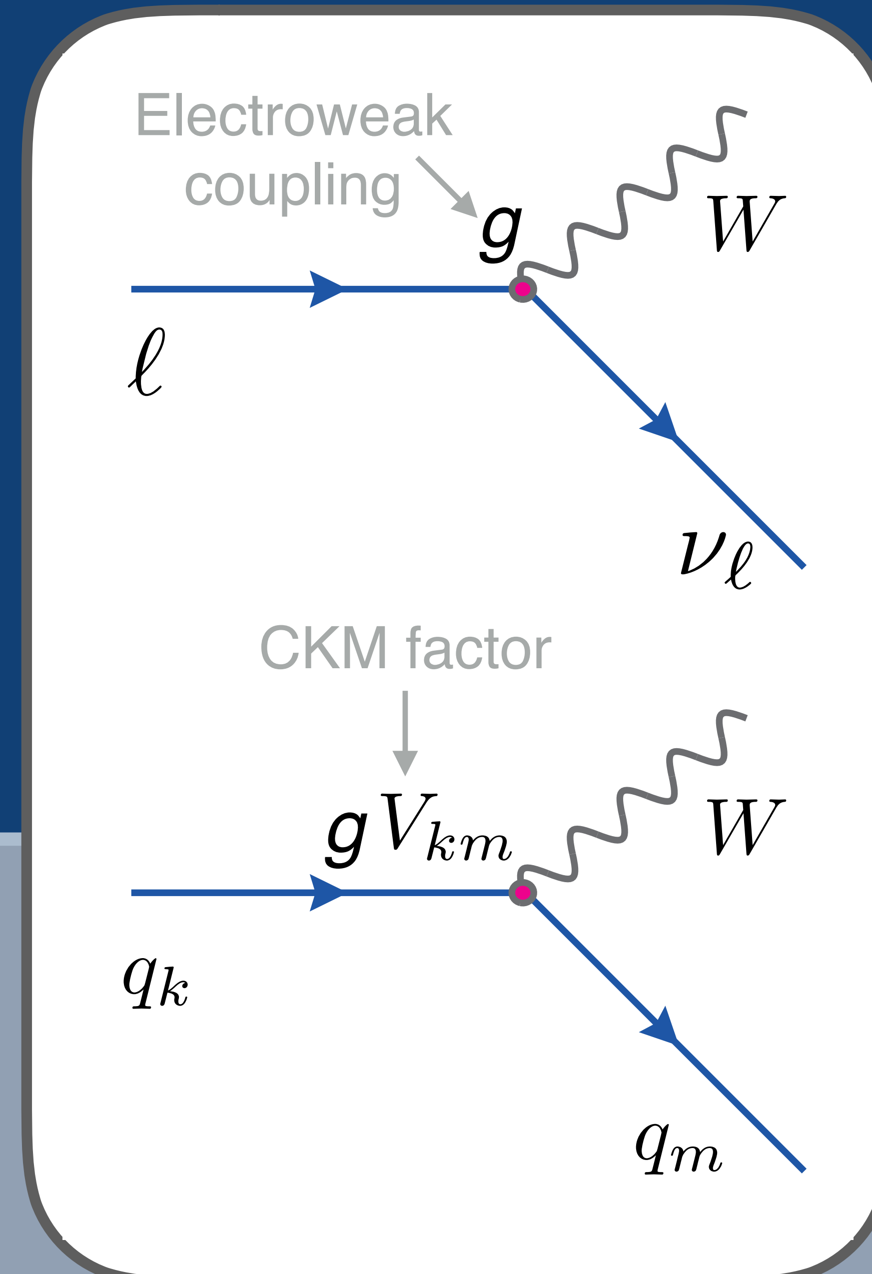
$$V_{\mathrm{CKM}} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & \boxed{V_{cb}} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$$



Nested hypothesis tests and $|V_{cb}|$

with Z. Ligeti, D. Robinson, M. Papucci

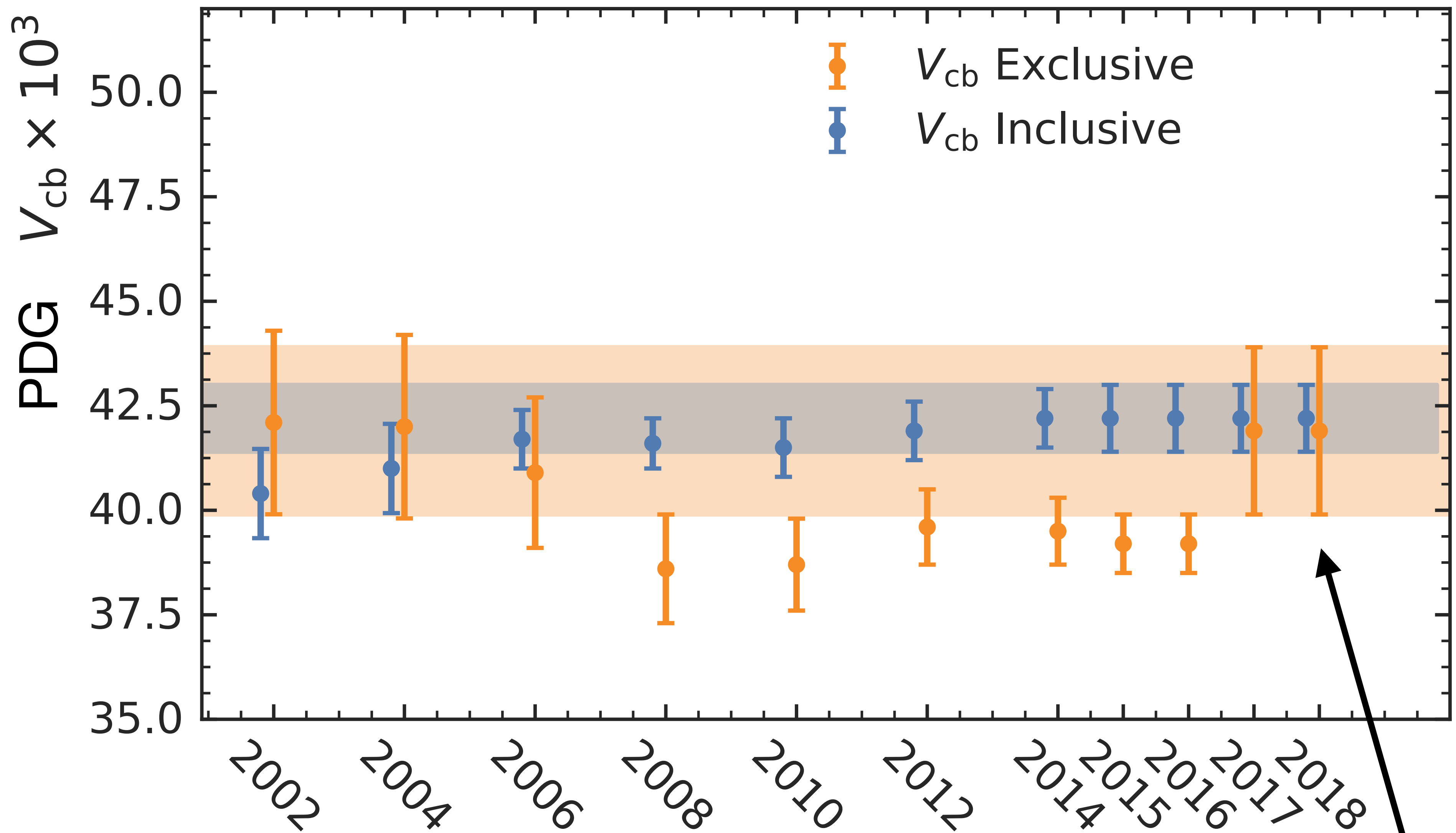
[arXiv:1902.09553, accepted by PRD]

