

# Off-shell Jacobi currents within the loop-tree duality

*William J. Torres Bobadilla*

*Institut de Física Corpuscular*

*Universitat de València - CSIC*

*In collaboration with:*

*P. Mastrolia, A. Primo, U. Schubert; J. Llanes, G. Rodrigo.*

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# Outline

- Colour decomposition
- Colour-Kinematics duality
  - C/K duality @ tree-level in  $d$
  - Integral relations @1L
- Conclusions/Outlook

# *Amplitudes?*

What are they?

Where do they appear?

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Electromagnetism

Electric and magnetic field

Optics

Intensity of light (wave)

Quantum Mechanics

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Quantum Field theory

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**Zur Quantenmechanik der Stoßvorgänge** (In German)

On the quantum mechanics of collisions

Max Born. 1926. 5 pp.

Published in **Z.Phys.** 37 (1926) no.12, 863-867

DOI: [10.1007/BF01397477](https://doi.org/10.1007/BF01397477)

[References](#) | [BibTeX](#) | [LaTeX\(US\)](#) | [LaTeX\(EU\)](#) | [Harvmac](#) | [EndNote](#)

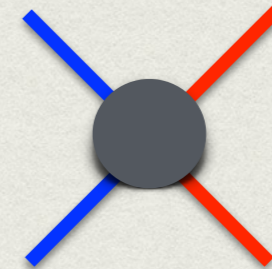
[Detailed record](#) - [Cited by 73 records](#) 50+

# Scattering Amplitudes

- \* Particle interactions

$$1 + 2 \rightarrow 3 + 4$$

2-→2 scattering



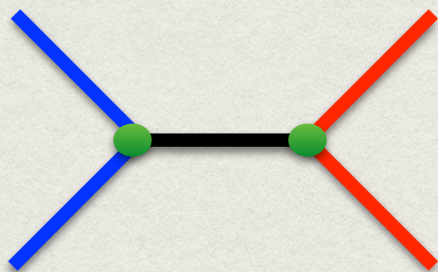
The simplest process

- \* Quantum probability

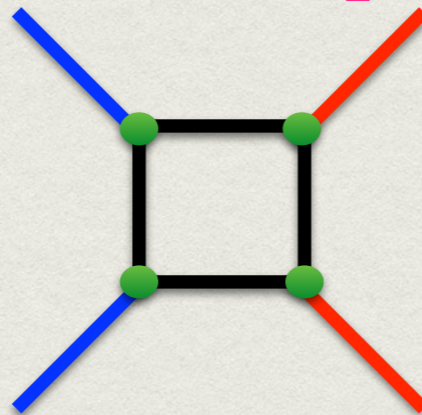
$$\sim \left| \text{diagram} \right|^2$$

- \* Amplitudes ~ Feynman diagrams

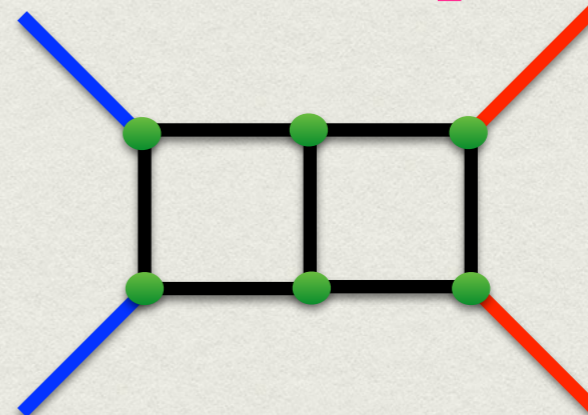
Tree-level



one-loop



two-loop



...

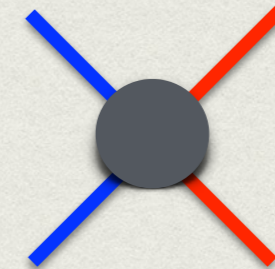
Perturbation expansion

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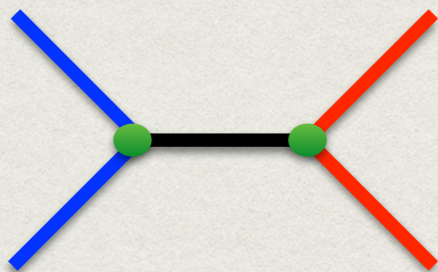
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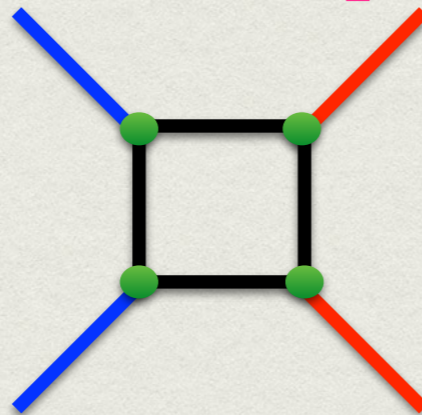
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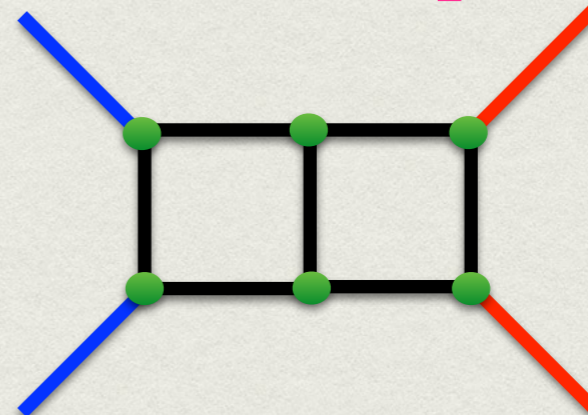
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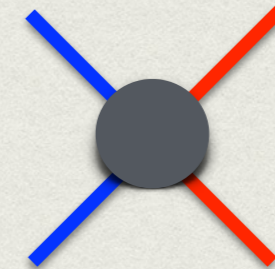
...

precision depends on the  
number of couplings = ●

Perturbation expansion

# Scattering Amplitudes

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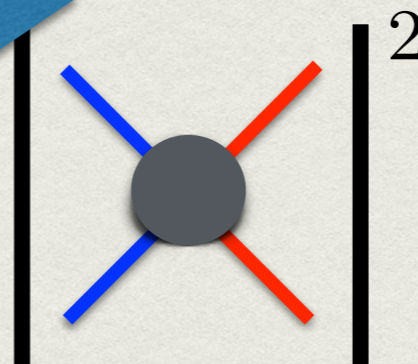


