Transport from the Fluid/Gravity Correspondence

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What is hydrodynamics?

Late time effective theory of interacting theories near thermal equilibrium.

A more general definition abandons the concept of late time. Instead we just restrict the effective theory to dynamics of conserved currents only. (This assumes that the algebra of conserved currents is closed on the subspace of near-thermal states)

For a conserved $U(1)$ current: $$\partial_\mu \mathbf{J}^\mu = \mathbf{0}$$

Continuity equation (charge conservation): $$\partial_t \mathbf{J}^0 = \vec{\nabla} \cdot \vec{\mathbf{J}}$$

can be viewed as initial value problem for time evolution of the charge density $\mathbf{J}^0$ with some initial condition $\mathbf{J}^0(\mathbf{t} = \mathbf{t}_0)$.

Yet, the continuity equation cannot be solved as such. We need extra inputs and that are three functions $\vec{\mathbf{J}}(\mathbf{t})$.

Similarly for the energy-momentum conservation: $$\partial_\mu \mathbf{T}^{\mu\nu} = \mathbf{0}, \quad \text{or} \quad \partial_t \mathbf{T}^{0\,\nu} = \nabla_i \mathbf{T}^{i\,\nu}$$

$\mathbf{T}^{00} = \epsilon$ – energy density of the fluid and $\mathbf{T}^{0i} \sim \mathbf{u}^i$ is the velocity of the fluid, while $\mathbf{T}^{ij}$ are missing data. Equation of state relates energy density to pressure $\mathbf{P} = \mathbf{T}^{ii}$, thus reducing the number of required inputs.

Constitutive Relations for $U(1)$ current

Data have to be provided on $\vec{\mathbf{J}}$ $(\mathbf{T}^{ij})$ in order to close the dynamical equations.

So, $\vec{\mathbf{J}}$ has to be related to $\mathbf{J}^0$ (and $\mathbf{T}^{ij}$ to $\mathbf{T}^{0j}$)

Linear relation (in the absence of external fields): $\vec{\mathbf{J}} \sim \vec{\nabla}\mathbf{J}^0$ ($\vec{\nabla}$ is the only vector)

Most general linear constitutive relation

$$\vec{\mathbf{J}}(\mathbf{t},\,\mathbf{x}) = \int_{\mathbf{t}',\mathbf{x}'} \mathbf{D}(\mathbf{t}-\mathbf{t}',\,\mathbf{x}-\mathbf{x}')\,\vec{\nabla}\mathbf{J}^0(\mathbf{t}',\,\mathbf{x}')\,, \qquad \vec{\mathbf{J}}(\omega,\,\mathbf{q}) = \mathbf{D}(\omega,\,\mathbf{q}^2)\,\vec{\mathbf{q}}\,\mathbf{J}^0(\omega,\,\mathbf{q})$$

$\mathbf{D}$ is called "Memory Function". It contain a wealth (infinite amount) of info about transport coefficients to be determined from the microscopic theory. It generalises the concept of diffusion constant.

Causality: $\mathbf{D}(\mathbf{t}-\mathbf{t}') \sim \theta(\mathbf{t}-\mathbf{t}')$

Causality is related to short time scales, which is probed by hydro evolution at early times. Early times are important. There have been a lot of talk about early "hydronisation" without isotropisation. Most of the entropy is also produced at early times.
Causality constraint is a necessary UV completion of the usual late time hydrodynamics.

Constitutive Relation for Conformal Fluids

Define Π^{ij} as a traceless part of T^{ij} (also assume Landau frame)

$$\Pi^{ij}(t,\,x) \;=\; \int_{t',x'} \eta(t-t',\,x-x')\,\nabla^{\{i}T^{0j\}}(t',\,x') + \int_{t',x'} \zeta(t-t',\,x-x')\,\nabla^{i}\,\nabla^{j}\,\nabla^{k}T^{0k}(t',\,x')$$

$$\Pi^{ij} \;=\; \eta(\omega,\,q^2)\,q_{\{i}\,u_{j\}}(\omega,\,q) \;+\; \zeta(\omega,\,q^2)\,q_i\,q_j\,q_k u_k(\omega,\,q)$$

