Bridging mathematics and physics: models of the evolution of dynamic aperture in hadron colliders and applications to LHC

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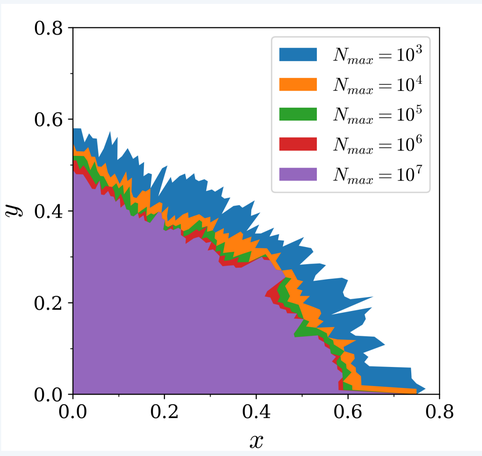
Abstract

In recent years, models for the time-evolution of the dynamic aperture have been proposed and applied to the analysis of non-linear betatronic motion in circular accelerators. We present 3 new models, which are built from a rigorous inversion from the Nekhoroshev theorem. We show that these agree to high accuracy with our data, allowing for a more precise extrapolation and opening up possibilities for tracking simulations beyond what is achieveable with the current CPU resources.

Definition of Dynamic Aperture

The dynamic aperture $DA(N)$ is:

- smallest simply connected volume in phase space, stable for at least N turns
- parameterised as **radius** of sphere with equivalent volume



Evolution of DA from the Nekhoroshev theorem: four models

The Nekhoroshev theorem provides an estimate for the number of turns $N(r)$ for which the orbit of an initial condition of amplitude r remains bounded:

$$\frac{N(r)}{N_0} = \sqrt{\frac{r}{r^*}} \exp\left\{\left(\frac{r^*}{r}\right)^{\frac{1}{\kappa}}\right\} \qquad N_0,\ r^*,\ \kappa > 0$$

If we interpret the radius r as the DA, then its evolution is given by the inverse of the theorem. The latter is an asymptotic estimate, so we can neglect the factor $\sqrt{r}$:

$$DA(N) = b\left[\ln\frac{N}{N_0}\right]^{-\kappa} \qquad \textbf{Model 2}$$

If we set $N_0 = 1$, and add an asymptotic term inspired by the Kolmogorov-Arnold-Moser theory, we recover the original model for the evolution of DA:

$$DA(N) = D_\infty + b\ln^{-\kappa} N \qquad \textbf{Model 1 (deprecated)}$$

It is possible to make an exact inversion of the Nekhoroshev theorem, using the so-called Lambert-W function. It is the inverse of the product exponential:

$$y = x\,\mathrm{e}^x \quad \Leftrightarrow \quad x = \mathcal{W}(y)$$

It has an infinite set of complex branches; we will need the -1 branch. Then we get another formulation for the evolution of DA:

$$DA(N) = \rho\left[-\mathcal{W}_{-1}\left(-(\mu N)^{-\frac{2}{\kappa}}\right)\right]^{-\kappa} \qquad \textbf{Model 4}$$

where $\rho = b\left(\frac{\kappa}{2}\right)^{-\kappa}$ and $\mu = \frac{8}{7}\sqrt{\frac{6}{\rho}}$. Finally, if one wants to avoid the Lambert-W function, we can approximate it by its asymptotic series expansion, giving:

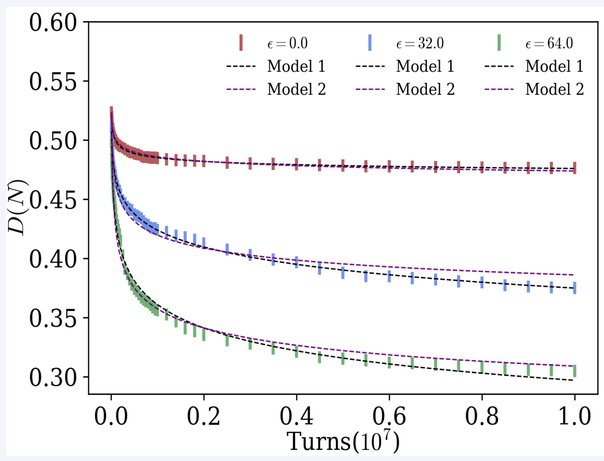
$$DA(N) = b\left[\ln\mu N + \frac{\kappa}{2}\ln\left(\frac{2}{\kappa}\ln\mu N\right)\right]^{-\kappa} \qquad \textbf{Model 3}$$

