Quark Correlations in the CGC

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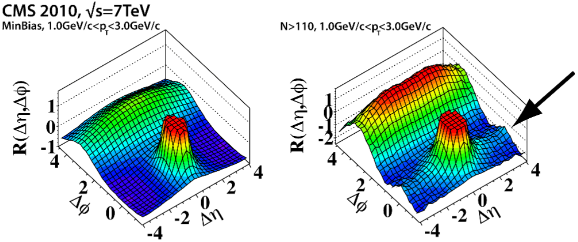
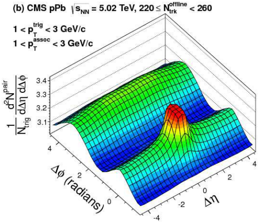
Motivation: The Ridge Structure





Particle Correlations in the CGC - II

Idea: Final state correlations carry the imprint of the partonic correlations that exist in the initial state:

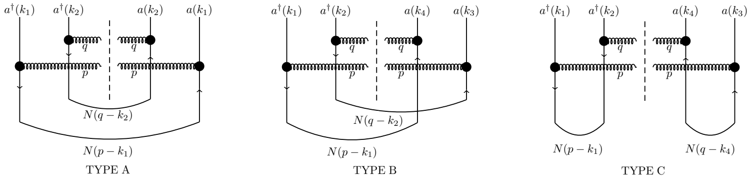
1. Local anisotropy of target fields
   - A. Kovner, M. Lublinsky, Phys.Rev. D83 (2011) 034017
   - A. Kovner, M. Lublinsky, Phys.Rev. D84 (2011) 094011
   - A. Kovner, M. Lublinsky, Int. J. Mod. Phys. E Vol. 22 (2013) 1330001
   - A. Dumitru, L. McLerran, V. Skokov, Phys. Lett. B 743 (2015)
   - A. Dumitru, V. Skokov, Phys.Rev. D91 (2015) 074006
2. Spatial variation of partonic density
   - E. Levin, A. Rezaeian, Phys.Rev. D84 (2011) 034031
3. Recent work: elliptic flow from color dipole orientation
   - E. Iancu, A. Rezaeian, Phys.Rev. D95 (2017) no.9, 094003
4. "Glasma Graph" contributions to particle production:
   - A. Dumitru, F. Gelis, J. Jalilian-Marian, T. Lappi, Phys.Lett. B697 (2011) 21
   - K. Dusling, R. Venugopalan, Phys.Rev.Lett. 108 (2012) 262001
   - K. Dusling, R. Venugopalan, Phys.Rev. D87 (2013) 051502
   - K. Dusling, R. Venugopalan, Phys.Rev. D87 (2013) 054014

WHAT IS THE PHYSICS BEHIND THE GLASMA GRAPH CALCULATIONS?

BOSE ENHANCEMENT of gluons in the hadronic wave function!

[T. A., N. Armesto, G. Beuf, A. Kovner, M. Lublinsky, Phys.Lett. B751 (2015) 448-452]

Consider inclusive two particle production and assume parton-hadron duality.



$N(k) \equiv$dipole scattering probability : $N(k) = -\int d^2x\, e^{i\mathbf{k}\cdot\mathbf{x}} \left\langle \frac{1}{N_c} tr\left[S^\dagger(x)S(0)\right]\right\rangle_T$

$$\text{Type A} \propto \int_{k_1,k_2} \underbrace{\langle in|a_a^{\dagger i}(k_1)a_b^{\dagger j}(k_2)a_a^k(k_1)a_b^l(k_2)|in\rangle_P}_{D(k_1,k_2)} N(p-k_1)\,N(q-k_2)$$

CGC hadronic wave-function is boost invariant $\Rightarrow$ $a_a^i(k) \equiv \frac{1}{\sqrt{Y}}\int_{|\eta<Y/2|}\frac{d\eta}{2\pi}\, a_a^i(\eta,k).$

$$D(k_1,k_2) = S^2(N_c^2-1)^2\,\frac{k_1^i k_1^k k_2^j k_2^l}{k_1^2 k_2^2}\,\frac{g^4\mu^2(k_1)\mu^2(k_2)}{k_1^2k_2^2}\left\{1+\frac{1}{S(N_c^2-1)}\left[\delta^{(2)}(k_1-k_2)+\delta(2)(k_1+k_2)\right]\right\}$$

The first term is the "classical" term (the square of the number of particles)

The second term is the typical Bose enhancement term!