The simplest process

## \* Quantum mechanics

$$1 + 2 \rightarrow 3 + 4$$

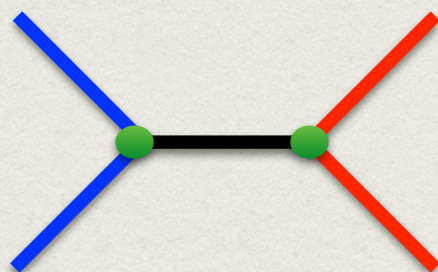
scattering



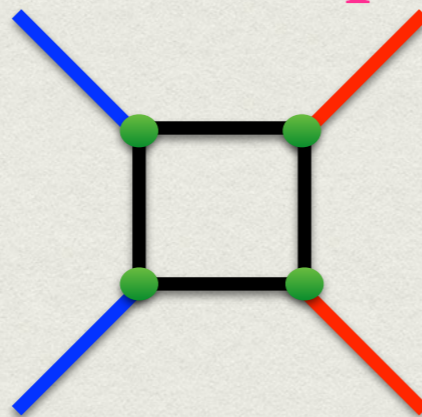
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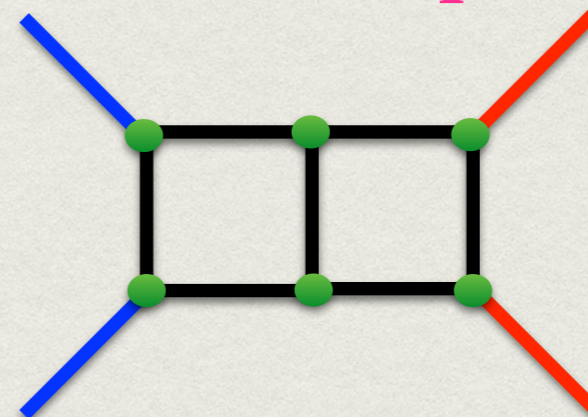
Tree-level



one-loop



two-loop



...

Dyson series

Perturbation expansion

# Colour decomposition

In QCD any amplitude can be decomposed as

$$\mathcal{A}_n(\{k_i, a_i, h_i\}) = A_n(1^{h_1}, 2^{h_2}, \dots, n^{h_n}) \times \text{Colour Structure} + \text{all non-cyclic permutations}$$

Diagram illustrating the decomposition of an amplitude  $\mathcal{A}_n(\{k_i, a_i, h_i\})$  into a sum of terms. The first term is  $A_n(1^{h_1}, 2^{h_2}, \dots, n^{h_n}) \times \text{Colour Structure}$ , followed by "+ all non-cyclic permutations". Arrows point from the variables in the first term to their descriptions:  $k_i$  to "Momenta",  $a_i$  to "Color",  $h_i$  to "Helicities", the first  $A_n$  to "Primitive amplitudes depends on Lorentz variables only", and "Colour Structure" to "Can contain traces or products of generators T".

## At tree-level

For the **n-gluon** tree-level amplitude, the **colour decomposition** is

$$\mathcal{A}_n^{\text{tree}}(\{k_i, a_i, h_i\}) = g^{n-2} \text{Tr}(T^{a_1} T^{a_2} \dots T^{a_n}) A_n^{\text{tree}}(1^{h_1}, 2^{h_2}, \dots, n^{h_n}) + \text{all non-cyclic permutations}$$

### Properties between amplitudes

- Reflection invariance
- Cyclic invariance

→ **(n-1)!** Independent amplitudes

$$\mathcal{A}_n^{\text{tree}}(\{p_i, h_i, a_i\}) = g^{n-2} \sum_{\sigma \in S_n / Z_n} \text{Tr}(T^{a_{\sigma(1)}} \dots T^{a_{\sigma(n)}}) A_n^{\text{tree}}(\sigma(1^{h_1}), \dots, \sigma(n^{h_n}))$$

# Colour decomposition

In QCD any amplitude can be decomposed as

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Momenta → Color → Helicities → Primitive amplitudes depends on Lorentz variables only

Can contain traces or products of generators T

## At tree-level

An alternative representation

$$\mathcal{A}_n^{\text{tree}}(\{p_i, h_i, a_i\}) = (ig)^{n-2} f^{a_1 a_2 x_1} f^{x_1 a_3 x_2} \dots f^{x_{n-3} a_{\sigma_{n-1}} a_b} A_n^{\text{tree}}(1^{h_1}, \sigma(2^{h_2}), \dots, n^h) + \text{all non-cyclic permutations}$$

## Properties between amplitudes



Kleiss-Kuijf relations

$$A_n^{\text{tree}}(1, \alpha_1, \dots, \alpha_j, n, \beta_1, \dots, \beta_{n-2-j}) = (-1)^{n-2-j} \sum_{\sigma \in \vec{\alpha} \sqcup \vec{\beta}^T} A_n^{\text{tree}}(1, \sigma_1, \dots, \sigma_{n-2-j}, n)$$

→ **(n-2)!** Independent amplitudes

[Kleiss and Kuijf (1989)]

$$\mathcal{A}_n^{\text{tree}}(\{p_i, h_i, a_i\}) = (ig)^{n-2} \sum_{\sigma \in S_{n-2}} f^{a_1 a_2 x_1} f^{x_1 a_3 x_2} \dots f^{x_{n-3} a_{\sigma_{n-1}} a_b} A_n^{\text{tree}}(1^{h_1}, \sigma(2^{h_2}), \dots, n^h)$$

# Colour-Kinematics duality

## Jacobi Relation (colour)

$$c_s = c_t - c_u$$

$$f^{a_1 a_2 b} f^{a_3 a_4 b} = f^{a_4 a_1 b} f^{a_2 a_3 b} - f^{a_1 a_3 b} f^{a_2 a_4 b}$$

$$f^{a_1 a_2 b} T^b = T^{a_1} T^{a_2} - T^{a_2} T^{a_1}$$

Write QCD amplitudes in terms of cubic graphs

$$\mathcal{A}_n = g^{n-2} \sum \frac{n_i c_i}{D_i}$$

$$\mathcal{A}_4(p_1, p_2, p_3, p_4) = c_1 \frac{n_1}{P_{23}^2 - \mu^2} + c_2 \frac{n_2}{P_{12}^2} + c_3 \frac{n_3}{P_{24}^2 - \mu^2}$$

- Satisfy automatically for 4-point tree amplitudes  $n_s = n_t - n_u$  [Zhu (1980)]
- For higher multiplicity, is not trivially satisfied [Bern, Carrasco, Johansson (2008),(2010)]
- Bern-Carrasco-Johansson relations [Bern, Dennen, Huang, Kiermaier (2010)], [Boels, Isermann (2012)]
- ... [Mastrolia, Primo, Schubert, W.J.T. (2015)]

$$\sum_{i=3}^n \left( \sum_{j=3}^i s_{2j} \right) A_n^{\text{tree}}(1, 3, \dots, i, 2, i+1, \dots, n) = 0$$