Bulk viscosity is zero

ζ is a second shear viscosity

introduced by E. Shuryak, M.L., D80 (2009) 065026 $\quad \zeta = 0$

Yanyan Bu and M.L., arXiv:1406.7222 (PRD), arXiv:1409.3095 (JHEP), $\quad \zeta \neq 0$

Non-linear constitutive relation:

$$\vec{J}(t,\, x) \;=\; D\vec{\nabla}J^0 \;+\; \#(\vec{\nabla}J^0)^2 \;+\; \text{infinitely many terms}$$

$$\vec{J}(t) \;=\; \int^{t} D(t-t')\,\vec{\nabla}J^0(t')\,dt'$$

What about the low limit? We pretend to know J^0 at $t \geq t_0$. So, we have no choice but to limit the integral

$$\vec{J}(t) \;=\; \vec{J}(t_0) \;+\; \int_{t_0}^{t} D(t-t')\,\vec{\nabla}J^0(t')\,dt'$$

Here $\vec{J}(t_0)$ is an initial condition for the current.

When D is not known from a microscopic theory or experiment, it has to be modelled.

Hydro Model A: Diffusion constant/Navier-Stokes

Instantaneous response

$$\mathbf{D}(\mathbf{t}-\mathbf{t}',\,\mathbf{x}-\mathbf{x}') \;=\; \mathbf{D_0}(\mathbf{x}-\mathbf{x}')\,\delta(\mathbf{t}-\mathbf{t}')$$

Then

$$\vec{\mathbf{J}}(\mathbf{t},\,\mathbf{x}) \;=\; \int_{\mathbf{x}'} \mathbf{D_0}(\mathbf{x}-\mathbf{x}')\,\vec{\nabla}\mathbf{J}^0(\mathbf{t},\,\mathbf{x}')$$

If we further assume that $\mathbf{D_0}$ is local $\mathbf{D_0}(\mathbf{x}-\mathbf{x}') \;=\; \mathbf{D_0}\,\delta(\mathbf{x}-\mathbf{x}')$,

$$\vec{\mathbf{J}}(\mathbf{t},\,\mathbf{x}) \;=\; \mathbf{D_0}\,\vec{\nabla}\mathbf{J}^0(\mathbf{t},\,\mathbf{x})$$

which is the usual diffusive approximation. Notice that $\vec{\mathbf{J}}(\mathbf{t}=\mathbf{t_0}) \;\neq\; \mathbf{0}$

For the spatial components of the energy-momentum tensor:

$$\mathbf{\Pi}^{\mathbf{ij}}(\mathbf{t},\,\mathbf{x}) \;=\; \eta_0\,\nabla^{\{\mathbf{i}}\mathbf{u}^{\mathbf{j}\}}(\mathbf{t},\,\mathbf{x}) \qquad \mathbf{i} \neq \mathbf{j} \qquad \text{Navier Stokes}$$

Hydro Model B: Relaxation time approximation

$$\mathbf{D}(\mathbf{t}-\mathbf{t}', \mathbf{x}-\mathbf{x}') = \frac{\mathbf{D_0}}{\tau}\, \mathbf{e}^{-(\mathbf{t}-\mathbf{t}')/\tau}\, \delta(\mathbf{x}-\mathbf{x}')\,, \qquad \mathcal{D}(\omega) \sim \frac{\mathbf{1}}{\omega - \mathbf{i}/\tau}$$

$$\partial_{\mathbf{t}} \vec{\mathbf{J}}(\mathbf{t}, \mathbf{x}) = \frac{\mathbf{1}}{\tau}\left[\mathbf{D_0}\, \vec{\nabla} \mathbf{J}^{\mathbf{0}}(\mathbf{t}, \mathbf{x}) - \vec{\mathbf{J}}(\mathbf{t}, \mathbf{x}) + \vec{\mathbf{J}}(\mathbf{t_0}, \mathbf{x})\right]$$

$\vec{\mathbf{J}}(\mathbf{t_0}, \mathbf{x})$ is usually assumed to be zero.