What about quark correlations: Pauli blocking?

Are quarks in the CGC state subject to correlations?

Within the "glasma graph" approach:

- gluons:
  - final state correlations are due to the Bose enhancement of the gluons in the projectile wave function.
  - This effect is long range in rapidity since the CGC wave function is dominated by the rapidity integrated mode of the soft gluon field.
- quarks:
  - quarks should experience Pauli blocking $\Rightarrow$ the probability to find two identical quarks with the same quantum numbers in the CGC state should be supressed.
  - Is the effect short or long range in rapidity?

Quark contribution to the wave function - I

The free part of the Light Cone Hamiltonian:

$$H_0 = \int_{k^+,k} \frac{k^2}{2k^+}\ a_i^{\dagger a}(\underline{k})\ a_i^a(\underline{k}) + \sum_s \int_{p^+,p} \frac{p^2}{2p^+}\left[d^\dagger_{\alpha\, s}(\underline{p})\, d_{\alpha\, s}(\underline{p}) + \bar{d}^\dagger_{\alpha\, s}(\underline{p})\, \bar{d}_{\alpha\, s}(\underline{p})\right]$$

The full Hamiltonian contains several types of perturbations:

$$\delta H = \delta H^{\rho} + \delta H^{g\, qq} + \cdots$$

Interaction with the background field: $\delta H^{\rho} = \delta H^{\rho\, g} + \delta H^{\rho\, qq} + \delta H^{\rho\, gg}$

Quark-gluon interaction: $\delta H^{g\, qq}$

the relevant matrix elements: $\langle g|\delta H^{\rho\, g}|0\rangle, \quad \langle q\bar{q}|\delta H^{\rho qq}|0\rangle, \quad \langle q\bar{q}|\delta H^{g\, qq}|g\rangle$

With all these ingredients one can calculate the dressed quark wave function:

$$|v\rangle_4^D = g^4 \int_{\underline{p},\underline{\bar{p}},\underline{q},\underline{\bar{q}}}\left[\zeta^{\epsilon\iota}_{s_1 s_2}(k^+,p,\bar{p},\alpha)\zeta^{\gamma\delta}_{r_1 r_2}(\bar{k}^+,q,\bar{q},\beta)\ d^\dagger_{\epsilon,s_1}(\underline{p})\,\bar{d}^\dagger_{\iota,s_2}(\underline{\bar{p}})\ d^\dagger_{\gamma,r_1}(\underline{q})\,\bar{d}^\dagger_{\delta,r_2}(\underline{\bar{q}})\right]|v\rangle$$

The amplitude $\zeta^{\gamma\delta}_{s_1 s_2}(k^+,p,q,\alpha) = \frac{\tau^a_{\gamma\delta}}{k^+}\int_k \rho^a(k)\ \phi_{s_1 s_2}(k,p,q;\alpha)$

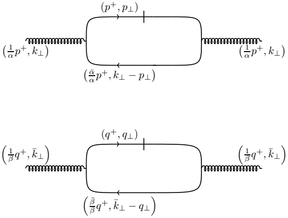
Average number of quark pairs in the wave function - I

Average number of quark pairs in the wave function is defined as

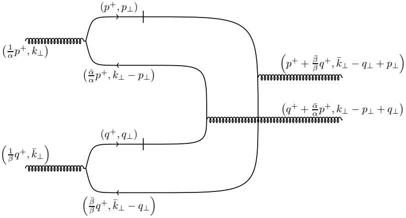
$$\frac{dN}{dp^+ d^2p\, dq^+ d^2q} = \frac{1}{(2\pi)^6} \left\langle {}_4^D\langle v| d^\dagger_{\alpha,s_1}(p^+,p) d^\dagger_{\beta,s_2}(q^+,q) d_{\alpha,s_1}(p^+,p) d_{\beta,s_2}(q^+,q) |v\rangle_4^D \right\rangle_P$$