[arXiv:1708.07134, PRD]

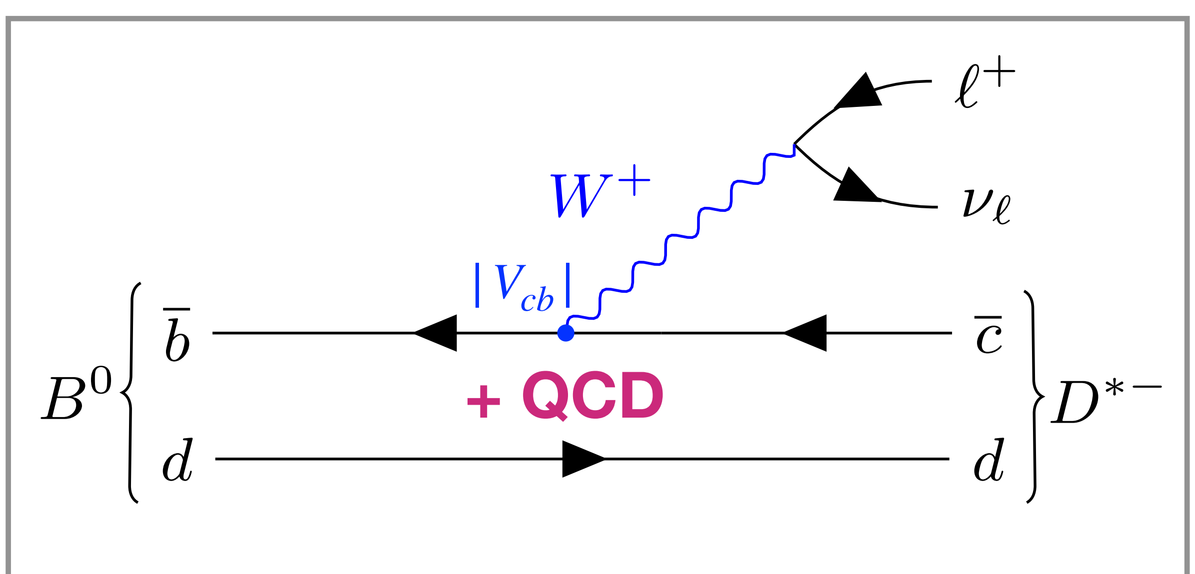
Karlsruher Institut für Technologie

BERKELEY LAB Lawrence Berkeley National Laboratory

Bundesministerium für Bildung und Forschung



Prel. 'Belle Tagged'
[arXiv:1702.01521]

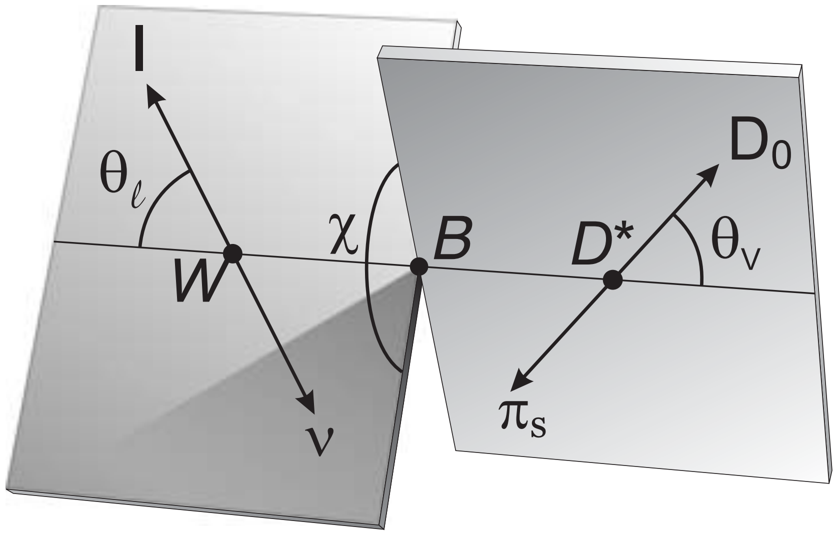


$$|V_{cb}|_{\rm CLN} = (38.2 \pm 1.5) \times 10^{-3}\,,$$
$$|V_{cb}|_{\rm BGL_{332}} = (41.7^{+2.0}_{-2.1}) \times 10^{-3}\,,$$
$$|V_{cb}|_{\rm BGL_{222}} = (41.9^{+2.0}_{-1.9}) \times 10^{-3}\,,$$

The Boyd-Grinstein-Lebed Parametrization

- Decay rate described by 3 form factors
  (in zero lepton mass limit)

$$\langle D^*(\varepsilon,p')|\bar{c}\gamma^\mu b|\bar{B}(p)\rangle = i g \epsilon^{\mu\nu\alpha\beta}\varepsilon^*_\nu p_\alpha p'_\beta,$$
$$\langle D^*(\varepsilon,p')|\bar{c}\gamma^\mu\gamma^5 b|\bar{B}(p)\rangle = f\varepsilon^{*\mu} + (\varepsilon^*\cdot p)[a_+(p+p')^\mu + a_-(p-p')^\mu],$$



- **BGL method:** Expand form factors using dispersion relations & unitarity

$$g(z) = \frac{1}{P_g(z)\phi_g(z)}\sum_{n=0}^{N} a_n z^n, \qquad f(z) = \frac{1}{P_f(z)\phi_f(z)}\sum_{n=0}^{N} b_n z^n, \qquad \mathcal{F}_1(z) = \frac{1}{P_{\mathcal{F}_1}(z)\phi_{\mathcal{F}_1}(z)}\sum_{n=0}^{N} c_n z^n,$$

Combination of f and a_+

Conformal variable z:

$$z = \frac{\sqrt{w+1}-\sqrt{2a}}{\sqrt{w+1}+\sqrt{2a}}$$

QCD encoded in coefficients:

$$\{a_n, b_n, c_n\}$$

$c_0 =$ **constants** $\times\, b_0$

The Problem at a glance

- At what order should you truncate the series?

$$g(z) = \frac{1}{P_g(z)\phi_g(z)} \sum_{n=0}^{N} \boxed{a_n} z^n, \qquad f(z) = \frac{1}{P_f(z)\phi_f(z)} \sum_{n=0}^{N} \boxed{b_n} z^n, \qquad \mathcal{F}_1(z) = \frac{1}{P_{\mathcal{F}_1}(z)\phi_{\mathcal{F}_1}(z)} \sum_{n=0}^{N} \boxed{c_n} z^n,$$

- Truncate too soon:
  - Model dependence in extracted result for $|V_{cb}|$?
- Truncate too late:
  - Unnecessarily increase variance on $|V_{cb}|$?

What is the **ideal** truncation order?

Can get intertwined, as three form factors are involved

Careful: [arXiv:1905.08209, PLB] introduced an identical notation, but with another meaning!