→

(n-3)! Independent amplitudes

# Colour-Kinematics duality

$$\begin{array}{c} 2 \\ \diagdown \\ \bullet \\ \diagup \\ 1 \end{array} \begin{array}{c} 3 \\ \diagup \\ \bullet \\ \diagdown \\ 4 \end{array} = \begin{array}{c} 2 \\ \diagdown \\ \bullet \\ \diagup \\ 1 \end{array} \begin{array}{c} 3 \\ \diagup \\ \bullet \\ \diagdown \\ 4 \end{array} - \begin{array}{c} 2 \\ \diagdown \\ \bullet \\ \diagup \\ 1 \end{array} \begin{array}{c} 3 \\ \diagup \\ \bullet \\ \diagdown \\ 4 \end{array}$$

$c_s \qquad c_t \qquad c_u$

[Bern, Carrasco, Johansson (2008),(2010)]

[Johansson, Ochirov (2014),(2015)]

[de la Cruz, Kniss, Weinzierl (2015),(2016)]

...

- ✓ Relations between kinematic numerators
- ✓ Provides symmetries among amplitudes
- ✓ Strong Connection between gravity and Yang-Mills amplitudes
- ✓ Construction of gravity from knowledge of Yang-Mills amplitudes

## Construct an off-shell current

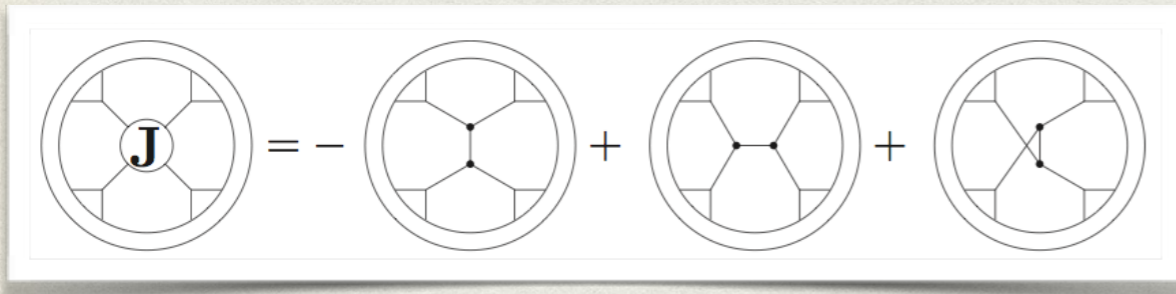
[Llanes, Rodrigo, W.J.T. (2017)]

$$\begin{array}{l}
 \begin{array}{c} 1 \\ \diagup \\ \bullet \\ \diagdown \\ 4 \end{array} \begin{array}{c} 2 \\ \diagdown \\ \bullet \\ \diagup \\ 3 \end{array} \begin{array}{c} J \end{array} = -n \left( \begin{array}{c} 1 \\ \diagup \\ \bullet \\ \diagdown \\ 4 \end{array} \begin{array}{c} 2 \\ \diagdown \\ \bullet \\ \diagup \\ 3 \end{array} \right) - n \left( \begin{array}{c} 1 \\ \diagdown \\ \bullet \\ \diagup \\ 4 \end{array} \begin{array}{c} 2 \\ \diagup \\ \bullet \\ \diagdown \\ 3 \end{array} \right) + n \left( \begin{array}{c} 1 \\ \diagdown \\ \bullet \\ \diagup \\ 4 \end{array} \begin{array}{c} 2 \\ \diagup \\ \bullet \\ \diagdown \\ 3 \end{array} \right) \\
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 \end{array}$$

$\longrightarrow p_1^{\mu_1} \left( \begin{array}{c} p_2 \\ \diagup \\ \bullet \\ \diagdown \\ p_3 \end{array} \right) + (2 \leftrightarrow 3)$   
 $\longrightarrow p_1^{\mu_1} \left( \begin{array}{c} p_2 \\ \diagup \\ \bullet \\ \diagdown \\ p_3 \end{array} \right) + \not{p}_2 \left( \begin{array}{c} p_4, \mu_4 \\ \diagup \\ \bullet \\ \diagdown \\ p_1, \mu_1 \end{array} \right) + (\{1234\} \rightarrow \{4321\})$   
 $\longrightarrow p_1^{\mu_1} \left( \begin{array}{c} p_2, \mu_2 \\ \diagup \\ \bullet \\ \diagdown \\ p_3, \mu_3 \end{array} \right) \pm \text{cyc. perm.}$

# Colour-Kinematics duality

- At multi-loop level or higher-points



[Mastrolia, Primo, Schubert, W.J.T. (2015)]

**External** particles become **internal**

$$u(p_i), v(p_i) \rightarrow \not{p}_i$$

$$\varepsilon^{\mu_i}(p_i; q_i) \rightarrow \Pi^{\mu_i \nu_i}(p_i; q_i)$$

**Propagator in axial gauge**

- Numerator built from the J-block is decomposed in terms of squared momenta

$$(N_g^{\text{loop}})_{\alpha_1 \dots \alpha_4} = J^{\mu_1 \dots \mu_4} \Pi_{\mu_1 \alpha_1}(p_1, q_1) \Pi_{\mu_2 \alpha_2}(p_2, q_2) \Pi_{\mu_3 \alpha_3}(p_3, q_3) \Pi_{\mu_4 \alpha_4}(p_4, q_4),$$

$$(N_g^{\text{loop}})_{\alpha_1 \dots \alpha_4} = \sum_{i=1}^4 p_i^2 (A_g^i)_{\alpha_1 \dots \alpha_4} + \sum_{\substack{i,j=1 \\ i \neq j}}^4 p_i^2 p_j^2 (C_g^{ij})_{\alpha_1 \dots \alpha_4}.$$

$$A_g = A_g(\{p_i\})$$

$$C_g = C_g(\{p_i\})$$

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Propagator in axial gauge

- Numerator built from the J-block is decomposed in terms of squared momenta

The diagram shows the J-block (a circle with four internal lines forming a cross) equal to a sum of eight diagrams. The first four diagrams are labeled  $A_g^1, A_g^2, A_g^3, A_g^4$  and each has one internal line removed. The next four diagrams are labeled  $C_g^{12}, C_g^{13}, C_g^{14}, C_g^{23}, C_g^{24}, C_g^{34}$  and each has two internal lines removed, representing the squared momenta of the remaining lines.

- Any loop diagram built from the  $J$ -block can be written as the sum of diagrams with one or two propagators less.

