Hydrodynamics of vortical and polarized fluids

Amaresh Jaiswal

National Institute of Science Education and Research, Jatni, India

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Introduction & Motivation

- Non-central heavy-ion collisions create fireballs with large global angular momenta.
- This may generate a spin polarization of the hot and dense matter in a way similar to the Einstein-de Haas and Barnett effects.
- Much effort has recently been invested in studies of polarization and spin dynamics of particles produced in relativistic heavy ion collisions from both experimental and theoretical point of view; see earlier talks by Rainer, Takafumi and Francesco.
- L. Adamczyk et al. (**STAR**), Global Λ hyperon polarization in nuclear collisions: evidence for the most vortical fluid , **Nature 548, 62 (2017)**.
- www.sciencenews.org/article/smashing-gold-ions-creates-most-swirly-fluid-ever (Record-making vorticity found in QGP).
- Motivation to develop hydrodynamic framework with angular momentum conservation taken into account.

Energy-momentum tensor and angular momentum: system of scalar particles

- For a system of scalar particles, the conservation of energy-momentum and angular momentum reads

$$\partial_\mu T^{\mu\nu} = 0, \quad \partial_\lambda J^{\lambda,\mu\nu} = 0$$

- The total angular momentum can be written as

$$J^{\lambda,\mu\nu} = L^{\lambda,\mu\nu} = x^\mu T^{\lambda\nu} - x^\nu T^{\lambda\mu}$$

where L represents the orbital angular momentum.

- $\partial_\lambda J^{\lambda,\mu\nu} = T^{\mu\nu} - T^{\nu\mu} \quad \Rightarrow$ for a symmetric energy-momentum tensor, the orbital angular momentum is automatically conserved.
- Therefore angular momentum conservation leads to no new constraints for evolution.

Energy-momentum tensor and angular momentum: system of particles with intrinsic spin

- For a system of particles with intrinsic spin, we must consider angular momentum conservation along with energy and momentum conservation.

$$\partial_\mu T^{\mu\nu} = 0, \quad \partial_\lambda J^{\lambda,\mu\nu} = 0$$

- The total angular momentum can be written as

$$J^{\lambda,\mu\nu} = L^{\lambda,\mu\nu} + S^{\lambda,\mu\nu}$$

where L represents the orbital part and S represents the intrinsic spin part.

- For a symmetric energy-momentum tensor, the orbital angular momentum is automatically conserved and therefore one has to explicitly consider the conservation of spin

$$\partial_\lambda S^{\lambda,\mu\nu} = 0$$

Local distribution functions

- Phase-space distribution functions for spin-1/2 particles.
- Generalized to two by two spin density matrices for each value of the x and p. **F. Becattini et al., Annals Phys. 338 (2013) 32**

$$f^+_{rs}(x,p) = \frac{1}{2m}\bar{u}_r(p)X^+u_s(p), \quad f^-_{rs}(x,p) = -\frac{1}{2m}\bar{v}_s(p)X^-v_r(p)$$

- Here

$$X^\pm = \exp\left[-\beta_\mu(x)p^\mu \pm \xi(x)\right]M^\pm \quad , \quad M^\pm = \exp\left[\pm\frac{1}{2}\omega_{\mu\nu}(x)\hat{\Sigma}^{\mu\nu}\right]$$

- Notations:

$$\beta^\mu = u^\mu/T, \quad \xi = \mu/T, \quad \hat{\Sigma}^{\mu\nu} = (i/4)[\gamma^\mu,\gamma^\nu],$$

where, T: temperature, μ: chemical potential, u^μ: fluid four velocity, $\omega_{\mu\nu}$: the spin tensor, and $\hat{\Sigma}^{\mu\nu}$: the spin operator.

Spin/polarization tensor

- Express antisymmetric $\omega_{\mu\nu}$ as

$$\omega_{\mu\nu} \equiv k_\mu u_\nu - k_\nu u_\mu + \epsilon_{\mu\nu\beta\gamma} u^\beta \omega^\gamma \;\Rightarrow\; k_\mu = \omega_{\mu\nu} u^\nu, \;\; \omega_\mu = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\, \omega^{\nu\alpha} u^\beta.$$

  where $k \cdot u = \omega \cdot u = 0$, and $\frac{1}{2}\omega_{\mu\nu}\omega^{\mu\nu} = k \cdot k - \omega \cdot \omega$.