Our Notation

$$\left\{ a_{0,\ldots,n_a-1}, b_{0,\ldots,n_b-1}, c_{1,\ldots,n_c} \right\} \downarrow \mathbf{BGL}_{n_a\, n_b\, n_c}$$

Different approaches on the market

This work
[arXiv:1902.09553, accepted by PRD]

Use a **nested hypothesis test**
to determine optimal truncation order

Challenge nested fits

$\mathbf{BGL}_{n_a+1,n_b,n_c}$

$\mathbf{BGL}_{n_a,n_b,n_c}$ $\leftrightarrow$ $\mathbf{BGL}_{n_a,n_b+1,n_c}$

$\mathbf{BGL}_{n_a,n_b,n_c+1}$

Test statistics & Decision boundary

$\Delta\chi^2 = \chi^2_N - \chi^2_{N+1}$ $\qquad$ $\Delta\chi^2 > 1$

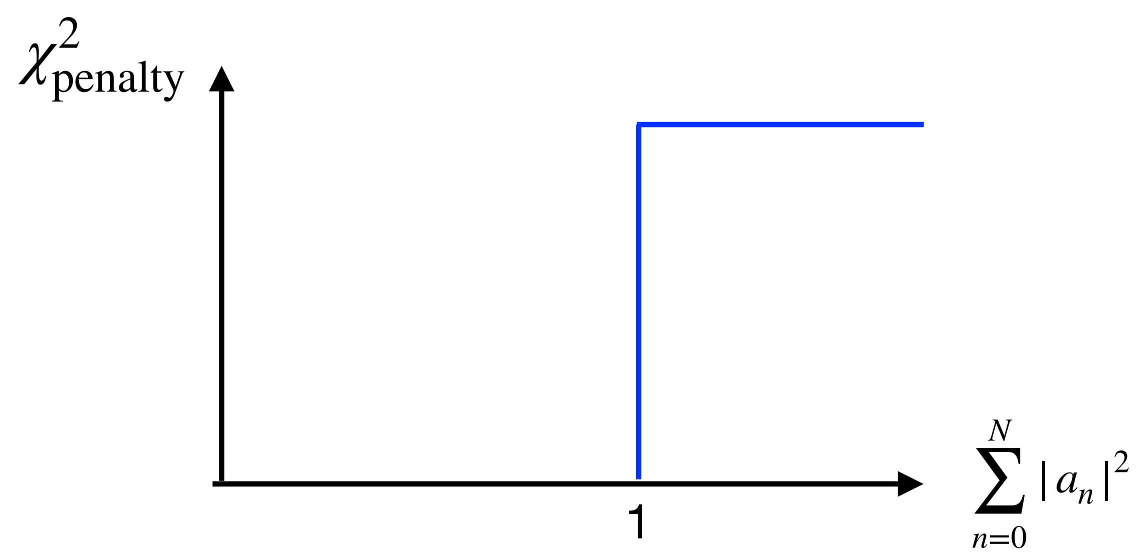
Distributed like a χ^2-distribution with 1 dof
(Wilk's theorem)

Gambino, Jung, Schacht
[arXiv:1905.08209, PLB]

Constrain contributions
from higher order coefficients
using **unitarity bounds**

$$\sum_{n=0}^{N} |a_n|^2 \leq 1 \qquad \sum_{n=0}^{N} \left(|b_n|^2 + |c_n|^2 \right) \leq 1$$

e.g. $\qquad \chi^2 \to \chi^2 + \chi^2_{\text{penalty}}$



Different approaches on the market

This work
[arXiv:1902.09553, accepted by PRD]

Use a **nested hypothesis test** to determine optimal truncation order

Challenge nested fits

$$\mathbf{BGL}_{n_a,n_b,n_c} \leftrightarrow \mathbf{BGL}_{n_a+1,n_b,n_c}$$
$$\mathbf{BGL}_{n_a,n_b,n_c} \leftrightarrow \mathbf{BGL}_{n_a,n_b+1,n_c}$$
$$\mathbf{BGL}_{n_a,n_b,n_c} \leftrightarrow \mathbf{BGL}_{n_a,n_b,n_c+1}$$

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$$\Delta\chi^2 = \chi^2_N - \chi^2_{N+1} \qquad \Delta\chi^2 > 1$$

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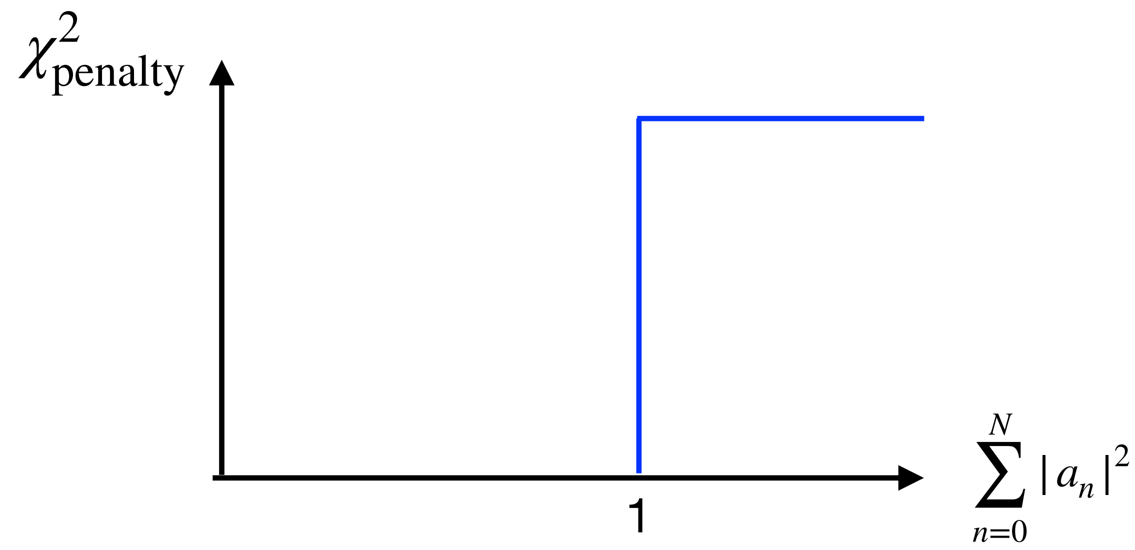

Nesting procedure

Steps:

1. Carry out nested fits with one parameter added
2. Accept descendant over parent fit, if $\Delta\chi^2 > 1$
3. Repeat 1 and 2 until you find **stationary** points
4. If multiple **stationary** points remain, choose the one with smallest N, then smallest χ^2

$\textbf{BGL}_{n_a,n_b,n_c}$

$\textbf{BGL}_{n_a+1,n_b,n_c}$

$\textbf{BGL}_{n_a,n_b+1,n_c}$

$\textbf{BGL}_{n_a,n_b,n_c+1}$

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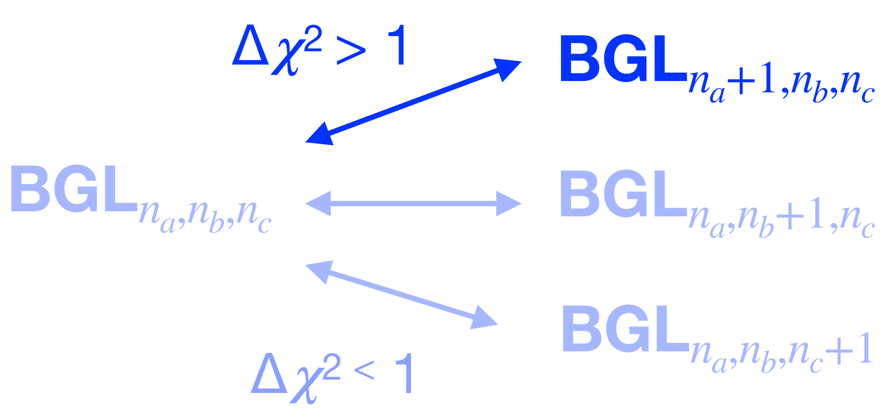


Nesting procedure

Steps:

1 Carry out nested fits with one parameter added

2 Accept descendant over parent fit, if $\Delta\chi^2 > 1$

3 Repeat 1 and 2 until you find **stationary** points

4 If multiple **stationary** points remain, choose the one with smallest N, then smallest χ^2

$\mathbf{BGL}_{n_a+1,n_b,n_c}$

stationary

$\mathbf{BGL}_{n_a+2,n_b,n_c}$

$\mathbf{BGL}_{n_a+1,n_b+1,n_c}$

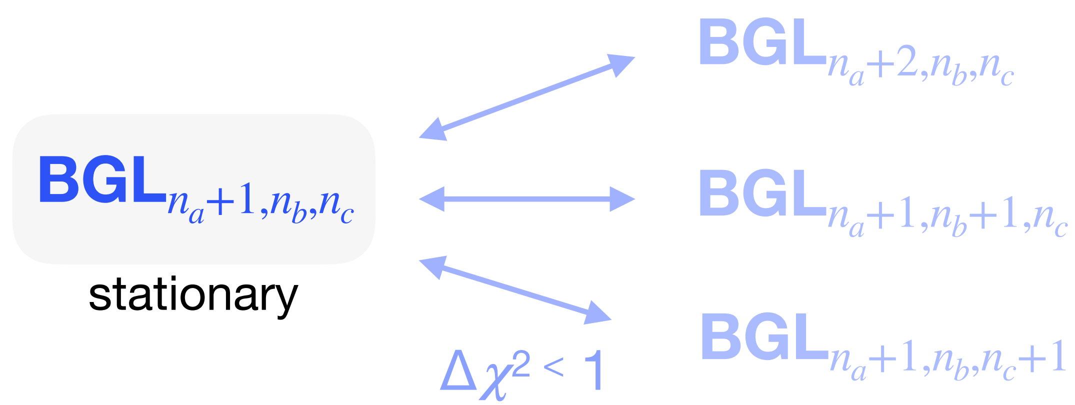
$\mathbf{BGL}_{n_a+1,n_b,n_c+1}$

$\Delta\chi^2 < 1$

Nesting procedure

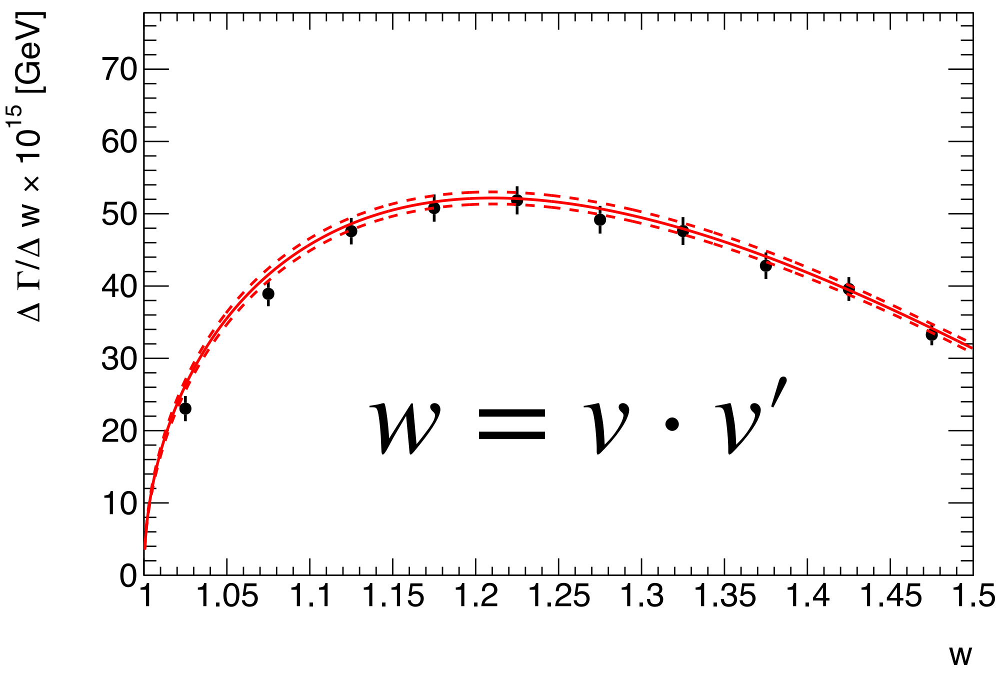
Steps:

1. Carry out nested fits with one parameter added
2. Accept descendant over parent fit, if $\Delta\chi^2 > 1$
3. Repeat 1 and 2 until you find **stationary** points
4. If multiple **stationary** points remain, choose the one with smallest N, then smallest χ^2



Applied to Belle Tagged result

[arXiv:1702.01521]



4 x 1D projections of kinematic variables + correlations

Unfolded for detector effects and migrations

Tagged Belle Measurement:

n_c \ n_a	1	2	3	1	2	3	1	2	3
1	33.2 38.7 ± 1.1	31.6 38.6 ± 1.1	31.2 38.7 ± 1.1	33.0 39.1 ± 1.6	29.1 40.8 ± 1.7	28.9 40.8 ± 1.7	30.4 40.8 ± 1.9	29.1 40.7 ± 1.9	28.9 40.7 ± 1.9
2	32.9 38.9 ± 1.2	31.3 38.8 ± 1.2	31.1 38.9 ± 1.2	32.7 39.6 ± 1.7	27.7 41.7 ± 1.9	27.7 41.7 ± 1.9	29.2 41.9 ± 2.1	27.7 41.8 ± 2.1	27.7 41.8 ± 2.1
3	31.7 39.1 ± 1.2	31.3 38.7 ± 1.4	31.0 38.7 ± 1.3	29.1 42.0 ± 2.1	27.7 41.9 ± 2.1	27.7 41.8 ± 2.1	29.2 41.9 ± 1.9	27.6 41.8 ± 2.0	23.2 41.5 ± 2.1
	$n_b = 1$			$n_b = 2$			$n_b = 3$		

$\boxed{\textbf{BGL}_{111}} \rightarrow \boxed{\textbf{BGL}_{211}} \rightarrow \boxed{\textbf{BGL}_{221}} \rightarrow \boxed{\textbf{BGL}_{222}}$ stationary

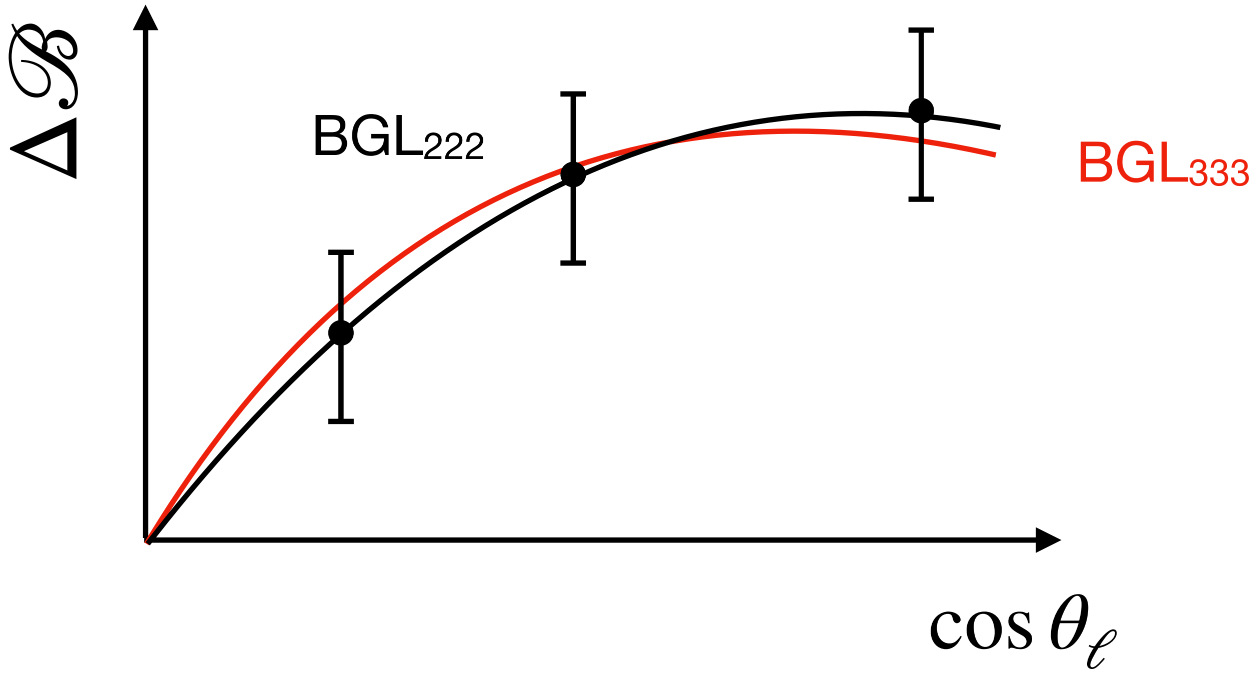
Toy study to illustrate the possible bias

