellation of UV log’s (as in DREG…)

Rodrigo et al, JHEP10(2016)162
FDU approach @ NLO

Example: Higgs decay at NLO

- Total decay rate for Higgs into a pair of massive quarks:
  - Agreement with the standard DREG result
  - Smoothly achieves the massless limit
  - Local version of UV counterterms successfully reproduces the expected behaviour
  - Efficient numerical implementation

Rodrigo et al, JHEP10(2016)162
FDU approach @ NLO

- The total decay-rate can be expressed using purely four-dimensional integrands
- We recover the total NLO correction, while avoiding dealing with DREG

Main advantages:
- Direct **numerical** implementation (integrable functions for $\varepsilon=0$)
  - Finite integral for $\varepsilon=0$ \(\leftrightarrow\) Integrability with $\varepsilon=0$
  - With FDU is true!
- No need of tensor reduction (**avoids the presence of** Gram determinants, which could introduce numerical instabilities)
- **Smooth transition** to the massless limit (due to the efficient treatment of quasi-collinear configurations)
- **Mapped real-contribution used as a fully local IR counter-term for the dual contribution!**
Conclusions and perspectives

- Loop-tree duality allows to treat virtual and real contributions simultaneously (loop measure expressed in Euclidean space)
- Physical interpretation of IR/UV singularities in loop integrals
- More transparent description of thresholds
- Combined virtual-real terms are integrable in four space-time dimensions!! FDU

- Perspectives:
  - Automation of multileg processes @ NLO (ongoing)
  - Extension of the local IR formalism to NNLO
  - Exploit simplifications due to easier asymptotic expansions @ NNLO (ongoing)
  - Careful comparison with other schemes

2-loop examples available!!! (Felix’s talk)

“Workstop-Thinkstart meeting”
UZH, Zurich, Sep. 2016
Thanks for the attention!!!
LTD/FDU approach: multileg

Real-virtual momentum mapping (GENERAL)

- Real-virtual momentum mapping with massive particles:
  - Consider 1 the emitter, r the radiated particle and 2 the spectator
  - Apply the PS partition and restrict to the only region where $1//r$ is allowed (i.e. $\mathcal{R}_1 = \{y'_{1r} < \min y'_{kj}\}$)
  - Propose the following mapping:
    \[
    \begin{align*}
    p_{r}^{\mu} &= q_{1}^{\mu} \\
    p_{1}^{\mu} &= (1 - \alpha) \hat{p}_{1}^{\mu} + (1 - \gamma) \hat{p}_{2}^{\mu} - q_{1}^{\mu} \\
    p_{2}^{\mu} &= \alpha \hat{p}_{1}^{\mu} + \gamma \hat{p}_{2}^{\mu}
    \end{align*}
    \]
    with $\hat{p}_{i}$ massless four-vectors build using $p_{i}$ (simplify the expressions)
  - Express the loop three-momentum with the same parameterization used for describing the dual contributions!

Impose on-shell conditions to determine mapping parameters

Repeat in each region of the partition...

Rodrigo et al, JHEP 10(2016)162
LTD/FDU approach: renormalization

UV counterterms and local renormalization

- LTD must be applied to deal with **UV singularities** by building **local** versions of the usual UV counterterms.

  - **1:** **Expand** internal propagators around the “**UV propagator**”

    \[
    \frac{1}{q_i^2 - m_i^2 + i0} = \frac{1}{q_{UV}^2 - \mu_{UV}^2 + i0} \\
    \times \left[ 1 - \frac{2q_{UV} \cdot k_{i,UV} + k_{i,UV}^2 - m_i^2 + \mu_{UV}^2}{q_{UV}^2 - \mu_{UV}^2 + i0} + \frac{(2q_{UV} \cdot k_{i,UV})^2}{(q_{UV}^2 - \mu_{UV}^2 + i0)^2} \right] + \mathcal{O} \left( (q_{UV}^2)^{-5/2} \right)
    \]


- **2:** Apply LTD to get the **dual representation** for the expanded UV expression, and **subtract** it from the **dual+real** combined integrand.

  LTD extended to deal with multiple poles
  (use residue formula to obtain the dual representation)

- **3:** Take into account **wave-function and vertex renormalization** constants
  (not trivial in the massive case!)

Rodrigo et al, JHEP10(2016)162
Self-energy corrections with **on-shell renormalization** conditions

\[
\Sigma_R(\phi_1 = M) = 0 \quad \frac{d\Sigma_R(\phi_1)}{d\phi_1} \bigg|_{\phi=M} = 0
\]

**Wave-function renormalization constant** *(both IR and UV poles)*:

\[
\Delta Z_2(p_1) = -g_S^2 C_F \int \phi_1 \phi_2 \left( (d-2) \frac{q_1 \cdot p_2}{p_1 \cdot p_2} + 4M^2 \left(1 - \frac{q_1 \cdot p_2}{p_1 \cdot p_2}\right) \right) G_F(q_3)
\]

**Vertex renormalization (only UV):**

\[
\Gamma^{(1)}_{A,UV} = g_S^2 C_F \int (G_F(q_{UV}))^3 \left[ \gamma^\nu \phi_{UV} \Gamma^{(0)}_A \phi_{UV} \gamma^\nu - d_{A,UV} \mu_{UV}^2 \Gamma^{(0)}_A \right]
\]

**Important features:**

- Integrated results agrees with standard UV counter-terms!
- **Smooth massless limit!**