# Colour-Kinematics duality

- At multi-loop level or higher-points

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External particles become internal

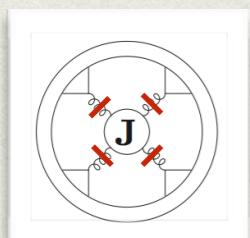
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Propagator in axial gauge

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$$= 0$$

- By imposing on-shellness of the four particles

# Colour-Kinematics duality

[Llanes, Rodrigo, W.J.T. (2017)]

Decompose **off**- into on-shell momenta

$$p_i^\alpha = r_i^\alpha + \frac{p_i^2}{2q \cdot r_i} q^\alpha \longrightarrow$$

Extract full dependence on the **off-shell** momenta

$$\begin{aligned} \sum_{\lambda=1}^{d_s-2} \varepsilon_{\lambda(d_s)}^\alpha(p_i) \varepsilon_{\lambda(d_s)}^{*\beta}(p_i) &= \sum_{\lambda_i=1}^{d_s-2} \varepsilon_i^\alpha \varepsilon_i^{*\beta} + \frac{p_i^2}{(r_i \cdot q)^2} q^\alpha q^\beta, \\ \sum_{\lambda=1}^{2^{(d_s-2)/2}} u_{\lambda(d_s)}(p_i) \bar{u}_{\lambda(d_s)}(p_i) &= \sum_{\lambda_i=1}^{2^{(d_s-2)/2}} u_i \bar{u}_i + \frac{p_i^2}{2(r_i \cdot q)} \not{q}. \end{aligned}$$

Completeness relations

Construct multi-loop numerator

$$N_g = N_{g\mu_1 \dots \mu_4} X^{\mu_1 \dots \mu_4}, \quad N_{g\mu_1 \dots \mu_4} = J_g^{\nu_1 \dots \nu_4} \Pi_{\mu_1 \nu_1}(p_1, q) \dots \Pi_{\mu_4 \nu_4}(p_4, q).$$

Residual kinematic dependence

Numerator is decomposed in product of squared momenta

$$N_g^{\nu_1 \dots \nu_4} = \frac{1}{2} \sum_{i,j,k,l=1}^4 \epsilon_{ijkl} p_i^2 \left( A_{ijkl} \varepsilon_{ij}^{\nu_i \nu_j} \varepsilon_{kl}^{\nu_k \nu_l} + B_{ijkl} \varepsilon_{jk}^{\nu_j \nu_k} Q_l^{\nu_i \nu_l} + C_{ijkl} p_j^2 q^{\nu_i \nu_j} \varepsilon_{kl}^{\nu_k \nu_l} \right),$$

**A, B** and **C** are completely independent of  $p_i^2$

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**A, B** and **C** are completely independent of  $p_i^2$

What about  $p_i^2 p_j^2 p_k^2$  and  $p_i^2 p_j^2 p_k^2 p_l^2$  contributions?

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Residual kinematic dependence

Same reference momentum **q**  
for all internal gluons!

Numerator is decomposed in product of squared momenta

$$N_g^{\nu_1 \dots \nu_4} = \frac{1}{2} \sum_{i,j,k,l=1}^4 \epsilon_{ijkl} p_i^2 \left( A_{ijkl} \varepsilon_{ij}^{\nu_i \nu_j} \varepsilon_{kl}^{\nu_k \nu_l} + B_{ijkl} \varepsilon_{jk}^{\nu_j \nu_k} Q_l^{\nu_i \nu_l} + C_{ijkl} p_j^2 q^{\nu_i \nu_j} \varepsilon_{kl}^{\nu_k \nu_l} \right),$$

**A, B** and **C** are completely independent of  $p_i^2$

What about  ~~$p_i^2 p_j^2 p_k^2$~~  and  ~~$p_i^2 p_j^2 p_k^2 p_l^2$~~  contributions?

# Colour-Kinematics duality

## One-loop example

[Llanes, Rodrigo, W.J.T. (2017)]

$$\int \frac{d^d \ell}{(2\pi)^d} \frac{n(\text{Diagram 1})}{D_0 D_1 D_2} = \int \frac{d^d \ell}{(2\pi)^d} \frac{1}{D_0 D_1 D_2} \left[ -n(\text{Diagram 2}) - n(\text{Diagram 3}) + n(\text{Diagram 4}) \right] = \int \frac{d^d \ell}{(2\pi)^d} \left[ \tilde{A}_{11} \mathcal{I} \left[ \text{Diagram 5} \right] + \tilde{A}_{12} \mathcal{I} \left[ \text{Diagram 6} \right] + \tilde{C}_{11} \mathcal{I} \left[ \text{Diagram 7} \right] \right] = 0$$

$$\int \frac{d^d \ell}{(2\pi)^d} \frac{n(\text{Diagram 8})}{D_0 D_1} = \int \frac{d^d \ell}{(2\pi)^d} \frac{1}{D_0 D_1} \left[ -n(\text{Diagram 9}) - n(\text{Diagram 10}) + n(\text{Diagram 11}) \right] = \int \frac{d^d \ell}{(2\pi)^d} \left[ \tilde{A}_{21} \mathcal{I} \left[ \text{Diagram 12} \right] + \tilde{A}_{22} \mathcal{I} \left[ \text{Diagram 13} \right] + \tilde{C}_{11} \mathcal{I} \left[ \text{Diagram 14} \right] \right] = 0$$

## From string theory

$$\int \frac{d^d \ell}{(2\pi)^d} \left[ \frac{1}{\ell^2(\ell+p_{12})^2(\ell-p_4)^2} n \left( \text{Diagram 15} \right) - \frac{1}{\ell^2(\ell+p_2)^2(\ell+p_{23})^2} n \left( \text{Diagram 16} \right) \right. \\ \left. + \frac{1}{\ell^2(\ell+p_2)^2(\ell-p_4)^2} n \left( \text{Diagram 17} \right) + \frac{1}{s_{12}\ell^2(\ell+p_{12})^2} n \left( \text{Diagram 18} \right) \right. \\ \left. - \frac{1}{s_{23}(\ell+p_1)^2(\ell-p_4)^2} n \left( \text{Diagram 19} \right) + \frac{1}{s_{24}(\ell+p_2)^2(\ell-p_4)^2} n \left( \text{Diagram 20} \right) \right] = 0$$

[Tourkine, Vanhove (2016)]

[Ochirov, Tourkine, Vanhove (2017)]

- ☑ Satisfied automatically for 4-point one-loop amplitudes
- ☑ Off-shell decomposition eliminates redundant terms
- ☑ Interesting integral relations at one-loop level
- ☑ Straightforward application with Loop-Tree duality formalism

# Colour-Kinematics duality

[Llanes, Rodrigo, W.J.T. (2017)]