Use the central values of the **$\mathbf{BGL_{222}}$ fit** as a starting point to add **fine structure**

	‘1-times’	’10-times’
Parameter	Value $\times\ 10^2$	Value $\times\ 10^2$
$\tilde{a}_2$	2.6954	26.954
$\tilde{b}_2$	-0.2040	-2.040
$\tilde{c}_3$	0.5350	5.350

Create a “true” higher order Hypothesis of order **$\mathbf{BGL_{333}}$**

Has fine structure element the **current data cannot resolve**



Toy study to illustrate the possible bias

Use the central values of the **BGL_{222} fit** as a starting point to add **fine structure**

	'1-times'	'10-times'
Parameter	Value $\times 10^2$	Value $\times 10^2$
$\tilde{a}_2$	2.6954	26.954
$\tilde{b}_2$	−0.2040	−2.040
$\tilde{c}_3$	0.5350	5.350

Create a "true" higher order Hypothesis of order **BGL_{333}**

Has fine structure element the **current data cannot resolve**

Toy Test

Produce **ensemble** of toy measurements using **untagged covariance** & **BGL_{333}** central values

Each toy is fitted to build the descendant tree and carry out a **nested hypo. test** to select its preferred **$BGLn_an_bn_c$**

Construct Pulls

$$\textbf{Pull} = \frac{\left|V_{cb}\right|_{\textbf{true}} - \left|V_{cb}\right|_{\textbf{toy}}}{\Delta \left|V_{cb}\right|_{\textbf{toy}}}$$

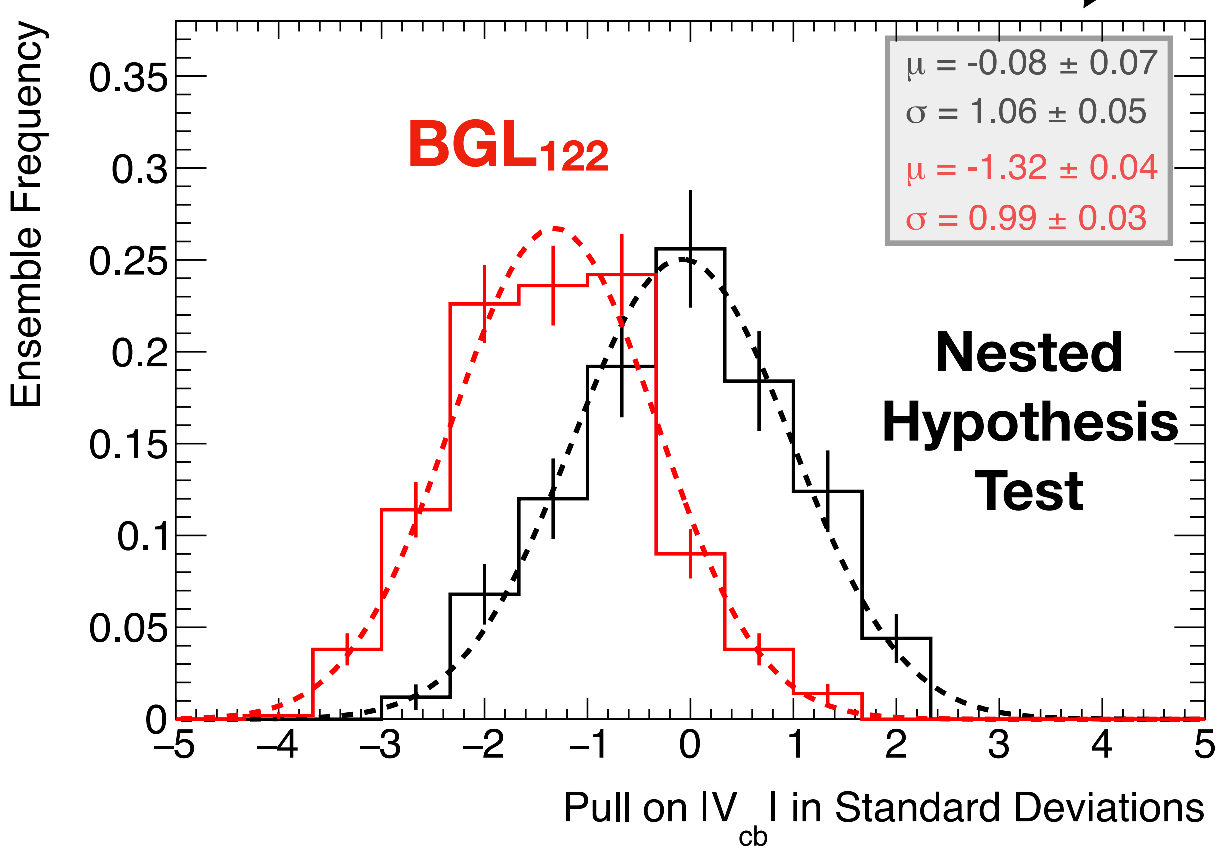
As calculated from selected $BGLn_an_bn_c$ fit of each toy

If methodology unbiased, should follow a standard normal distribution (mean 0, width 1)