- use explicit expression for $|v\rangle_4^D$:

$$\frac{dN}{d\eta_1 d^2p\, d\eta_2 d^2q} = \frac{1}{(2\pi)^4} g^8 \int d^2k\, d^2\bar{k}\, d^2l\, d^2\bar{l}\, \langle \rho^a(k)\rho^c(\bar{k})\rho^b(l)\rho^d(\bar{l})\rangle_P$$
$$\times \Big\{ \mathrm{tr}(\tau^a\tau^b)\,\mathrm{tr}(\tau^c\tau^d)\Phi_2(k,l;p)\Phi_2(\bar{k},\bar{l};q) - \mathrm{tr}(\tau^a\tau^b\tau^c\tau^d)\Phi_4(k,l,\bar{k},\bar{l};p,q) \Big\}$$



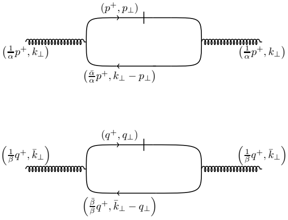
$\Phi_2\Phi_2$



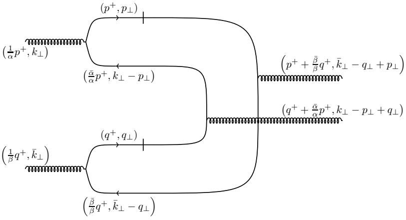
Φ_4

Average number of quark pairs in the wave function - II

Two different contributions to the wave function:



$\Phi_2\Phi_2$



Φ_4

The amplitudes:

$$\Phi_2(k,p) \equiv \int_0^1 d\alpha \int_q \sum_{s_1\, s_2} \phi_{s_1,s_2}(k,p,q;\alpha)\, \phi^*_{s_1,s_2}(k,p,q;\alpha)$$

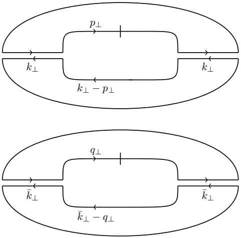
$$\Phi_4(k,l,\bar{k},\bar{l};p,q) \equiv \sum_{s_1\, s_2, \bar{s}_1, \bar{s}_2} \int_0^1 \frac{d\alpha\, d\beta}{(\beta+\bar{\beta}e^{\eta_1-\eta_2})(\alpha+\bar{\alpha}e^{\eta_2-\eta_1})}$$
$$\times \int_{\bar{p}\bar{q}} \phi_{s_1 s_2}(k,p,\bar{p};\alpha)\, \phi_{\bar{s}_1\bar{s}_2}(\bar{k},q,\bar{q};\beta)\, \phi^*_{s_1\bar{s}_2}(l,p,\bar{q};\beta)\, \phi^*_{\bar{s}_1 s_2}(\bar{l},q,\bar{p};\alpha)$$

- The rapidity dependence in Φ_4 is due to the mixing of different pairs.

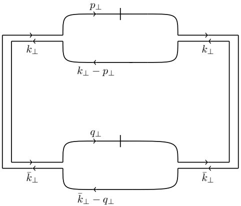
Projectile averaging and N_c counting - I

$\left\langle \rho^a(k)\rho^c(\bar{k})\rho^b(l)\rho^d(\bar{l})\right\rangle_P$: pair wise contraction.

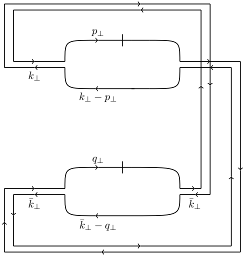
$\Phi_2\Phi_2$ contribution after contracting the color charge densities:



Uncorrelated $O(N_c^4)$



Correlated $O(N_c^2)$



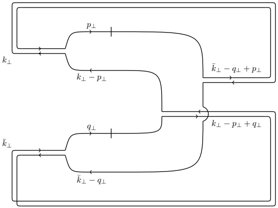
Correlated $O(N_c^2)$

$\Phi_2\Phi_2$ contribution to the correlated quark pair density is $O(N_c^2)$.

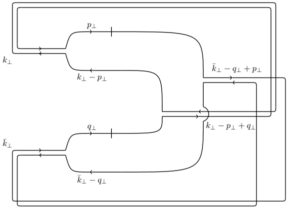
Projectile averaging and N_c counting - II

$\left\langle \rho^a(k)\rho^c(\bar{k})\rho^b(l)\rho^d(\bar{l})\right\rangle_P$: pair wise contraction.