- We assume the constraint

$$k \cdot \omega = 0$$

  we find the compact form

$$M^{\pm} = \exp\left[\pm\frac{1}{2}\omega_{\mu\nu}(x)\hat{\Sigma}^{\mu\nu}\right] \to \cosh(\zeta) \pm \frac{\sinh(\zeta)}{2\zeta}\,\omega_{\mu\nu}\hat{\Sigma}^{\mu\nu},$$

  where

$$\zeta \equiv \frac{1}{2}\sqrt{k \cdot k - \omega \cdot \omega}.$$

- We further assume that $k \cdot k - \omega \cdot \omega \geq 0 \;\;\Rightarrow\;\; \zeta$ is real.

Charge current

The **charge current** [S. de Groot, W. van Leeuwen, and C. van Weert]

$$N^\mu = \int \frac{d^3p}{2(2\pi)^3 E_p} p^\mu \left[\mathrm{tr}(\mathrm{X}^+) - \mathrm{tr}(\mathrm{X}^-)\right] = n u^\mu$$

where 'tr' denotes the trace over spinor indices and n is the charge density

$$n = 4\cosh(\zeta)\sinh(\xi)\, n_{(0)}(T)$$

Here $n_{(0)}(T) = \langle (u \cdot p) \rangle_0$ is the number density of spin 0, neutral Boltzmann particles, obtained using the thermal average

$$\langle \cdots \rangle_0 \equiv \int \frac{d^3p}{(2\pi)^3 E_p} (\cdots)\, e^{-\beta \cdot p},$$

where $E_p = \sqrt{m^2 + \mathbf{p}^2}$.

Energy-momentum tensor tensor

The **energy-momentum tensor** for a perfect fluid has the form

$$T^{\mu\nu} = \int \frac{d^3p}{2(2\pi)^3 E_p} p^\mu p^\nu \left[\mathrm{tr}(X^+) + \mathrm{tr}(X^-)\right] = (\varepsilon + P)u^\mu u^\nu - Pg^{\mu\nu},$$

where the energy density and pressure are given by

$$\varepsilon = 4\,\cosh(\zeta)\cosh(\xi)\,\varepsilon_{(0)}(T)$$

and

$$P = 4\,\cosh(\zeta)\cosh(\xi)\,P_{(0)}(T),$$

respectively. In analogy to the density $n_{(0)}(T)$, we define the auxiliary quantities

$$\varepsilon_{(0)}(T) = \langle (u\cdot p)^2\rangle_0, \quad P_{(0)}(T) = -(1/3)\langle\left[p\cdot p - (u\cdot p)^2\right]\rangle_0$$

Entropy current

The **entropy current** is given by an obvious generalization of the Boltzmann expression

$$S^\mu = -\int \frac{d^3p}{2(2\pi)^3 E_p}\, p^\mu \left(\mathrm{tr}\left[\mathrm{X}^+ (\ln \mathrm{X}^+ - 1) \right] + \mathrm{tr}\left[\mathrm{X}^- (\ln \mathrm{X}^- - 1) \right] \right)$$

This leads to the following entropy density

$$s = u_\mu S^\mu = \frac{\varepsilon + P - \mu\, n - \Omega w}{T},$$

where Ω is defined through the relation $\zeta = \Omega/T$ and

$$w = 4\, \sinh(\zeta) \cosh(\xi)\, n_{(0)}.$$

This suggests that Ω should be used as a thermodynamic variable of the grand canonical potential, in addition to T and μ. Taking the pressure P to be a function of T, μ and Ω, we find

$$s = \left.\frac{\partial P}{\partial T}\right|_{\mu,\Omega}, \quad n = \left.\frac{\partial P}{\partial \mu}\right|_{T,\Omega}, \quad w = \left.\frac{\partial P}{\partial \Omega}\right|_{T,\mu}.$$

Basic conservation laws

- The conservation of energy and momentum requires that

$$\partial_\mu T^{\mu\nu} = 0.$$

- The longitudinal component with respect to u^μ:

$$\partial_\mu[(\varepsilon + P)u^\mu] = u^\mu \partial_\mu P.$$

- Rearranging the terms, we get

$$T\,\partial_\mu(su^\mu) + \mu\,\partial_\mu(nu^\mu) + \Omega\,\partial_\mu(wu^\mu) = 0.$$

- The middle term vanishes due to charge conservation,

$$\partial_\mu(nu^\mu) = 0.$$

- Therefore to have conserved entropy current for perfect-fluid

$$\partial_\mu(wu^\mu) = 0.$$

Basic conservation laws contd.