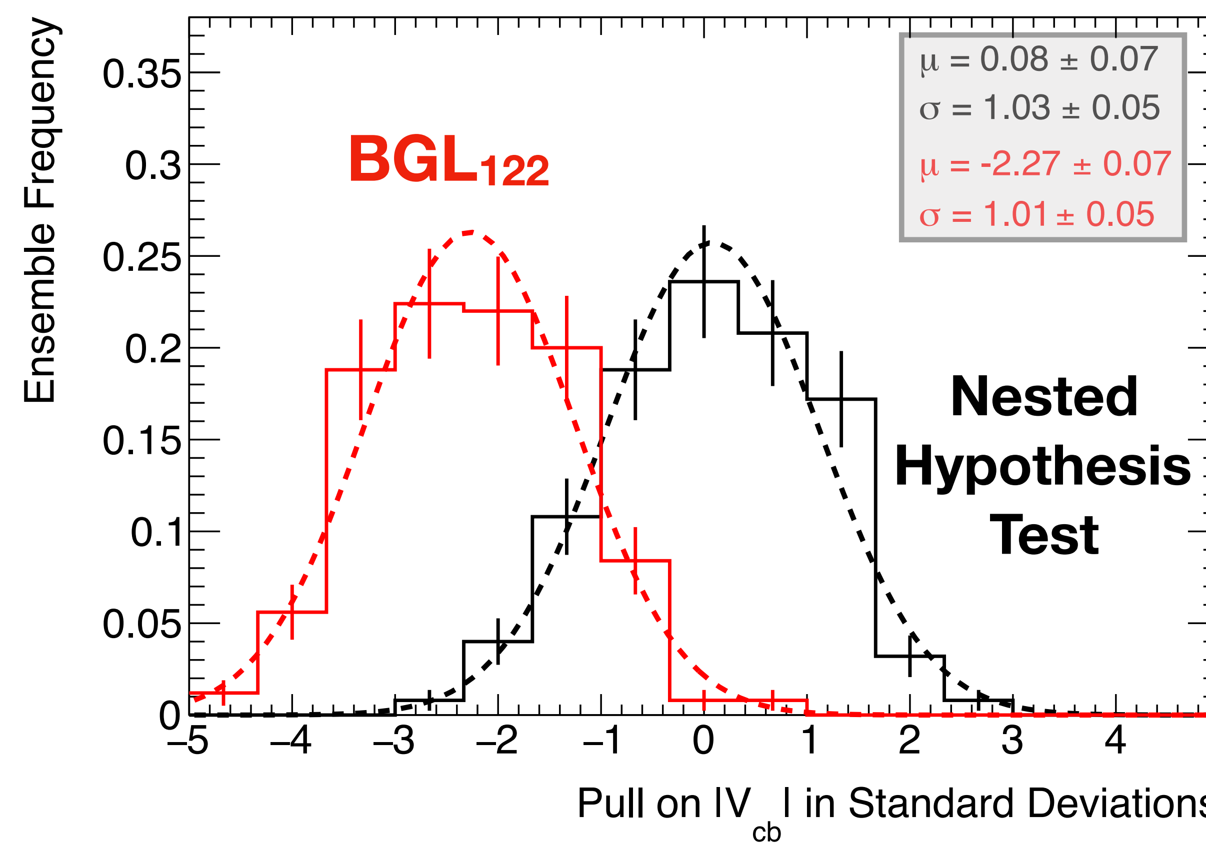
Pull Results

Mean and Variance of Pulls

‘1-times’



’10-times’

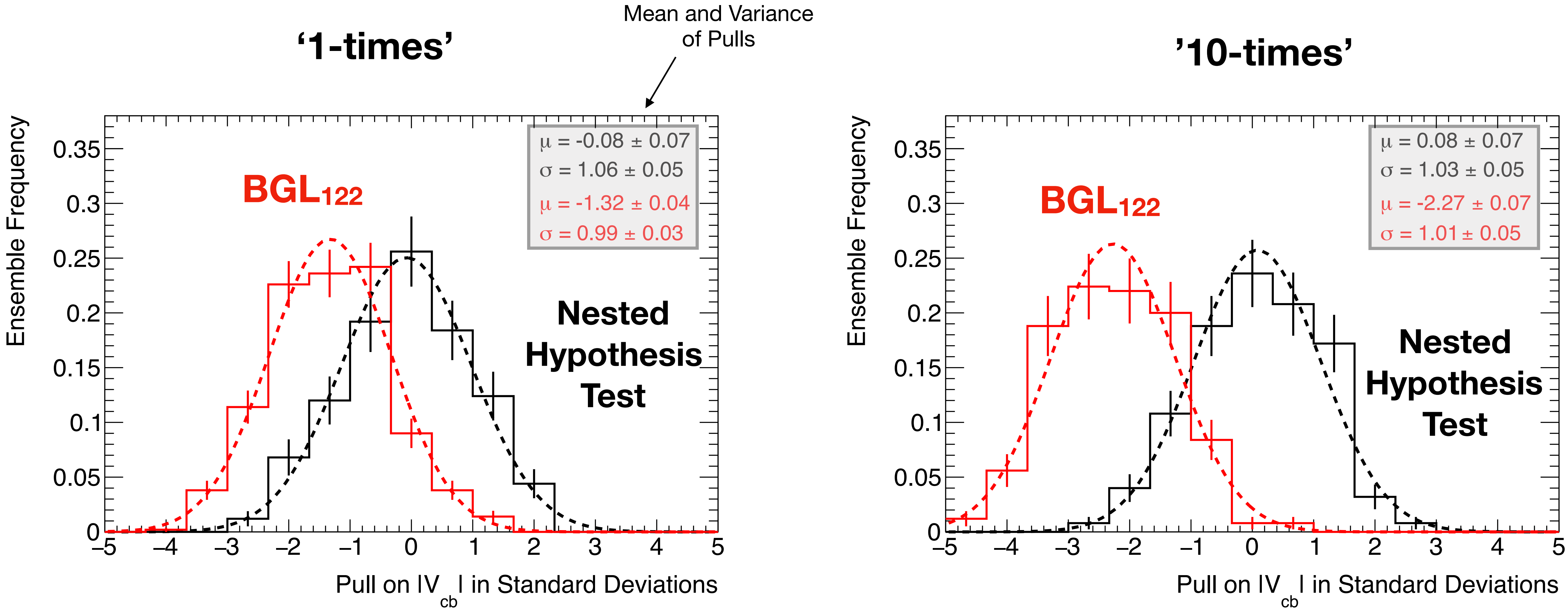


→ Procedure produces **unbiased** $|V_{cb}|$ values, just picking a hypothesis (BGL_{122}) does not

Relative Frequency of selected Hypothesis:

	BGL_{122}	BGL_{212}	BGL_{221}	BGL_{222}	BGL_{223}	BGL_{232}	BGL_{322}	BGL_{233}	BGL_{323}	BGL_{332}	BGL_{333}
1-times	6%	0%	37%	27%	6%	6%	11%	0%	2%	4%	0.4%
10-times	0%	0%	8%	38%	14%	8%	16%	3%	4%	8%	1%