## One-loop example

$$\int \frac{d^d \ell}{(2\pi)^d} \frac{n(\text{Diagram 1})}{D_0 D_1 D_2} = \int \frac{d^d \ell}{(2\pi)^d} \frac{1}{D_0 D_1 D_2} \left[ -n(\text{Diagram 2}) - n(\text{Diagram 3}) + n(\text{Diagram 4}) \right] = \int \frac{d^d \ell}{(2\pi)^d} \left[ \tilde{A}_{11} \mathcal{I} \left[ \text{Diagram 5} \right] + \tilde{A}_{12} \mathcal{I} \left[ \text{Diagram 6} \right] + \tilde{C}_{11} \mathcal{I} \left[ \text{Diagram 7} \right] \right] = 0$$

$$\int \frac{d^d \ell}{(2\pi)^d} \frac{n(\text{Diagram 8})}{D_0 D_1} = \int \frac{d^d \ell}{(2\pi)^d} \frac{1}{D_0 D_1} \left[ -n(\text{Diagram 9}) - n(\text{Diagram 10}) + n(\text{Diagram 11}) \right] = \int \frac{d^d \ell}{(2\pi)^d} \left[ \tilde{A}_{21} \mathcal{I} \left[ \text{Diagram 12} \right] + \tilde{A}_{22} \mathcal{I} \left[ \text{Diagram 13} \right] + \tilde{C}_{11} \mathcal{I} \left[ \text{Diagram 14} \right] \right] = 0$$

>> LTD's talks

## From string theory

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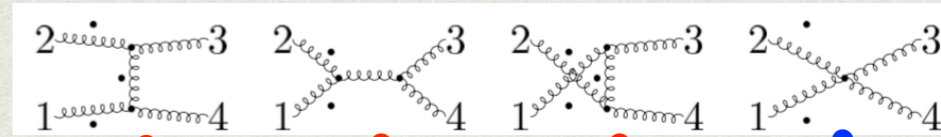
# BCJ relations in DimReg

[Mastrolia, Primo, Schubert, W.J.T. (2015)]

## Four-dimensional formulation of FDH

[Fazio, Mastrolia, Mirabella, W.J.T. (2014)]

Consider the 4-point amplitude



$$\mathcal{A}_4(p_1, p_2, p_3, p_4) = c_1 \frac{n_1}{P_{23}^2 - \mu^2} + c_2 \frac{n_2}{P_{12}^2} + c_3 \frac{n_3}{P_{24}^2 - \mu^2}$$

$$-c_1 + c_2 + c_3 = 0$$

and the Jacobi identity

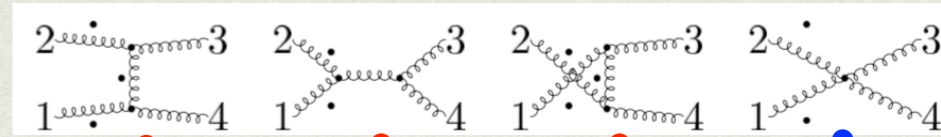
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Solving for  $c_2$

$$\mathcal{A}_4(p_1, p_2, p_3, p_4) = c_1 K_1 + c_3 K_3$$

being

$$K_1 = \frac{n_1}{P_{23}^2 - \mu^2} + \frac{n_2}{P_{12}^2},$$

$$K_3 = \frac{n_3}{P_{24}^2 - \mu^2} - \frac{n_2}{P_{12}^2}.$$

Colour-ordered amplitudes

$$K_1 = A(1, 2, 3, 4)$$

$$K_3 = A(2, 1, 3, 4)$$

and the Jacobi identity

Kinematic numerators obey Jacobi identity

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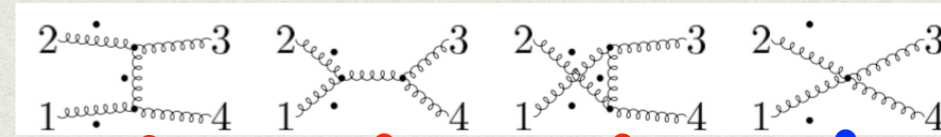
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$$\begin{pmatrix} \frac{1}{P_{23}^2 - \mu^2} & \frac{1}{P_{12}^2} & 0 \\ 0 & -\frac{1}{P_{12}^2} & \frac{1}{P_{24}^2 - \mu^2} \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = \begin{pmatrix} K_1 \\ K_3 \\ 0 \end{pmatrix}$$

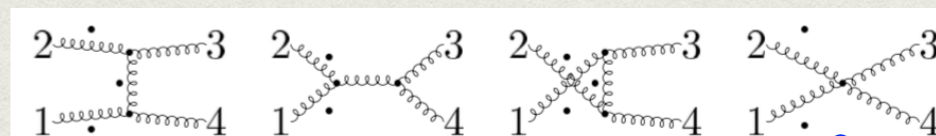
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**4-pt C/K-relations**

$$A(2, 1, 3, 4) = \frac{P_{23}^2 - \mu^2}{P_{24}^2 - \mu^2} A(1, 2, 3, 4).$$

# BCJ relations in DimReg

Four-dimensional formulation of FDH

[Fazio, Mastrolia, Mirabella, W.J.T. (2014)]

[Mastrolia, Primo, Schubert, W.J.T. (2015)]

As well, for the 5-point

$$\begin{aligned} A_5(1, 3, 4, 2, 5) &= \frac{-P_{12}^2 P_{45}^2 A_5(1, 2, 3, 4, 5) + (P_{14}^2 - \mu^2)(P_{24}^2 + P_{25}^2 - 2\mu^2) A_5(1, 4, 3, 2, 5)}{(P_{13}^2 - \mu^2)(P_{24}^2 - \mu^2)}, \\ A_5(1, 2, 4, 3, 5) &= \frac{-(P_{14}^2 - \mu^2)(P_{25}^2 - \mu^2) A_5(1, 4, 3, 2, 5) + P_{45}^2 (P_{12}^2 + P_{24}^2 - \mu^2) A_5(1, 2, 3, 4, 5)}{P_{35}^2 (P_{24}^2 - \mu^2)}, \\ A_5(1, 4, 2, 3, 5) &= \frac{-P_{12}^2 P_{45}^2 A_5(1, 2, 3, 4, 5) + (P_{25}^2 - \mu^2)(P_{14}^2 + P_{25}^2 - 2\mu^2) A_5(1, 4, 3, 2, 5)}{P_{35}^2 (P_{24}^2 - \mu^2)}, \\ A_5(1, 3, 2, 4, 5) &= \frac{-(P_{14}^2 - \mu^2)(P_{25}^2 - \mu^2) A_5(1, 4, 3, 2, 5) + P_{12}^2 (P_{24}^2 + P_{45}^2 - \mu^2) A_5(1, 2, 3, 4, 5)}{(P_{13}^2 - \mu^2)(P_{24}^2 - \mu^2)}. \end{aligned}$$