Fit to toy model: Hénon map

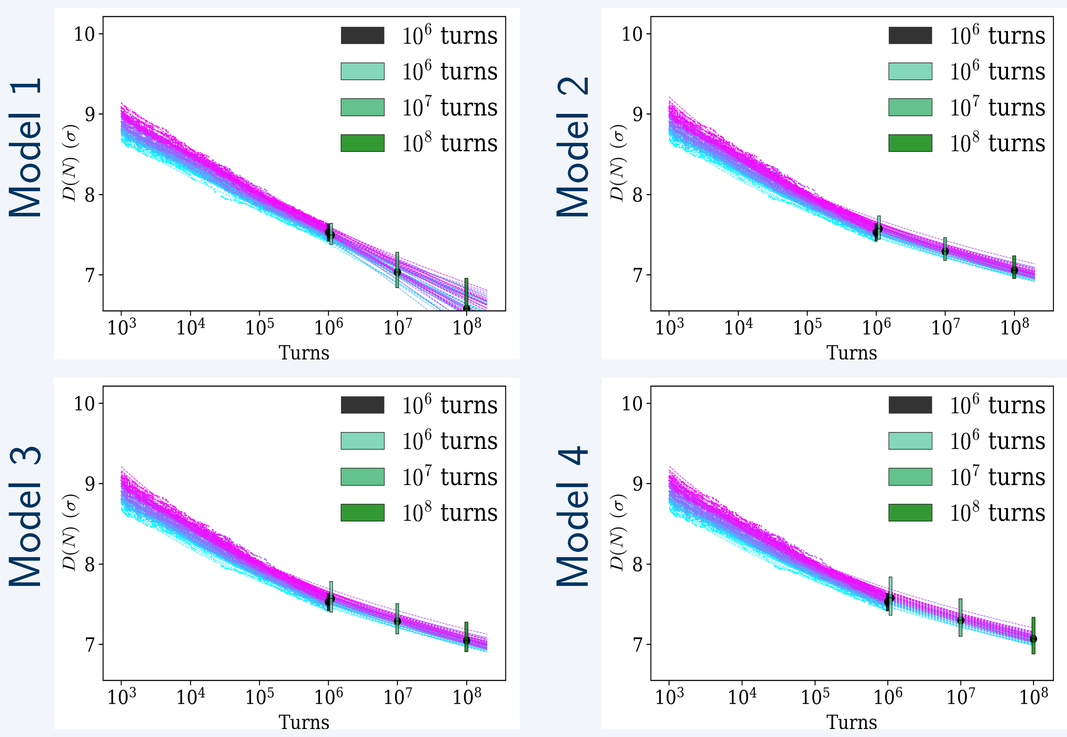
As a first try, we used the modulated 4D Hénon map:

$$\begin{pmatrix} x^{(n+1)} \\ p_x^{(n+1)} \\ y^{(n+1)} \\ p_y^{(n+1)} \end{pmatrix} = \mathbf{L}\begin{pmatrix} x^{(n)} \\ p_x^{(n)} + [x^{(n)}]^2 - [y^{(n)}]^2 \\ y^{(n)} \\ p_y^{(n)} - 2x^{(n)}y^{(n)} \end{pmatrix}$$

where $\mathbf{L}$ is the outer product of two 2D rotations, with frequencies that are perturbed by a small parameter ϵ.



Fit to LHC simulations



Next we tested our models on more relevant data: tracking simulations for the LHC at collision energy. These are done up to 10^6 turns (~ 80s) for 60 random realisations.

DA	10^6	10^7	10^8
Data	7.52	/	/
Model 1	7.495	7.033	6.575
Model 2	7.572	7.292	7.058
Model 3	7.568	7.286	7.048
Model 4	7.57	7.29	7.06

Parameter interdependence

After testing several functions, we observed that there exists a relation between b and κ, given by $b(\kappa) = \alpha\,\mathrm{e}^{\beta\kappa}$ with $\alpha \approx 7.18$ and $\beta \approx 3.07$ for the LHC simulation data. The Nekhoroshev theorem can then be recast as:

$$\frac{N(r)}{\tilde{N}_0} = \sqrt{\frac{r}{\tilde{r}^*}} \exp\left\{\frac{1}{\mathcal{B}}\left(\frac{\tilde{r}^*}{r}\right)^{\frac{1}{\kappa}}\right\}$$

with $\mathcal{B} = \mathrm{e}^{-\beta}$, $\tilde{r}^* = r^*\mathcal{B}$, $\tilde{N}_0 = N_0\sqrt{\mathcal{B}}^{\kappa}$. The four models can then also be redefined accordingly.

Conclusions & impact

- Models 2, 3, and 4 offer good description
- Model 1 gives slightly different results and sometimes unphysical parameters
- Models 2, 3, and 4 can be used to extrapolate within good accuracy.
- Model 1 should hence no longer be used
- It is shown that b has an exponential dependence on κ. This can be used to redefine the models.