- In the absence of net spin polarization, i.e., for $\zeta = 0$, we obtain the standard expression: $n = 4 \sinh(\xi)\, n_{(0)}$.
- On the other hand, $\partial_\mu(n u^\mu) = 0$ and $\partial_\mu(w u^\mu) = 0$ implies

$$\partial_\mu \left[(n \pm w) u^\mu\right] = 0$$

- We find $n \pm w = 4 \sinh[(\mu \pm \Omega)/T]\, n_{(0)}$: thermodynamic quantities corresponding to charge and spin couple.
- Ω can be interpreted as a chemical potential related with spin.
- A system of spin-1/2 particles = A two component mixture of scalar particles.
- The present framework can be regarded as a minimal extension of the standard perfect-fluid hydrodynamics of charged particles.
- We may first solve these equations and subsequently use this solution as the **dynamic background for the spin evolution**.

Spin dynamics

- Since we use a symmetric form of the energy-momentum tensor $T^{\mu\nu}$, the spin tensor $S^{\lambda,\mu\nu}$ satisfies the conservation law,

$$\partial_\lambda S^{\lambda,\mu\nu} = 0.$$

- For $S^{\lambda,\mu\nu}$ we use the form taken from Becattini, Tinti, Annals Phys. 325 (2010) 1566-1594,

$$S^{\lambda,\mu\nu} = \int \frac{d^3p}{2(2\pi)^3 E_p}\, p^\lambda \,\mathrm{tr}\left[(X^+ - X^-)\hat{\Sigma}^{\mu\nu}\right] = \frac{wu^\lambda}{4\zeta}\omega^{\mu\nu}$$

- Using the conservation law for the spin density, $\partial_\lambda(wu^\lambda) = 0$, and introducing the rescaled spin tensor $\bar{\omega}^{\mu\nu} = \omega^{\mu\nu}/(2\zeta)$, we obtain

$$u^\lambda \partial_\lambda\, \bar{\omega}^{\mu\nu} = \frac{d\bar{\omega}^{\mu\nu}}{d\tau} = 0,$$

with the normalization condition $\bar{\omega}_{\mu\nu}\, \bar{\omega}^{\mu\nu} = 2$.

Spin dynamics contd.

- The tensor $\bar{\omega}_{\mu\nu}$ can be decomposed as

$$\bar{\omega}_{\mu\nu} = \bar{k}_\mu u_\nu - \bar{k}_\nu u_\mu + \epsilon_{\mu\nu\beta\gamma} u^\beta \bar{\omega}_\mu, \quad \bar{k}_\mu = k_\mu/(2\zeta), \quad \bar{\omega}_\mu = \omega_\mu/(2\zeta),$$

satisfying the constraints

$$\bar{k} \cdot u = 0, \;\; \bar{\omega} \cdot u = 0, \;\; \bar{k} \cdot \bar{\omega} = 0, \;\; \bar{k} \cdot \bar{k} - \bar{\omega} \cdot \bar{\omega} = 1,$$

which leave only four independent components in $\bar{k}_\mu$ and $\bar{\omega}_\mu$.

- The last condition is fulfilled by employing the parameterization

$$\bar{k}_\mu = m_\mu \sinh(\psi), \quad \bar{\omega}_\mu = n_\mu \cosh(\psi).$$

- The four-vectors m_μ and n_μ are space-like and normalized to -1,

$$m_\mu m^\mu = -1, \quad n_\mu n^\mu = -1.$$

and satisfy the transversality conditions

$$m \cdot u = n \cdot u = m \cdot n = 0.$$

Spin dynamics contd.