Making use of the photon decoupling identity

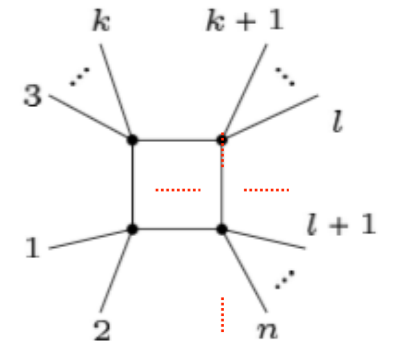
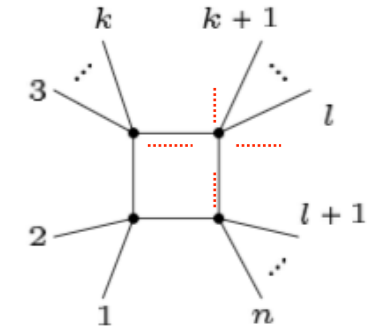
$$A_5(1, 2, 4, 3, 5) = \frac{(P_{14}^2 + P_{45}^2 - \mu^2) A_5(1, 2, 3, 4, 5) + (P_{14}^2 - \mu^2) A_5(1, 2, 3, 5, 4)}{(P_{24}^2 - \mu^2)}$$

# BCJ relations @ 1-loop

[Primo, W.J.T. (2016)]

Inspired by the generalised unitarity

$$C_{12|3\dots k|(k+1)\dots l|(l+1)\dots n}^{\pm} = A_4^{\text{tree}}(-l_1^{\pm}, 1, 2, l_3^{\pm}) A_k^{\text{tree}}(-l_3^{\pm}, P_{3\dots k}, l_{k+1}^{\pm}) \\ \times A_{l-k+2}^{\text{tree}}(-l_{k+1}^{\pm}, P_{k+1\dots l}, l_{l+1}^{\pm}) A_{n-l+2}^{\text{tree}}(-l_{l+1}^{\pm}, P_{l+1\dots n}, l_1^{\pm})$$



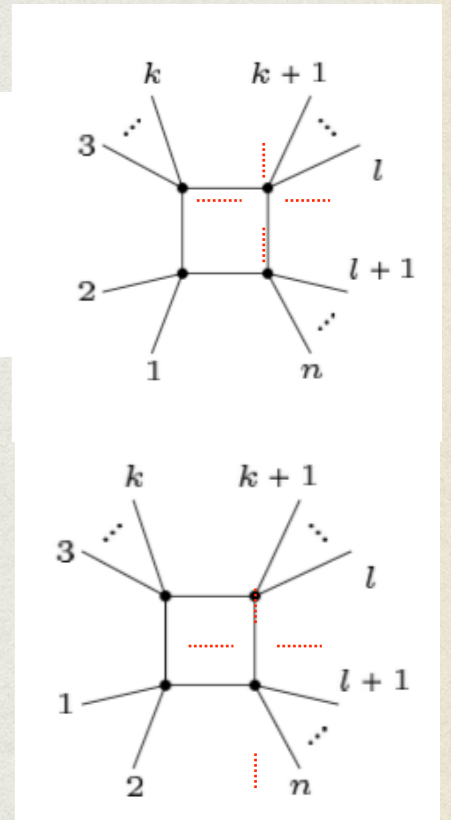
$$C_{21|3\dots k|(k+1)\dots l|(l+1)\dots n}^{\pm} = \frac{P_{l_3^{\pm}2}^2 - \mu^2}{P_{-l_1^{\pm}2}^2 - \mu^2} C_{12|3\dots k|(k+1)\dots l|(l+1)\dots n}^{\pm}$$

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C/K relation

$$A(2, 1, 3, 4) = \frac{P_{23}^2 - \mu^2}{P_{24}^2 - \mu^2} A(1, 2, 3, 4).$$

— One-loop amplitudes in N=4 sYM

[Bjerrum-Bohr, Damgaard, Sondergaard, Vanhove (2010)]

— Cut constructible part of One-loop QCD amplitudes

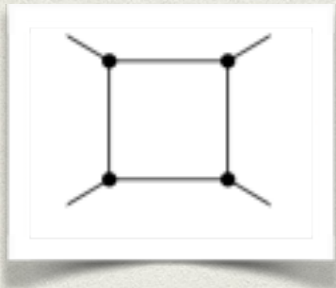
[Chester (2016)]

— One-loop QCD amplitudes

[Primo, W.J.T. (2016)]

# One-loop scattering amplitudes

Deal with with integrals of the form



$$I_{i_1 \dots i_k} [\mathcal{N}(\bar{l}, p_i)] = \int d^d \bar{l} \frac{\mathcal{N}_{i_1 \dots i_k}(\bar{l}, p_i)}{D_{i_1} \dots D_{i_k}}$$

$$\bar{l}^2, \bar{l} \cdot p_i, \bar{l} \cdot \varepsilon_i$$

Numerator and denominators are polynomials in the integration variable

Tensor reduction

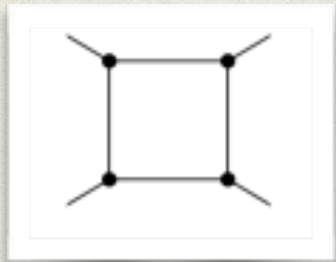
$$A_n^{(1), D=4}(\{p_i\}) = \sum_{K_4} C_{4;K_4}^{[0]} \text{ (square) } + \sum_{K_3} C_{3;K_3}^{[0]} \text{ (triangle) } + \sum_{K_2} C_{2;K_2}^{[0]} \text{ (bubble) } + \sum_{K_1} C_{1;K_1}^{[0]} \text{ (tadpole) }$$

[Passarino - Veltman (1979)]

- ☑ Cut-constructible amplitude -> determined by its branch cuts
- ☑ All one-loop amplitudes are cut-constructible in dimensional regularisation.
- ☑ Master integrals are known

# One-loop scattering amplitudes

Deal with with integrals of the form



$$I_{i_1 \dots i_k} [\mathcal{N}(\bar{l}, p_i)] = \int d^d \bar{l} \frac{\mathcal{N}_{i_1 \dots i_k}(\bar{l}, p_i)}{D_{i_1} \dots D_{i_k}}$$

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[Passarino - Veltman (1979)]