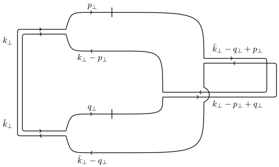
Φ_4 contribution after contracting the color charge densities:



(Φ_4^A) Correlated $O(N_c^3)$



(Φ_4^B) Correlated $O(N_c^3)$



Correlated $O(N_c)$

Φ_4 contribution to the correlated quark pair density is $O(N_c^3)$.

In the large N_c-limit, the only contribution to the correlated quark pair density comes from two diagrams of Φ_4.

Correlated quarks in the wave function

The leading N_c contribution to the correlated quark pair density in the projectile wave function is given by

$$\left[\frac{dN^P(p,q;\eta_1,\eta_2)}{d^2p\, d^2q\, d\eta_1\, d\eta_2}\right]_{\rm correlated} = -\int_{k\bar{k}l\bar{l}} \left\langle \rho^a(k)\rho^c(\bar{k})\rho^b(l)\rho^d(\bar{l})\right\rangle \Phi_4(k,l,\bar{k},\bar{l};p,q)\, {\rm tr}\{\tau^a\tau^b\tau^c\tau^d\}$$

- Adopt MV model for the averaging over the color charges:

$$\left\langle \rho^a(k)\rho^b(p)\right\rangle = (2\pi)^2\mu^2(k)\,\delta^{ab}\,\delta^{(2)}(k+p)$$

- estimate the results in the following kinematics:
  (i) rapidity difference between the quarks is relatively large: $\eta_1 - \eta_2 \gg 1$
  (ii) the two transverse momenta to be of the same order and much larger than saturation momentum: $|p| \sim |q| \gg Q_s$
- with this estimate answer the basic questions:
  (i) what is the sign of the correlation?
  (ii) how far in rapidity difference does it extend?

Correlated quarks in the wave function and Pauli blocking

The final result:

$$\left[\frac{dN^P(p,q;\eta_1,\eta_2)}{d^2pd^2qd\eta_1 d\eta_2}\right]_{\text{correlated}} \simeq -Se^{\eta_2-\eta_1}(\eta_1-\eta_2)^2\frac{\mu^4}{p^4q^4}\frac{g^8N_c^3}{4}\left\{\frac{25\pi^2}{2}q^4\left[\eta_1-\eta_2+\ln\frac{p^2}{Q_s^2}\right]^2\delta^{(2)}(q-p)\right.$$
$$\left.+\,\pi\left[3\frac{(p^2+q^2)\left[5p^2q^2-3(p\cdot q)^2-(p^2+q^2)p\cdot q\right]}{(q-p)^4}\ln\frac{(p-q)^2}{Q_s^2}+4(\eta_1-\eta_2)p\cdot q\right]\right\}$$

- the correlated contribution is negative! confirms our expectation based on the physics of the Pauli blocking
- the correlation is formally short range in rapidity since it decreases exponentially as a function of the rapidity difference.
- Note that the rate of the decrease in rapidity is tampered by the fourth power of $\eta_1 - \eta_2$, so that in practical terms the correlation may extend fairly far in rapidity.

Quark pair production cross section

The formal expression for the inclusive quark pair production emission reads

$$\frac{d\sigma}{d\eta_1 d^2 k_1 d\eta_2 d^2 k_2} = \langle 0|\Omega\, \hat{S}^\dagger\, \Omega^\dagger d^\dagger_{\alpha,s_1}(\underline{k_1})\, d_{\alpha,s_1}(\underline{k_1})\, d^\dagger_{\beta,s_2}(\underline{k_2})\, d_{\beta,s_2}(\underline{k_2})\, \Omega\, \hat{S}\, \Omega^\dagger|0\rangle$$

- $\hat{S}$: eikonal S-matrix operator
- Ω: unitary operator that (perturbatively) diagonalizes the QCD Hamiltonian.