A similar construction for $\mathbf{T}^{\mathbf{ij}}(\mathbf{t}, \mathbf{x})$, $\quad \partial_{\mathbf{t}}\, \mathbf{\Pi}^{\mathbf{ij}}(\mathbf{t}, \mathbf{x}) = \cdots$ is frequently referred to as Israel-Stewart hydrodynamics. It reduces to NS in the limit $\tau \to \mathbf{0}$

Is there a "better" model ?

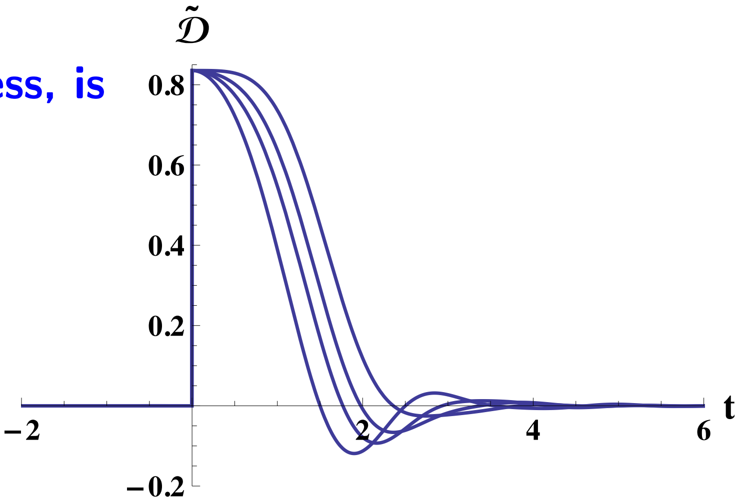
M. P. Heller, R. A. Janik, M. Spalinski and P. Witaszczyk, Phys. Rev. Lett. 113, no. 26, 261601 (2014) proposed 2-pole approximation (Model C):

$$\mathbf{D}(t-t') = d_1\, e^{-(t-t')\omega_1} + d_2\, e^{-(t-t')\omega_2}$$

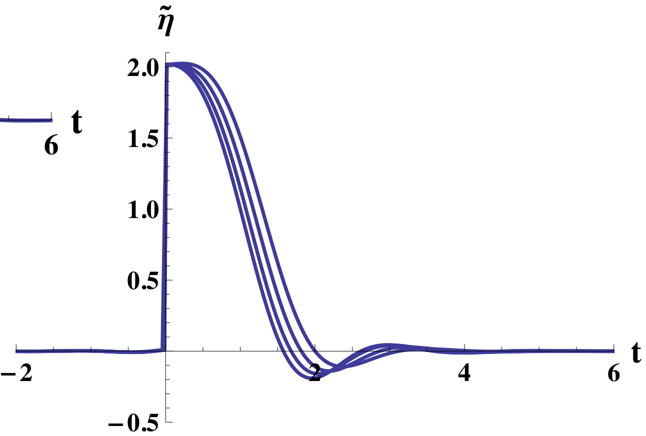
What is the right memory function $\mathbf{D}$? How to compute it from a microscopic theory?

For a holographically defined microscopic theory (not QCD), $\mathbf{D}$ can be computed using the fluid-gravity correspondence.
The correct structure, more or less, is

$$\mathbf{D}(t) = \theta(t) \sum_{n=0}^{\infty} d_n\, e^{-t\,\omega_n}$$



Y. Bu, M. Lublinsky and A. Sharon, JHEP 1604, 136 (2016)



Y. Bu and M. L, JHEP 1504, 136 (2015)

What about initial conditions $\mathbf{J^0(t_0)}$ and $\vec{\mathbf{J}}(\mathbf{t_0})$? So far, we pretended that $J^0(t_0)$ is provided as an experimental input (or should be fitted to experiment). At the same time $\vec{\mathbf{J}}(\mathbf{t_0})$ is usually simply modelled.