- We find two coupled equations

$$\frac{dm_\mu}{d\tau}\sinh(\psi) + m_\mu \cosh(\psi)\frac{d\psi}{d\tau} + m_\nu a^\nu \sinh(\psi) u_\mu + \epsilon_{\mu\nu\beta\gamma} u^\nu a^\beta n^\gamma \cosh(\psi) = 0,$$
$$\frac{dn_\mu}{d\tau}\cosh(\psi) + n_\mu \sinh(\psi)\frac{d\psi}{d\tau} + n_\nu a^\nu \cosh(\psi) u_\mu + \epsilon_{\mu\nu\alpha\beta} u^\nu a^\beta m^\alpha \sinh(\psi) = 0.$$

- Here $a^\mu = du^\mu/d\tau$ is the acceleration of the fluid element.
- Conditions on m_μ and n_μ are satisfied during the time evolution of the system, if they are satisfied on the initial hypersurface and if the following equation is fulfilled by the variable

$$\frac{d\psi}{d\tau} = \epsilon_{\mu\nu\beta\gamma} m^\mu u^\nu a^\beta n^\gamma.$$

Vortex solution 1

- Consider a rigidly rotating fluid with flow proflie given by

$$u^0 = \gamma, \quad u^1 = -\gamma\,\tilde{\Omega}\,y, \quad u^2 = \gamma\,\tilde{\Omega}\,x, \quad u^3 = 0,$$

  where $\tilde{\Omega}$ is a constant, $\gamma = 1/\sqrt{1-\tilde{\Omega}^2 r^2}$, and $r \equiv \sqrt{x^2+y^2}$.

- The assumed flow profile can be realised only within a cylinder with the radius $R < 1/\tilde{\Omega}$.

- The total time (convective) derivative takes the form

$$\frac{d}{d\tau} = u^\mu \partial_\mu = -\gamma\tilde{\Omega}\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right).$$

- The above equation can be used to find the fluid acceleration

$$a^\mu = \frac{du^\mu}{d\tau} = -\gamma^2\tilde{\Omega}^2(0, x, y, 0).$$

- The spatial part of the four-acceleration points towards the centre of the vortex, as it describes the centripetal acceleration.

Spin/polarization tensor

- Equations of the hydrodynamic background $\partial_\mu T^{\mu\nu} = 0$, $\partial_\mu(nu^\mu) = 0$, and $\partial_\mu(wu^\mu) = 0$ are satisfied for

$$T = T_0\gamma, \quad \mu = \mu_0\gamma, \quad \Omega = \Omega_0\gamma.$$

- One possibility is that the vortex represents an unpolarized fluid with $\omega_{\mu\nu} = 0$ and thus, with $\Omega_0 = 0$. Trivial solution!
- Another possibility is that $\Omega_0 \neq 0$. In this case

$$\omega_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\tilde{\Omega}/T_0 & 0 \\ 0 & \tilde{\Omega}/T_0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

- This yields $k^\mu = \tilde{\Omega}^2(\gamma/T_0)[0, x, y, 0]$ and $\omega^\mu = \tilde{\Omega}(\gamma/T_0)[0, 0, 0, 1]$.
- As a consequence, we find $\zeta = \tilde{\Omega}/(2T_0)$, which, for consistency with the hydrodynamic background equations, implies

$$\tilde{\Omega} = 2\,\Omega_0.$$

Thermal vortex

- With $\tilde{\Omega} = 2\,\Omega_0$, it is straightforward to show that our spin-evolution equation $u^\lambda \partial_\lambda [\omega^{\mu\nu}/(2\zeta)] = 0$ is satisfied.
- We also note that the spin tensor agrees with the thermal vorticity

$$\varpi_{\mu\nu} = -\frac{1}{2}\left(\partial_\mu \beta_\nu - \partial_\nu \beta_\mu\right)$$

as emphasised in the works by Becattini and collaborators.

- Therefore the above thermal vortex in global equilibrium satisfy the rigidly rotating vortex solution of our spin evolution equation.
- Important to consider evolution of vortices within hydrodynamic framework to study polarization observables.
- One of course needs initial condition with vortices to start the evolution.

Conclusions and Summary

- In this work we have introduced a hydrodynamic framework, which includes the evolution of the spin density.
- Equations that determine the dynamics of the system follow solely from conservation laws.
- They can be regarded as a minimal extension of the well established perfect-fluid picture.
- Our framework can be used to determine the space-time dynamics of fluid variables including the spin tensor.
- This property makes them useful for practical applications in studies of polarization evolution in high-energy nuclear collisions.
- In particular, the possibility to study the dynamics of systems in local thermodynamic equilibrium represents an important advance compared to studies, where global equilibrium was assumed.

Thank you for your attention!