The same N_c counting also holds for production cross section:

$$\frac{d\sigma}{d\eta_1\, d^2p\, d\eta_2\, d^2q} = \frac{g^8}{(2\pi)^6}\int -\frac{1}{2}\, \langle \rho^a(x)\rho^b(\bar{x})\rho^c(y)\rho^d(\bar{y})\rangle$$
$$\times\Phi_4(x,y,\bar{x},\bar{y};z_1,z_2,\bar{z}_1,\bar{z}_2;\bar{z},\bar{w};p,q)$$
$$\times\,\mathrm{tr}\Big\{[\tau^a - S_A^{a\bar{a}}(x)S_F(z_1)\tau^{\bar{a}}S_F^\dagger(\bar{z})][\tau^c - S_A^{c\bar{c}}(y)S_F(\bar{w})\tau^{\bar{c}}S_F^\dagger(z_2)]$$
$$\times\, [\tau^b - S_A^{b\bar{b}}(\bar{x})S_F(\bar{z}_1)\tau^{\bar{b}}S_F^\dagger(\bar{w})][\tau^d - S_A^{d\bar{d}}(\bar{y})S_F(\bar{z})\tau^{\bar{d}}S_F^\dagger(\bar{z}_2)]\Big\}$$

- $S(x)=\exp\{igt^a\alpha^a(x)\}\to$ Expand each S in α $(\alpha^a(x)=\frac{1}{\nabla^2}(x,y)\rho^a_T(y))$
- Use MV model for $\rho_T\rho_T$ correlator:

$$\langle\rho^a_T(k)\rho^b_T(p)\rangle = (2\pi)^2\lambda^2(k)\delta^{ab}\delta^{(2)}(k+p)$$

- contract both the target and the projectile color charge densities!

Estimates for particle production

Kinematics:

- rapidity difference between the quarks is relatively large: $\eta_1 - \eta_2 \gg 1$
- $|p| \sim |q| \sim |p - q| \gg Q_s$
- $Q_s^2 = g^4 \mu^2$ and $Q_T^2 = g^4 \lambda^2$
- saturation momentum of the target is smaller than than that of projectile: $Q_T < Q_s$.

$\Downarrow$

This is the regime where correlations existing in the projectile wave function are not strongly distorted by the momentum transfer from the target.

$$\left[\frac{d\sigma}{d^2p\,d^2q\,d\eta_1\,d\eta_2}\right]_{\rm correlated} \simeq -Sg^{12}N_c^5\frac{\mu^4\lambda^4}{Q_s^2Q_T^2}e^{\eta_2-\eta_1}(\eta_1-\eta_2)^2\ln\left(\frac{Q_T^2}{\Lambda^2}\right)\frac{\pi^3}{p^4}$$
$$\times\left\{\frac{50\pi}{16}\ln\left(\frac{Q_s^4}{Q_T^2\Lambda^2}\right)\delta^{(2)}(q-p)+\frac{9Q_s^2}{q^4}\left[\frac{2(p^2+q^2)^2+p^2q^2}{(p-q)^4}\right]\ln\left[\frac{(p-q)^2}{Q_s^2}\right]\right.$$
$$\left.+\frac{9Q_s^2}{2q^4}\left[\ln\left(\frac{q^2}{Q_s^2}\right)+\ln\left(\frac{p^2}{Q_s^2}\right)\right]\right\}$$

Summary

Quark-quark correlations in the glasma graph calculations:

- Quarks experience Pauli blocking as opposed to gluons.
- Gluon-gluon correlations are long range rapidity where as quark-quark correlations are short range.
- The exponential decay with rapidity difference is tempered by a factor quadratic in rapidity difference, resulting in a dip at $\Delta\eta \sim 2$.
- Quark-quark correlated production turns out to be parametrically $O(\alpha_s^2 N_c)$ relative to gluon-gluon correlations, which for realistic values of $\alpha_s \sim 0.2$ and $N_c = 3$ results in a mild suppression factor.