Is it justified to treat them so differently and independently?

One option is to assume that the system emerges from far out of equilibrium and then "hydronises" with the initial conditions $\mathbf{J^0(t_0)}$ and $\vec{\mathbf{J}}(\mathbf{t_0})$ being random and totally independent.

From a holographic perspective one would have to consider black-hole formation, which is a process dual to hydronization, as a result of a collision/collapse.

many people/many papers

talks by Maximilian Attems and Jorge Casalderrey Solana

An alternative possibility is that the system "remembers" its $t < t_0$ history.

A way to study this is to start at $\mathbf{t} = -\infty$ with a system at equilibrium and turn on an external field which would take the system not too far out of equilibrium, that is the system all the time remains in the hydro regime

For e/m current we can turn on en external electric filed $\vec{\mathrm{E}}$.

In presence of external electric field, the constitutive relation generalises to

$$\vec{\mathbf{J}}(\mathbf{t}) = \int_{-\infty}^{\mathbf{t}} \mathbf{D}(\mathbf{t}-\mathbf{t}')\,\vec{\nabla}\mathbf{J}^0(\mathbf{t}')\,\mathbf{dt}' + \int_{-\infty}^{\mathbf{t}} \tilde{\sigma}_{\mathbf{e}}(\mathbf{t}-\mathbf{t}')\,\vec{\mathbf{E}}(\mathbf{t}')\,\mathbf{dt}'$$

with $\mathbf{J}^0(\mathbf{t} = -\infty) = \mathbf{0}$. The last term generalises the Ohm's law.

Switch off the external field at $\mathbf{t} = \mathbf{0}$ and let the system relax towards equilibrium

$$\vec{\mathbf{J}}(\mathbf{t} > \mathbf{0}) = \int_{\mathbf{0}}^{\mathbf{t}} \mathbf{D}(\mathbf{t}-\mathbf{t}')\,\vec{\nabla}\mathbf{J}^0(\mathbf{t}')\,\mathbf{dt}' + \vec{\mathbf{J}}_{\mathbf{H}}(\mathbf{t})$$

$$\vec{\mathbf{J}}_{\mathbf{H}}(\mathbf{t}) = \int_{-\infty}^{\mathbf{0}} \mathbf{D}(\mathbf{t}-\mathbf{t}')\,\vec{\nabla}\mathbf{J}^0(\mathbf{t}')\,\mathbf{dt}' + \int_{-\infty}^{\mathbf{0}} \tilde{\sigma}_{\mathbf{e}}(\mathbf{t}-\mathbf{t}')\,\vec{\mathbf{E}}(\mathbf{t}')\,\mathbf{dt}'$$

The "history" current depends on time for $\mathbf{t} > \mathbf{0}$ and it is not clear if $\vec{\mathbf{J}}_{\mathbf{H}}$ can be rendered into t-independent. Thus this hydro presumably cannot be solved as initial value problem.

Gradient expansion

$$\vec{\mathbf{J}}(\mathbf{t}) \;=\; \int_{\mathbf{t}'} \mathbf{D}(\mathbf{t}-\mathbf{t}')\,\vec{\nabla}\mathbf{J}^0(\mathbf{t}') \;+\; \int_{\mathbf{t}'} \tilde{\sigma}_{\mathrm{e}}(\mathbf{t}-\mathbf{t}')\,\vec{\mathbf{E}}(\mathbf{t}')$$

$$\mathbf{D}(\mathbf{t}-\mathbf{t}') \;=\; \int \frac{\mathbf{d}\omega}{\mathbf{2}\pi}\,\mathcal{D}(\omega)\,\mathbf{e}^{-\mathbf{i}\omega\,(\mathbf{t}-\mathbf{t}')} \;=\; \int \frac{\mathbf{d}\omega}{\mathbf{2}\pi}\,\mathcal{D}(\mathbf{i}\partial_{\mathrm{t}})\,\mathbf{e}^{-\mathbf{i}\omega\,(\mathbf{t}-\mathbf{t}')} \;=\; \mathcal{D}(\mathbf{i}\partial_{\mathrm{t}})\,\delta(\mathbf{t}-\mathbf{t}')$$

