As a first try, we used the modulated 4D Hénon map:

\[
(x^{(n+1)}, y^{(n+1)}, p_x^{(n+1)}, p_y^{(n+1)}) = L \begin{pmatrix} x^{(n)} \\ y^{(n)} \\ [x^{(n)}]^2 - [y^{(n)}]^2 \\ p_y^{(n)} - 2x^{(n)}y^{(n)} \end{pmatrix}
\]

where \(L\) is the outer product of two 2D rotations, with frequencies that are perturbed by a small parameter \(\epsilon\).

This model was used to simulate the evolution of the dynamic aperture (DA). The DA is defined as the smallest simply connected volume in phase space that is stable for at least \(N\) turns. We used the Nekhoroshev theorem to estimate the number of turns \(N\) for which the orbit remains bounded.

The Nekhoroshev theorem provides an estimate for the number of turns \(N(r)\) for which the orbit of an initial condition of amplitude \(r\) remains bounded:

\[
\frac{N(r)}{N_0} = \sqrt{\frac{r}{r^*}} \exp\left\{\frac{\kappa}{\kappa^2} \frac{r^*}{r^*} \right\} \quad N_0, r^*, \kappa > 0
\]

If we interpret the radius \(r\) as the DA, then its evolution is given by the inverse of the theorem. The latter is an asymptotic estimate, so we can neglect the factor \(\sqrt{r}\):

\[
DA(N) = b \left[ \ln \frac{N}{N_0} \right]^{-\kappa} \quad \text{Model 2}
\]

If we set \(N_0 = 1\), and add an asymptotic term inspired by the Kolmogorov-Arnold-Moser theory, we recover the original model for the evolution of DA:

\[
DA(N) = D_\infty + b \ln^{-\kappa} N \quad \text{Model 1 (deprecated)}
\]

It is possible to make an exact inversion of the Nekhoroshev theorem, using the so-called Lambert-W function. It is the inverse of the product exponential:

\[
y = x e^x \iff x = W(y)
\]

It has an infinite set of complex branches; we will need the -1 branch. Then we get another formulation for the evolution of DA:

\[
DA(N) = \rho \left[-W_{-1} \left(-\left(\mu N\right)^{-\frac{1}{2}}\right)\right]^{-\kappa} \quad \text{Model 4}
\]

where \(\rho = b \left(\frac{8}{\pi}\right)^{-\kappa}\) and \(\mu = \frac{8}{7\sqrt{\pi}}\). Finally, if one wants to avoid the Lambert-W function, we can approximate it by its asymptotic series expansion, giving:

\[
DA(N) = b \left[ \ln \mu N + \frac{\kappa}{2} \ln \left(\frac{2}{\kappa} \ln \mu N\right) \right]^{-\kappa} \quad \text{Model 3}
\]

As a fit to toy model: Hénon map

\[
\begin{pmatrix} x^{(n+1)} \\ y^{(n+1)} \\ p_x^{(n+1)} \\ p_y^{(n+1)} \end{pmatrix} = L \begin{pmatrix} x^{(n)} \\ y^{(n)} \\ [x^{(n)}]^2 - [y^{(n)}]^2 \\ p_y^{(n)} - 2x^{(n)}y^{(n)} \end{pmatrix}
\]

where \(L\) is the outer product of two 2D rotations, with frequencies that are perturbed by a small parameter \(\epsilon\).

We fitted this model to the LHC simulations.

Next we tested our models on more relevant data: tracking simulations for the LHC at collision energy. These are done up to \(10^6\) turns (~80s) for 60 random realisations.

The four models can then also be redefined accordingly.

Parameter interdependence

After testing several functions, we observed that there exists a relation between \(b\) and \(\kappa\), given by \(b(\kappa) = \alpha e^{3\kappa}\) with \(\alpha \approx 7.18\) and \(\beta \approx 3.07\) for the LHC simulation data. The Nekhoroshev theorem can then be recast as:

\[
\frac{N(r)}{N_0} = \sqrt{\frac{r}{r^*}} \exp\left\{\frac{1}{B} \left(\frac{r^*}{r}\right)^{\frac{1}{\kappa}}\right\}
\]

with \(B = e^{-\beta}\), \(r^* = r^* B\), \(N_0 = N_0 \sqrt{B}\). The four models can then also be redefined accordingly.

Conclusions & impact

- Models 2, 3, and 4 offer good description
- Model 1 gives slightly different results and sometimes unphysical parameters
- Models 2, 3, and 4 can be used to extrapolate within good accuracy
- Model 1 should hence no longer be used
- It is shown that \(b\) has an exponential dependence on \(\kappa\). This can be used to redefine the models.