BACK UP SLIDES

Explicit expressions - 1

$$\phi_{s_1 s_2}(k,p;\alpha) = \frac{1}{k^2\,[\bar{\alpha}p^2\,+\,\alpha(k-p)^2]}\left\{4\alpha\bar{\alpha}k^2 - [\bar{\alpha}k\cdot p + \alpha k\cdot(k-p)] + 2i\sigma^3 k\times p\right\}$$

$$\begin{aligned}\Delta = g^4\,\mathrm{tr}\,\Big[\;& \{\tau^a\tau^{a'}[\alpha^{a'}(x)-\alpha^{a'}(\bar{z})] - \tau^{a'}\tau^a[\alpha^{a'}(x)-\alpha^{a'}(z_1)]\}\\ \times\;& \{\tau^c\tau^{c'}[\alpha^{c'}(y)-\alpha^{c'}(z_2)] - \tau^{c'}\tau^c[\alpha^{c'}(y)-\alpha^{c'}(\bar{w})]\}\\ \times\;& \{\tau^b\tau^{b'}[\alpha^{b'}(\bar{x})-\alpha^{b'}(\bar{w})] - \tau^{b'}\tau^b[\alpha^{b'}(\bar{x})-\alpha^{b'}(\bar{z}_1)]\}\\ \times\;& \{\tau^d\tau^{d'}[\alpha^{d'}(\bar{y})-\alpha^{d'}(\bar{z}_2)] - \tau^{d'}\tau^d[\alpha^{d'}(\bar{y})-\alpha^{d'}(\bar{z})]\}\;\Big].\end{aligned}$$

$$\begin{aligned}\Delta_A = \frac{g^4 N_c^5}{16}\;& \Big\{\langle[\alpha(x)-\alpha(\bar{z})][\alpha(\bar{y})-\alpha(\bar{z})] + [\alpha(x)-\alpha(z_1)][\alpha(\bar{y})-\alpha(\bar{z}_2)]\rangle\Big\}\\ \times\;& \Big\{\langle[\alpha(\bar{x})-\alpha(\bar{w})][\alpha(y)-\alpha(\bar{w})] + [\alpha(\bar{x})-\alpha(\bar{z}_1)][\alpha(y)-\alpha(z_2)]\rangle\Big\}.\end{aligned}$$

$$\begin{aligned}\Delta_B = \frac{g^4 N_c^5}{16}\;& \Big\{\langle[\alpha(x)-\alpha(\bar{z})][\alpha(y)-\alpha(\bar{w})] + [\alpha(x)-\alpha(z_1)][\alpha(y)-\alpha(z_2)]\rangle\Big\}\\ \times\;& \Big\{\langle[\alpha(\bar{x})-\alpha(\bar{w})][\alpha(\bar{y})-\alpha(\bar{z})] + [\alpha(\bar{x})-\alpha(\bar{z}_1)][\alpha(\bar{y})-\alpha(\bar{z}_2)]\rangle\Big\}.\end{aligned}$$

Explicit expressions - 2

$$\delta H^{\rho\, g} = \int_0^\infty \frac{dk^+}{2\pi} \frac{d^2k}{(2\pi)^2} \frac{g\, k_i}{\sqrt{2}\, |k^+|^{3/2}} \left[a_i^{\dagger a}(k^+,\, k)\ \rho^a(-k) + a_i^a(k^+,\, k)\ \ \rho^a(k) \right]$$

$$\delta H^{\rho\, qq} = \sum_s \int \frac{dk^+ d^2k\, dp^+ d^2p}{(2\pi)^6} \frac{g^2}{(k^+)^2} \left[d^\dagger_{\alpha\, s}(p^+,\, p)\ \tau^a_{\alpha\beta}\ \bar{d}^\dagger_{\beta\, s}(k^+ - p^+,\, k-p)\ \rho^a(-k) + h.c. \right]$$

$$\begin{aligned} \delta H^{g\, qq} &= g\, \tau^a_{\alpha\beta} \sum_{s_1, s_2} \int \frac{dp^+\, d^2p\, dk^+\, d^2k}{2^{3/2}\, (2\pi)^6\, (k^+)^{1/2}}\, \theta(k^+ - p^+)\, \Gamma^i_{s_1\, s_2}(k^+, k, p^+, p) \\ &\times \left[a_i^a(k^+, k)\, d^\dagger_{\alpha,\, s_1}(p^+, p)\, \bar{d}^\dagger_{\beta, s_2}(k^+ - p^+, k-p) + h.c. \right] \end{aligned}$$

$$\Gamma^i_{s_1 s_2}(k^+, k, p^+, p) = \delta_{s_1 s_2} \left[2\frac{k_i}{k^+} - \left(\frac{p_i}{p^+} + \frac{k_i - p_i}{k^+ - p^+} \right) + 2 i s_1 \epsilon^{im} \left(\frac{p_m}{p^+} - \frac{k_m - p_m}{k^+ - p^+} \right) \right]$$