$$\vec{\mathbf{J}}(\mathbf{t},\mathbf{x}) \;=\; \mathcal{D}(\mathbf{i}\partial_{\mathrm{t}},\nabla^2)\,\vec{\nabla}\mathbf{J}^0(\mathbf{t},\mathbf{x}) \;+\; \sigma_{\mathrm{e}}(\mathbf{i}\partial_{\mathrm{t}},\nabla^2)\,\vec{\mathbf{E}}(\mathbf{t},\mathbf{x})$$

Gradient (small momenta) expansion:

$$\mathcal{D}(\mathbf{i}\partial_{\mathrm{t}},\nabla^2) \;=\; \mathbf{D_0}\,[\mathbf{1} \;+\; \mathbf{i}\tau\,\partial_{\mathrm{t}} \;+\; \lambda\,\nabla^2 \;+\; \ldots.], \qquad \sigma_{\mathrm{e}}(\mathbf{i}\partial_{\mathrm{t}},\nabla^2) \;=\; \sigma_{\mathbf{0}}\,[\mathbf{1} \;+\; \mathbf{i}\tau_{\sigma}\,\partial_{\mathrm{t}} \;+\; \ldots.]$$

σ_0 **is a DC conductivity.**

Another way of steering the system out of equilibrium is by magnetic field

$$\vec{\mathbf{J}}(\mathbf{t},\mathbf{x}) \;=\; \mathcal{D}(\mathbf{i}\partial_{\mathrm{t}},\nabla^2)\,\vec{\nabla}\mathbf{J}^0(\mathbf{t},\mathbf{x}) \;+\; \sigma_{\mathrm{e}}(\mathbf{i}\partial_{\mathrm{t}},\nabla^2)\,\vec{\mathbf{E}}(\mathbf{t},\mathbf{x}) \;+\; \sigma_{\mathrm{m}}(\mathbf{i}\partial_{\mathrm{t}},\nabla^2)\,\vec{\nabla}\times\vec{\mathbf{B}}(\mathbf{t},\mathbf{x})$$

This is the most general linear constitutive relation.

Neutral conformal fluids in a weakly curved 4d background

A way to steer a neutral fluid out of equilibrium is to shake it by an external metric perturbation

Most general constitutive relation with weakly curved metric $(\mathbf{h}_{\mu\nu} \sim \mathbf{u}_\mathbf{i})$

E. Shuryak, M.L., D80 (2009) 065026

Y. Bu and M. L, JHEP 1504, 136 (2015)

$$\mathbf{\Pi}_{\mu\nu} = -\eta\nabla_\mu \mathbf{u}_\nu - \zeta\nabla_\mu\nabla_\nu\nabla\mathbf{u} + \kappa\mathbf{u}^\alpha\mathbf{u}^\beta\mathbf{C}_{\mu\alpha\nu\beta} + \rho\mathbf{u}^\alpha\nabla^\beta\mathbf{C}_{\mu\alpha\nu\beta} + \xi\nabla^\alpha\nabla^\beta\mathbf{C}_{\mu\alpha\nu\beta} - \theta\mathbf{u}^\alpha\nabla_\alpha\mathbf{R}_{\mu\nu},$$