Unitarity based methods

$$\frac{i}{q_i^2 - m^2 - i\epsilon} \rightarrow 2\pi \delta^{(+)}(q_i^2 - m_i^2)$$

$$\text{Cut 4 diagram} = c_4 \text{ (square) } + c_3 \text{ (triangle) } + c_2 \text{ (bubble) }$$

$$\text{Cut 3 diagram} = c_4 \text{ (square) } + c_3 \text{ (triangle) }$$

$$\text{Cut 2 diagram} = c_4 \text{ (square) }$$

**cut-4** :: Britto Cachazo Feng

**cut-3** :: Forde

Bjerrum-Bohr, Dunbar, Ita, Perkins  
Mastrolia

**cut-2** :: Bern, Dixon, Dunbar, Kosower.  
Britto, Buchbinder, Cachazo, Feng.  
Britto, Feng, Mastrolia.

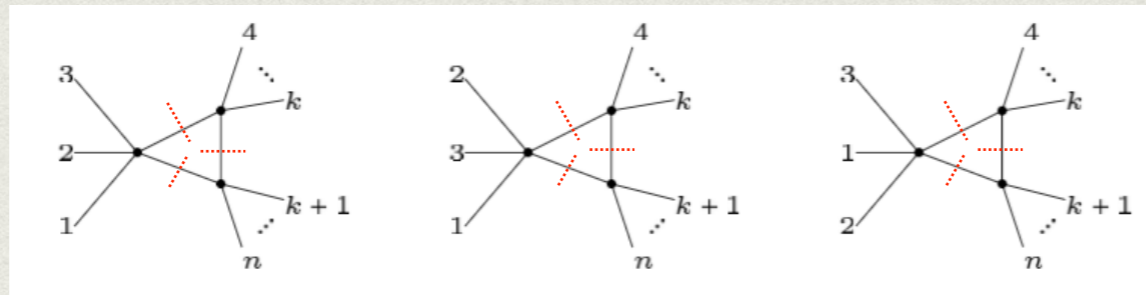
Isolate the leading discontinuity!

# BCJ relations @ 1-loop

[Primo, W.J.T. (2016)]

Same behaviour for lower topologies

$$C_{123|4\dots k|(k+1)\dots n}^{\pm} = A_5^{\text{tree}}(-l_1^{\pm}, 1, 2, 3, l_4^{\pm}) A_{k-1}^{\text{tree}}(-l_4^{\pm}, P_{4\dots k}, l_{k+1}^{\pm}) A_{n-k+2}^{\text{tree}}(-l_{k+1}^{\pm}, P_{k+1\dots n}, l_1^{\pm})$$



$$C_{213|4\dots k|(k+1)\dots n}^{\pm} = \frac{\left(P_{l_4^{\pm}2}^2 + P_{23}^2 - \mu^2\right) C_{123|4\dots k|(k+1)\dots n}^{\pm} + \left(P_{l_4^{\pm}2}^2 - \mu^2\right) C_{132|4\dots k|(k+1)\dots n}^{\pm}}{\left(P_{-l_1^{\pm}2}^2 - \mu^2\right)}$$

due to

$$A_5(1, 2, 4, 3, 5) = \frac{(P_{14}^2 + P_{45}^2 - \mu^2) A_5(1, 2, 3, 4, 5) + (P_{14}^2 - \mu^2) A_5(1, 2, 3, 5, 4)}{(P_{24}^2 - \mu^2)}$$

# BCJ relations + Unitarity @ work

[Primo, W.J.T. (2016)]

**Target ::** Reduce the number of independent residues needed to compute any colour-dressed one-loop amplitude

$$A_n^{1\text{-loop}} = \int d^d \bar{l} \frac{\mathcal{N}(l, \mu^2)}{D_0 D_1 \dots D_{n-1}},$$

$$D_i = (\bar{l} + p_i)^2 - m_i^2 = (l + p_i)^2 - m_i^2 - \mu^2.$$

$$\begin{aligned} \frac{N(l, \mu^2)}{D_0 D_1 \dots D_{n-1}} = & \sum_{i \ll m}^{n-1} \frac{\Delta_{ijklm}(l, \mu^2)}{D_i D_j D_k D_l D_m} + \sum_{i \ll l}^{n-1} \frac{\Delta_{ijkl}(l, \mu^2)}{D_i D_j D_k D_l} + \sum_{i \ll k}^{n-1} \frac{\Delta_{ijk}(l, \mu^2)}{D_i D_j D_k} \\ & + \sum_{i < j}^{n-1} \frac{\Delta_{ij}(l, \mu^2)}{D_i D_j} + \sum_i^{n-1} \frac{\Delta_i(l, \mu^2)}{D_i}, \end{aligned}$$

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**Ingredients ::** Residues @cut  $\rightarrow$  Keep under control their polynomial structure

$$\begin{aligned} \Delta_{ijklm} &= c\mu^2, \\ \Delta_{ijkl} &= c_0 + c_1 x_4 + c_2 \mu^2 + c_3 x_4 \mu^2 + c_4 \mu^4, \\ \Delta_{ijk} &= c_{0,0} + c_{1,0}^+ x_4 + c_{2,0}^+ x_4^2 + c_{3,0}^+ x_4^3 + c_{1,0}^- x_3 + c_{2,0}^- x_3^2 + c_{3,0}^- x_3^3 + c_{0,2} \mu^2 + c_{1,2}^+ x_4 \mu^2 + c_{1,2}^- x_3 \mu^2, \\ \Delta_{ij} &= c_{0,0,0} + c_{0,1,0} x_1 + c_{0,2,0} x_1^2 + c_{1,0,0}^+ x_4 + c_{2,0,0}^+ x_4^2 + c_{1,0,0}^- x_3 + c_{2,0,0}^- x_3^2 + c_{1,1,0}^+ x_1 x_4 \\ &\quad + c_{1,1,0}^- x_1 x_3 + c_{0,0,2} \mu^2, \\ \Delta_i &= c_{0,0,0,0} + c_{0,1,0,0} x_1 + c_{0,0,1,0} x_2 + c_{1,0,0,0}^- x_3 + c_{1,0,0,0}^+ x_4, \end{aligned}$$

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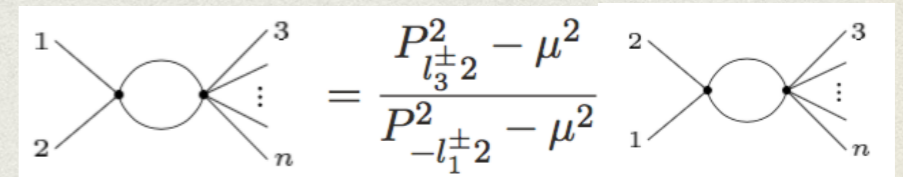
$$D_i = (\bar{l} + p_i)^2 - m_i^2 = (l + p_i)^2 - m_i^2 - \mu^2.$$

$$\begin{aligned} \frac{N(l, \mu^2)}{D_0 D_1 \dots D_{n-1}} = & \sum_{i \ll m}^{n-1} \frac{\Delta_{ijklm}(l, \mu^2)}{D_i D_j D_k D_l D_m} + \sum_{i \ll l}^{n-1} \frac{\Delta_{ijkl}(l, \mu^2)}{D_i D_j D_k D_l} + \sum_{i \ll k}^{n-1} \frac{\Delta_{ijk}(l, \mu^2)}{D_i D_j D_k} \\ & + \sum_{i < j}^{n-1} \frac{\Delta_{ij}(l, \mu^2)}{D_i D_j} + \sum_i^{n-1} \frac{\Delta_i(l, \mu^2)}{D_i}, \end{aligned}$$