Pull Results

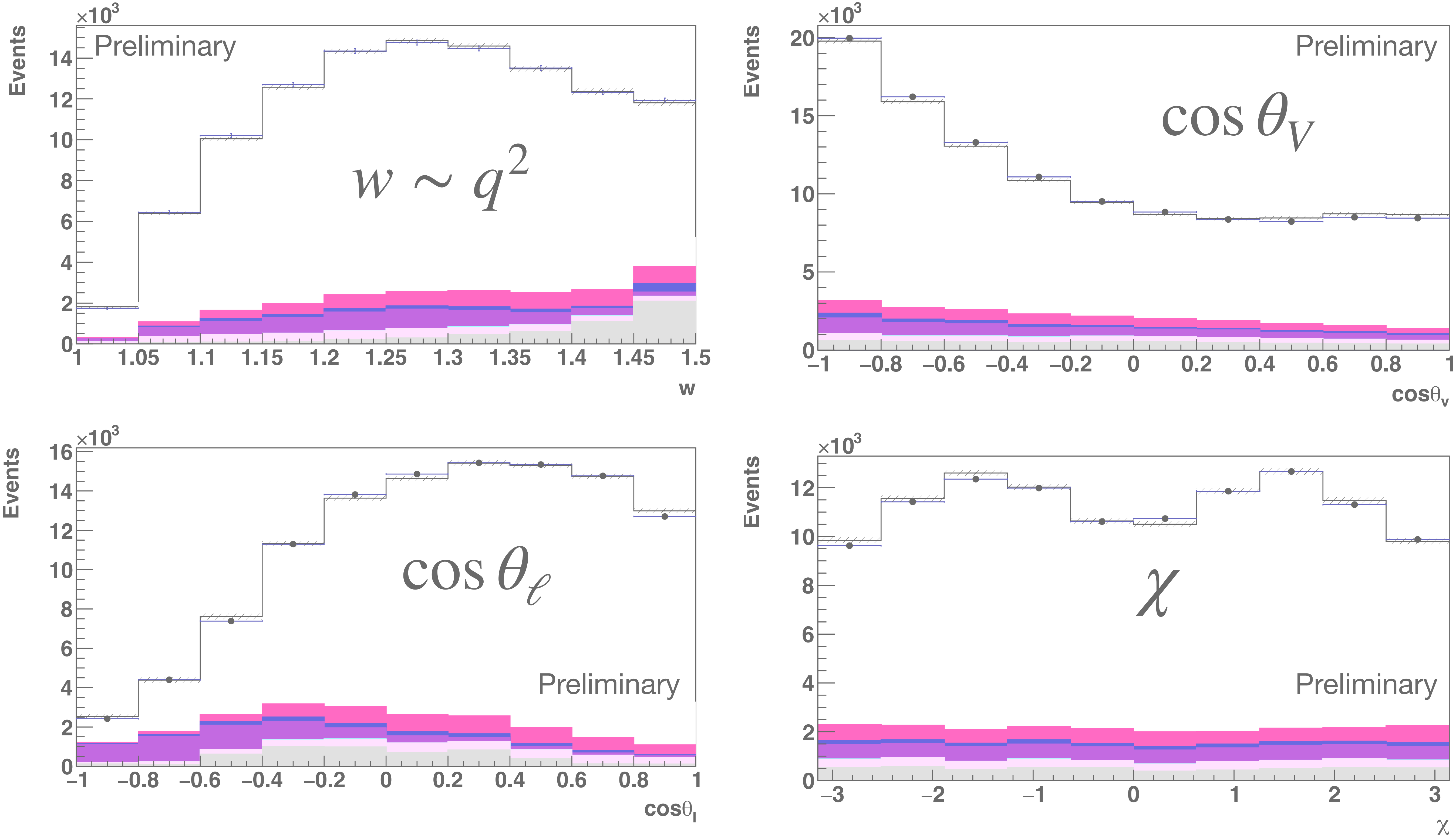


→ Procedure produces **unbiased** $|V_{cb}|$ values, just picking a hypothesis (BGL_{122}) does not

Relative Frequency of selected Hypothesis:

	BGL_{122}	BGL_{212}	BGL_{221}	BGL_{222}	BGL_{223}	BGL_{232}	BGL_{322}	BGL_{233}	BGL_{323}	BGL_{332}	BGL_{333}
1-times	6%	0%	37%	27%	6%	6%	11%	0%	2%	4%	0.4%
10-times	0%	0%	8%	38%	14%	8%	16%	3%	4%	8%	1%

Throwing the New Untagged Belle result into the mix



Unfolding the Untagged fit result

Measurement provides migration matrices and acceptance, so one can unfold the measured signal yields via

"True" Yield

Migration matrix

$$\chi^2 = \left(\mathbf{N}_{\mathrm{true}}\, \epsilon\, M - \mathbf{N}_{\mathrm{reco}}\right) C_{\mathrm{stat}}^{-1} \left(\mathbf{N}_{\mathrm{true}}\, \epsilon\, M - \mathbf{N}_{\mathrm{reco}}\right)$$

Acceptance / Efficiency matrix

statistical Covariance

recorded Signal events

Incorporating Systematic Uncertainties: $\sigma_{\mathbf{stat}} \sim \sigma_{\mathbf{syst}}$ $\longrightarrow$

See also Discussion by Gambino, Jung, Schacht in [arXiv:1905.08209, PLB] about this

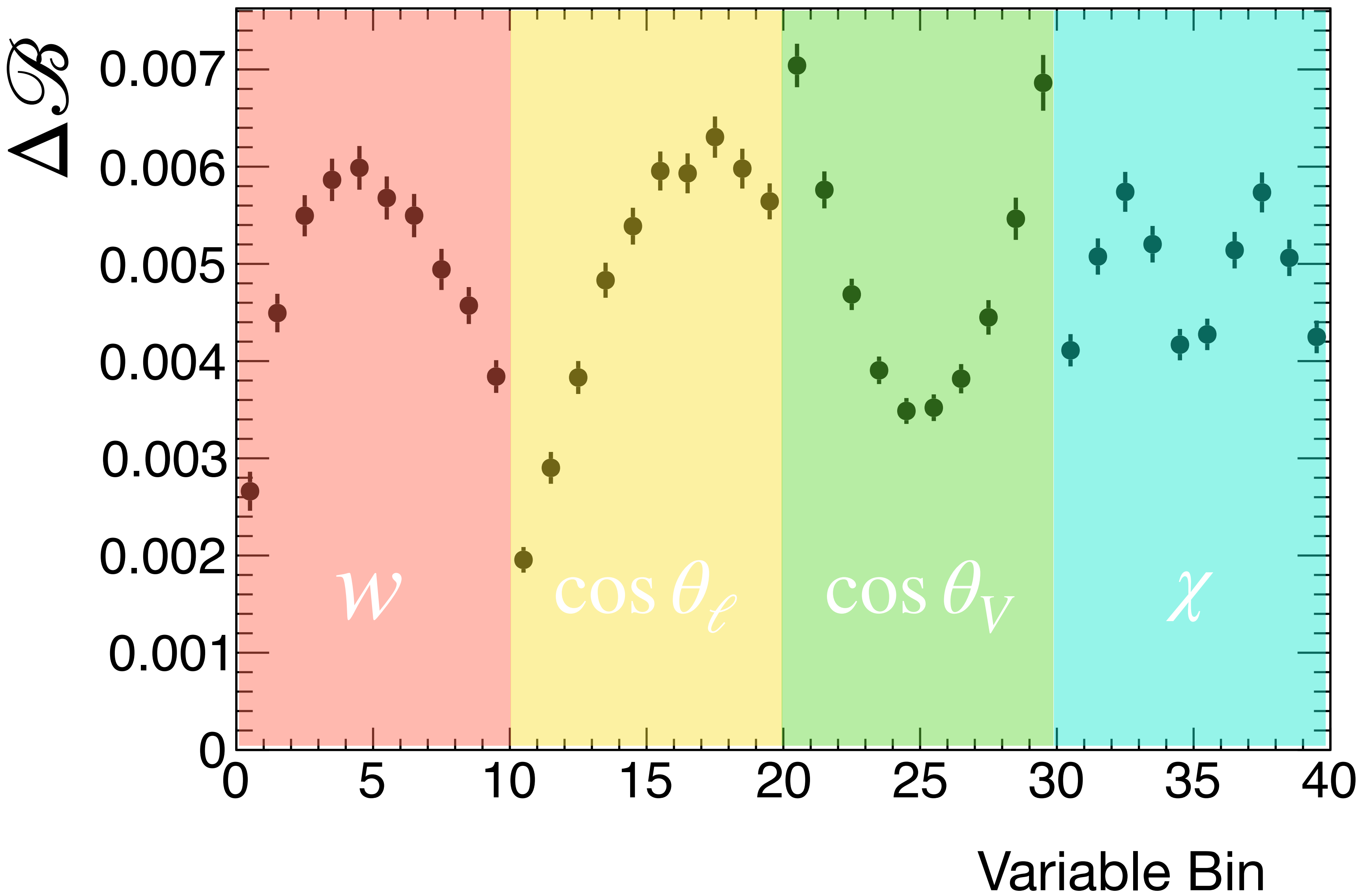
→ Measurement only provides relative errors, thus one has to be a bit careful here (**d'Agostini bias**)

Gaussian Constraint on Systematic Nuisance Parameter

$$\chi^2 \to \chi^2 + \sum_i \Theta_i^2 \qquad \mathbf{N}_{\mathrm{true}}{}^{j} \to \mathbf{N}_{\mathrm{true}}{}^{j} \prod_i \left(1 + \Theta_i \varepsilon_{ij}\right)$$

relative Error vector of a given source *i*

Unfolded result

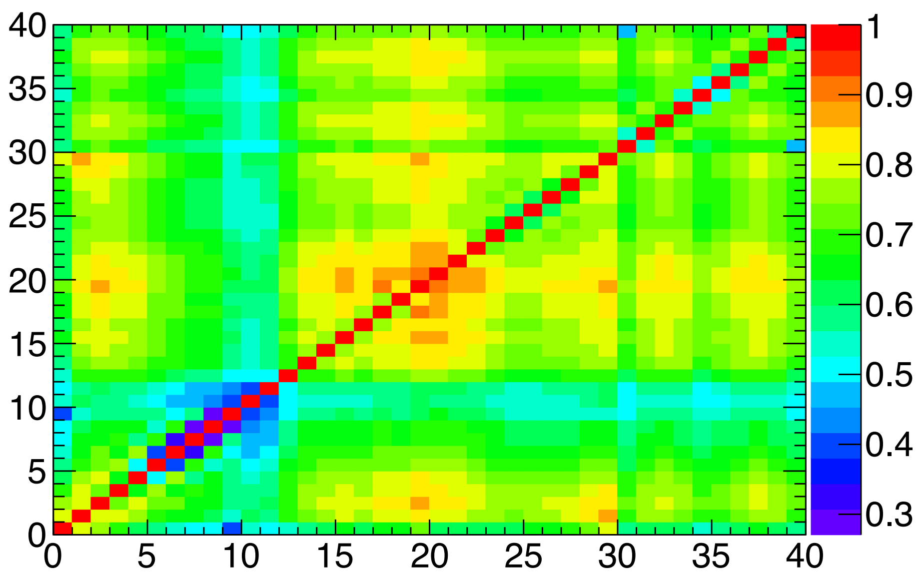


For some sources it would be necessary to know the correlation between bins (as they have a stat. component)