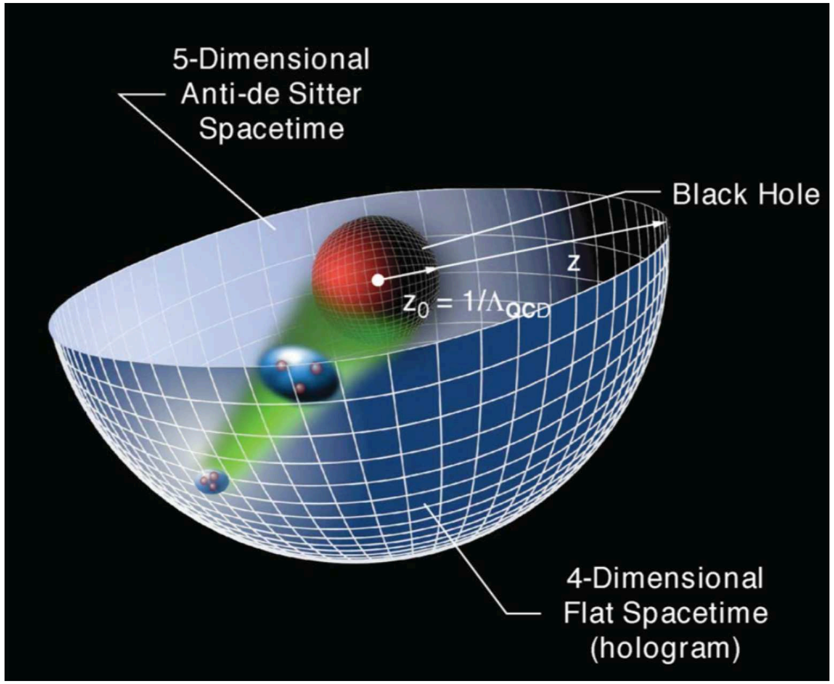
$\mathbf{C}_{\mu\alpha\nu\beta}$, $\mathbf{R}_{\mu\nu}$ are the Weyl and Ricci tensors of $h_{\mu\nu}$.

$\kappa(\omega, \mathbf{q}^2)$, $\rho(\omega, \mathbf{q}^2)$, $\xi(\omega, \mathbf{q}^2)$, $\theta(\omega, \mathbf{q}^2)$ - Gravitational Susceptibilities of the Fluid (GSFs).

All GSFs contribute to two-point correlators

AdS/CFT

QCD $\longrightarrow \mathcal{N}=4$ SYM (CFT). Strong coupling (and large N_c) $\to$ AdS/CFT $\to$ SUGRA on AdS$_5$. CFT at finite Temperature $\leftrightarrow$ AdS Black Hole



Bulk fields (gravitons, photons, etc) propagate signals from the horizon to the boundary, where the hologram is captured.

The bulk acts as a highly nonlinear dispersive medium.

There is no dissipation in the bulk.

All dissipative effects take place at the horizon.

5d Geometry

5d GR with negative cosmological constant:

$$\mathbf{S} \;=\; \frac{\mathbf{1}}{\mathbf{16\pi G_N}} \int \mathbf{d^5x} \sqrt{\mathbf{-g}}\; (\mathbf{R} \;+\; \mathbf{12})\,,$$

Einstein Equations

$$\mathbf{E_{MN}} \;\equiv\; \mathbf{R_{MN}} \;-\; \frac{\mathbf{1}}{\mathbf{2}}\mathbf{g_{MN}}\,\mathbf{R} \;-\; \mathbf{6}\,\mathbf{g_{MN}} \;=\; \mathbf{0}.$$

Schwarzschild-AdS_5 geometry (ingoing Eddington-Finkelstein coordinates)

$$\mathbf{ds^2 = g_{MN}dx^M dx^N = 2dtdr - r^2 f(r)dt^2 + r^2\delta_{ij}dx^i dx^j},$$

$\mathbf{f(r) = 1 - 1/r^4}$. **The horizon is at $\mathrm{r} = 1$, the Hawking temperature is $\pi\mathrm{T} = 1$.**

5d Maxwell field

Maxwell field in Schwarzschild-AdS_5 geometry (probe approximation)

$$\mathbf{S} \;=\; -\int \mathbf{d^5x}\,\sqrt{-\mathbf{g}}\,\frac{\mathbf{1}}{\mathbf{4}}\,\mathbf{e^2}\,(\mathbf{F^V})_{\mathbf{MN}}(\mathbf{F^V})^{\mathbf{MN}} + \mathbf{S_{c.t.}}$$

Maxwell equations

$$\mathbf{EQ^N} := \nabla_{\mathbf{M}}\mathbf{F^{MN}} \;=\; \mathbf{0}$$

4 dynamical eqns $\mathbf{EQ}^{\mu} \;=\; \mathbf{0} \rightarrow$ transport, $\mathbf{EQ^r} \;=\; \mathbf{0} \rightarrow$ current conservations.