Explicit expressions - 3

$$\langle g|\delta H^{\rho\, g}|0\rangle = \frac{\langle 0|a_i^a(k^+,k)\ \delta H^{\rho g}|0\rangle}{(2\pi)^{3/2}} = \frac{g k_i}{4\pi^{3/2}|k^+|^{3/2}}\rho^a(-k)$$

$$\langle q\bar{q}|\delta H^{\rho qq}|0\rangle = \frac{\langle 0|d_{\alpha s_1}(q^+,q)\ \bar{d}_{\beta s_2}(p^+,p)\delta H^{\rho\, qq}|0\rangle}{(2\pi)^3}$$

$$= \frac{g^2\tau^a_{\alpha\beta}}{(2\pi)^3(p^+ + q^+)^2}\rho^a(-p-q)\delta_{s_1 s_2}$$

$$\langle q\bar{q}|\delta H^{g\, qq}|g\rangle = \frac{\langle 0|d_{\alpha s_1}(p^+,p)\bar{d}_{\beta s_2}(q^+,q)\delta H^q a_i^{a\dagger}(k^+,k)|0\rangle}{(2\pi)^{9/2}}$$

$$= g\tau^a_{\alpha\beta}\frac{\Gamma^i_{s_1 s_2}(k^+,k,p^+,p)}{8\pi^{3/2}(k^+)^{1/2}}\delta^{(2)}(p+q-k)\delta(p^+ + q^+ - k^+)$$

Particle production and Pauli blocking

After all possible contractions at leading N_c, there are two types of contributions to the production cross section:

$$A = -\delta^{(2)}(0)\delta^{(2)}(p-q)\frac{g^{12}N_c^5}{16}\int_0^1 \frac{d\alpha\, d\beta}{(\beta+\bar{\beta}e^{\eta_1-\eta_2})(\alpha+\bar{\alpha}e^{\eta_2-\eta_1})}\int_{k,\bar{k},l,\bar{l}}\frac{\mu^2(k)\mu^2(\bar{k})\lambda^2(l)\lambda^2(\bar{l})}{l^4\bar{l}^4}$$
$$\times tr\Big\{\Big[\bar{\Psi}(k,l,p;\alpha)\bar{\Psi}^*(k,l,p;\alpha)+\Psi(k,l,p;\alpha)\Psi^*(k,l,p;\alpha)\Big]$$
$$\times\Big[\bar{\Psi}(\bar{k},\bar{l},p;\beta)\bar{\Psi}^*(\bar{k},\bar{l},p;\beta)+\Psi(\bar{k},\bar{l},p;\beta)\Psi^*(\bar{k},\bar{l},p;\beta)\Big]\Big\}$$

$$B = -\delta^{(2)}(0)\frac{g^{12}N_c^5}{16}\int_0^1 \frac{d\alpha\, d\beta}{(\beta+\bar{\beta}e^{\eta_1-\eta_2})(\alpha+\bar{\alpha}e^{\eta_2-\eta_1})}\int_{k,\bar{k},l,\bar{l}}\frac{\mu^2(k)\mu^2(\bar{k})\lambda^2(l)\lambda^2(\bar{l})}{l^4\bar{l}^4}$$
$$\times\delta^{(2)}(k+l-p-\bar{k}-\bar{l}+q)\,tr\Big\{\Big[\bar{\Psi}(k,l,p;\alpha)\bar{\Psi}^*(k,l,p;\beta)+\Psi(k,l,p;\alpha)\Psi^*(k,l,p;\beta)\Big]$$
$$\times\Big[\bar{\Psi}(\bar{k},\bar{l},q;\beta)\bar{\Psi}^*(\bar{k},\bar{l},q;\alpha)+\Psi(\bar{k},\bar{l},q;\beta)\Psi^*(\bar{k},\bar{l},q;\alpha)\Big]\Big\}$$

with

$$\Psi(k,l,p;\alpha)\equiv[\phi(k+l,p;\alpha)-\phi(k,p-l;\alpha)]$$
$$\bar{\Psi}(k,l,p;\alpha)\equiv[\phi(k+l,p;\alpha)-\phi(k,p;\alpha)]$$