**Ingredients ::** Residues @cut  $\rightarrow$  Keep under control their polynomial structure

$$\begin{aligned} \Delta_{ijklm} &= c\mu^2, \\ \Delta_{ijkl} &= c_0 + c_1 x_4 + c_2 \mu^2 + c_3 x_4 \mu^2 + c_4 \mu^4, \\ \Delta_{ijk} &= c_{0,0} + c_{1,0}^+ x_4 + c_{2,0}^+ x_4^2 + c_{3,0}^+ x_4^3 + c_{1,0}^- x_3 + c_{2,0}^- x_3^2 + c_{3,0}^- x_3^3 + c_{0,2} \mu^2 + c_{1,2}^+ x_4 \mu^2 + c_{1,2}^- x_3 \mu^2, \\ \Delta_{ij} &= c_{0,0,0} + c_{0,1,0} x_1 + c_{0,2,0} x_1^2 + c_{1,0,0}^+ x_4 + c_{2,0,0}^+ x_4^2 + c_{1,0,0}^- x_3 + c_{2,0,0}^- x_3^2 + c_{1,1,0}^+ x_1 x_4 \\ &\quad + c_{1,1,0}^- x_1 x_3 + c_{0,0,2} \mu^2, \\ \Delta_i &= c_{0,0,0,0} + c_{0,1,0,0} x_1 + c_{0,0,1,0} x_2 + c_{1,0,0,0}^- x_3 + c_{1,0,0,0}^+ x_4, \end{aligned}$$

**Procedure ::** C/K-relations @work  $\rightarrow$  Generate a system of equations that relates residues of different ordering through C/K-relations



$$\text{Diagram 1} = \frac{P_{l_3^\pm 2}^2 - \mu^2}{P_{-l_1^\pm 2}^2 - \mu^2} \text{Diagram 2}$$

# BCJ relations + Unitarity @ work

[Primo, W.J.T. (2016)]

**Target ::** Reduce the number of independent residues needed to compute any colour-dressed one-loop amplitude

$$A_n^{1\text{-loop}} = \int d^d \bar{l} \frac{\mathcal{N}(l, \mu^2)}{D_0 D_1 \dots D_{n-1}},$$

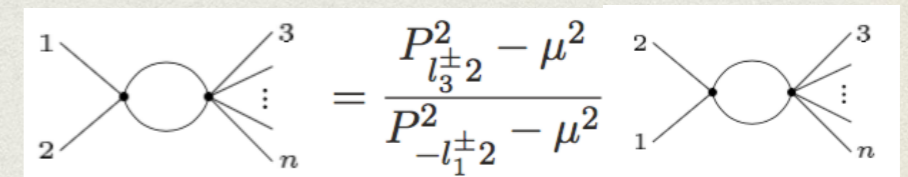
$$\frac{N(l, \mu^2)}{D_0 D_1 \dots D_{n-1}} = \sum_{i \ll m}^{n-1} \frac{\Delta_{ijklm}(l, \mu^2)}{D_i D_j D_k D_l D_m} + \sum_{i \ll l}^{n-1} \frac{\Delta_{ijkl}(l, \mu^2)}{D_i D_j D_k D_l} + \sum_{i \ll k}^{n-1} \frac{\Delta_{ijk}(l, \mu^2)}{D_i D_j D_k} \\ + \sum_{i < j}^{n-1} \frac{\Delta_{ij}(l, \mu^2)}{D_i D_j} + \sum_i^{n-1} \frac{\Delta_i(l, \mu^2)}{D_i},$$

$$D_i = (\bar{l} + p_i)^2 - m_i^2 = (l + p_i)^2 - m_i^2 - \mu^2.$$

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$$\begin{aligned} \Delta_{ijklm} &= c\mu^2, \\ \Delta_{ijkl} &= c_0 + c_1 x_4 + c_2 \mu^2 + c_3 x_4 \mu^2 + c_4 \mu^4, \\ \Delta_{ijk} &= c_{0,0} + c_{1,0}^+ x_4 + c_{2,0}^+ x_4^2 + c_{3,0}^+ x_4^3 + c_{1,0}^- x_3 + c_{2,0}^- x_3^2 + c_{3,0}^- x_3^3 + c_{0,2} \mu^2 + c_{1,2}^+ x_4 \mu^2 + c_{1,2}^- x_3 \mu^2, \\ \Delta_{ij} &= c_{0,0,0} + c_{0,1,0} x_1 + c_{0,2,0} x_1^2 + c_{1,0,0}^+ x_4 + c_{2,0,0}^+ x_4^2 + c_{1,0,0}^- x_3 + c_{2,0,0}^- x_3^2 + c_{1,1,0}^+ x_1 x_4 \\ &\quad + c_{1,1,0}^- x_1 x_3 + c_{0,0,2} \mu^2, \\ \Delta_i &= c_{0,0,0,0} + c_{0,1,0,0} x_1 + c_{0,0,1,0} x_2 + c_{1,0,0,0}^- x_3 + c_{1,0,0,0}^+ x_4, \end{aligned}$$

**Procedure ::** C/K-relations @work  $\rightarrow$  Generate a system of equations that relates residues of different ordering through C/K-relations



$$\text{Diagram 1} = \frac{P_{l_3^\pm 2}^2 - \mu^2}{P_{-l_1^\pm 2}^2 - \mu^2} \text{Diagram 2}$$

- The solution of the system gives us a reduce set of independent residues
- **Unitarity @work**  $\rightarrow$  Compute the independent residues through Unitarity Based Methods

$$\Delta^{(13\dots)} \equiv \sum_{l_i \in S} A_4(-l_1, 1, 2, l_2) \times A(\dots) \times \dots \times A(\dots),$$

# Conclusions/Outlook

- ☒ Further simplifications from Colour-Kinematics duality
- ☒ Most compact representation of the Jacobi identity for kinematic numerators
- ☒ New integral and integrand relations at one-loop level
- ☒ Unitarity + C/K-duality @ work
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- ☐ Provide integral relations at multi-loop level
- ☐ More applications to come in the near future

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