Near the conformal boundary $r = \infty$ the solution is expandable in a series ($A_r = 0$)

$$\mathbf{A}_{\mu}(\mathbf{r},\mathbf{x}_{\alpha}) = \mathbf{A}^{(0)}_{\mu}(\mathbf{x}_{\alpha}) + \frac{\mathbf{A}^{(1)}_{\mu}(\mathbf{x}_{\alpha})}{\mathbf{r}} + \frac{\mathbf{A}^{(2)}_{\mu}(\mathbf{x}_{\alpha})}{\mathbf{r^2}} + \frac{\mathbf{B}^{(2)}_{\mu}(\mathbf{x}_{\alpha})}{\mathbf{r^2}}\log\mathbf{r}^{-2} + \mathcal{O}\left(\frac{\log\mathbf{r}^{-2}}{\mathbf{r^3}}\right),$$

The boundary current (using the holographic dictionary)

$$\mathbf{J}^{\mu} = -\eta^{\mu\nu}\left(\mathbf{2A}^{(2)}_{\nu} + \mathbf{2B}^{(2)}_{\nu} + \eta^{\sigma\mathbf{t}}\partial_{\sigma}\mathbf{F}^{(0)}_{\mathbf{t}\nu}\right).$$

$\mathbf{EQ}^{\mu} = \mathbf{0}$ admit the most general static homogeneous solutions

$$\mathbf{A}_\mu = \mathbf{A}_\mu^{(0)} + \frac{\rho}{\mathbf{2r^2}}\delta_{\mu \mathrm{t}}, \qquad \mathbf{A}_\mu^{(0)} = \mathbf{const}, \quad \rho = \mathbf{const}$$

The boundary theory is a static uniformly charged plasma with no external fields

$$\mathbf{J}^0 = \rho, \qquad \vec{\mathbf{J}} = \mathbf{0}$$

Next, following the spirit of S. Bhattacharyya, V. E. Hubeny, S. Minwalla, and M. Rangamani, (2008)

$$\mathbf{A}_\mu^{(0)} \to \mathbf{A}_\mu^{(0)}(\mathbf{x}_\alpha), \qquad \rho \to \rho(\mathbf{x}_\alpha).$$

The solution has to be amended:

$$\mathbf{A}_\mu(\mathbf{r}, \mathbf{x}_\alpha) = \mathbf{A}_\mu^{(0)}(\mathbf{x}_\alpha) + \frac{\rho(\mathbf{x}_\alpha)}{\mathbf{2r^2}}\delta_{\mu \mathrm{t}} + \mathbf{a}_\mu(\mathbf{r}, \mathbf{x}_\alpha)$$

Solve for a (bulk-to-boundary propagator). $\mathbf{a}[\mathbf{A^0},\ \rho]$ is linear both in $\mathbf{A^0}$ and ρ

Different from approaches based on two-point correlators, which impose current conservation (on-shellness)

$U(1)$ vector current: Diffusion and Conductivity

$$\vec{\mathbf{J}}(\omega,\vec{\mathbf{q}}) = -\mathcal{D}\left(\omega,\mathbf{q}^2\right)\mathbf{i}\vec{\mathbf{q}}\,\rho(\omega,\vec{\mathbf{q}}) + \sigma_{\mathrm{e}}\left(\omega,\mathbf{q}^2\right)\vec{\mathbf{E}}(\omega,\vec{\mathbf{q}}) + \sigma_{\mathrm{m}}\left(\omega,\mathbf{q}^2\right)\mathbf{i}\vec{\mathbf{q}}\times\vec{\mathbf{B}}\,(\omega,\vec{\mathbf{q}})\,.$$