↓

Have to make the assumption, that neighbouring bins are fully correlated

Correlations



Preliminary New Untagged + Tagged Result for $|V_{cb}|$

Untagged + Tagged Belle Measurement:

n_c \ n_a	1	2	3	1	2	3	1	2	3
1	99.4 40.0 ± 0.7	99.1 40.1 ± 0.7	95.0 40.0 ± 0.7	98.1 39.8 ± 0.7	97.8 39.5 ± 0.9	94.4 39.5 ± 0.9	97.1 39.1 ± 0.9	97.0 39.1 ± 0.9	90.5 40.4 ± 1.0
2	95.7 39.8 ± 0.7	93.4 39.9 ± 0.7	93.4 39.9 ± 0.7	91.8 39.2 ± 0.7	91.8 39.1 ± 1.0	91.7 39.0 ± 1.0	91.7 39.0 ± 0.9	91.6 39.0 ± 1.0	85.7 39.7 ± 1.1
3	93.8 39.9 ± 0.7	93.4 39.8 ± 0.7	93.3 39.8 ± 0.8	91.7 39.0 ± 1.0	91.5 39.0 ± 1.1	90.6 38.4 ± 0.8	91.7 39.0 ± 0.8	90.0 38.4 ± 1.0	90.8 38.4 ± 1.1
	$n_b = 1$			$n_b = 2$			$n_b = 3$		

$\mathbf{BGL}_{111} \to \mathbf{BGL}_{112} \to \mathbf{BGL}_{212} \to \mathbf{BGL}_{222}$

$\mathbf{BGL}_{112} \to \mathbf{BGL}_{122}$ stationary

$\mathbf{BGL}_{112} \to \mathbf{BGL}_{113} \to \mathbf{BGL}_{123}$

Preliminary

$\mathbf{BGL}_{122}$ selected

$$\left|V_{cb}\right| = (39.2 \pm 0.7) \times 10^{-3}$$

Conclusions

- **Nested hypothesis tests** can determine the necessary truncation order in an unbiased way
  - Good **alternative** to **theory motivated priors**
  - Avoids overconstraining higher order coefficients in BGL expansion
    - These in turn might violate unitarity, but a priori not a conceptual problem (nature is unitary, i.e. a prior might introduce its own bias)
  - Tested “unbiasedness” of procedure via ensembles of pseudo-experiments (toys)

- **Preliminary combination of untagged and tagged measurements:**

Preliminary average of tagged & untagged meas.

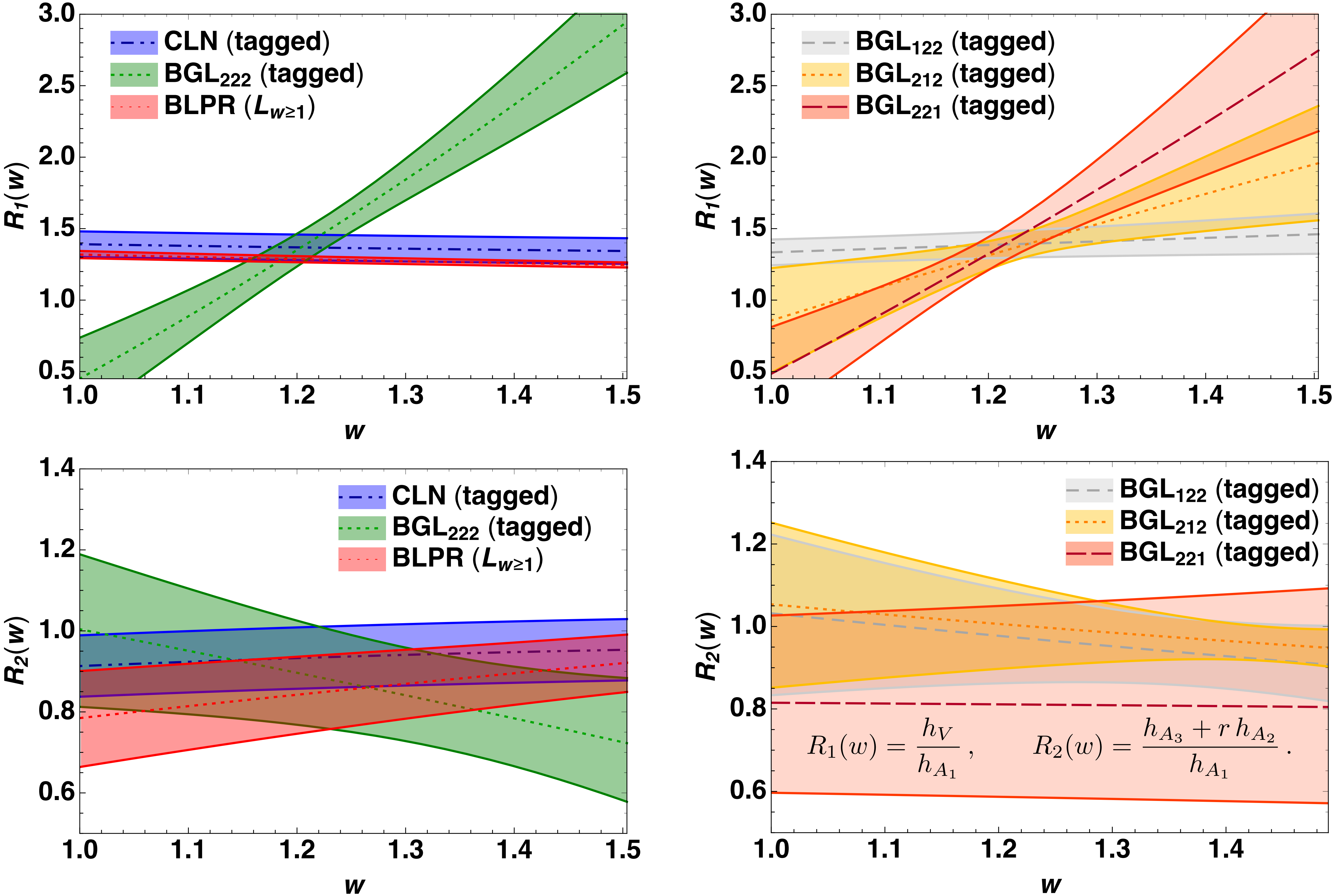
$$\left|V_{cb}\right| = (39.2 \pm 0.7) \times 10^{-3}$$

Inquired about correct covariance for Lepton ID systematics; stay tuned

→ **Plan to updated R(D/D*) predictions [arXiv:1703.05330]**

Backup

Consistency with Heavy Quark symmetry



$$R_1(w) = \frac{h_V}{h_{A_1}}\,, \qquad R_2(w) = \frac{h_{A_3} + r\,h_{A_2}}{h_{A_1}}\,.$$