$$\mathcal{D} = \mathbf{D_0}\left[\mathbf{1} + \mathbf{i}\tau\,\partial_{\mathrm{t}} + \lambda\,\nabla^2 + \ldots.\right] = \frac{\mathbf{1}}{\mathbf{2}} + \frac{\mathbf{1}}{\mathbf{8}}\pi\mathbf{i}\omega + \frac{\mathbf{1}}{\mathbf{48}}\left[-\pi^2\omega^2 + \mathbf{q}^2\left(\mathbf{6}\log\mathbf{2} - \mathbf{3}\pi\right)\right] + \cdots,$$

$$\sigma_{\mathrm{e}} = \sigma_0\left[\mathbf{1} + \mathbf{i}\tau_\sigma\,\partial_{\mathrm{t}} + \ldots.\right] = \mathbf{1} + \frac{\log\mathbf{2}}{\mathbf{2}}\mathbf{i}\omega + \frac{\mathbf{1}}{\mathbf{24}}\left[\pi^2\omega^2 - \mathbf{q}^2\left(\mathbf{3}\pi + \mathbf{6}\log\mathbf{2}\right)\right] + \cdots,$$

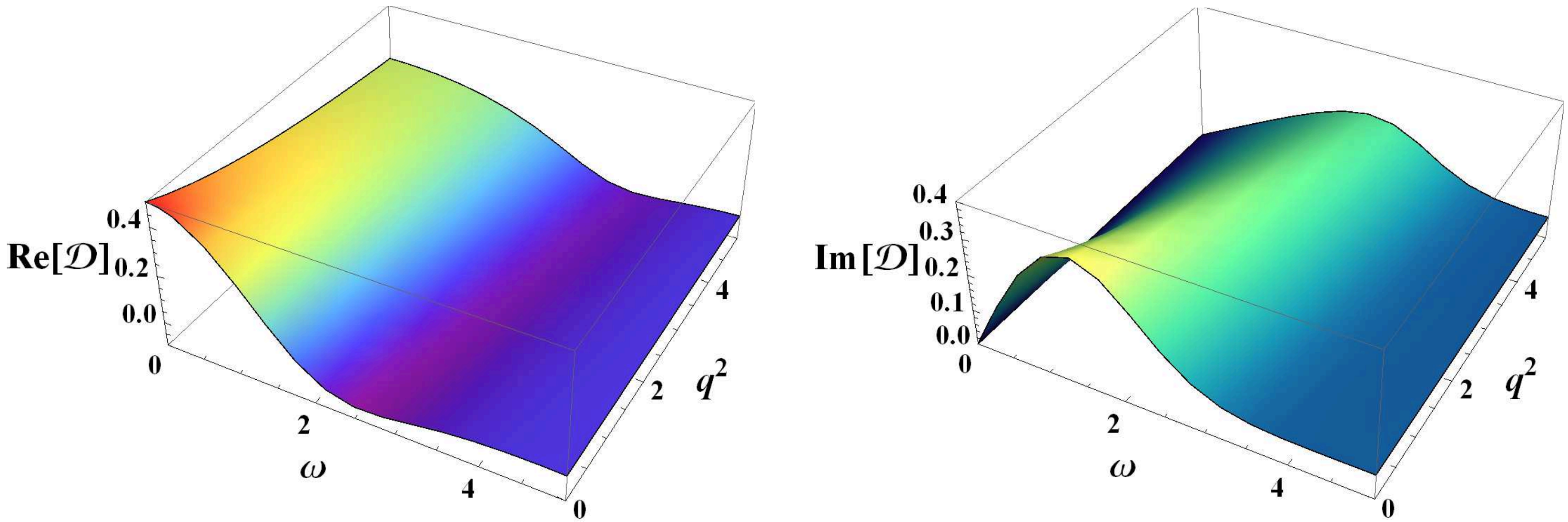
$$\sigma_{\mathrm{m}} = \mathbf{0} + \frac{\mathbf{1}}{\mathbf{16}}\mathbf{i}\omega\left(\mathbf{2}\pi - \pi^2 + \mathbf{4}\log\mathbf{2}\right) + \cdots.$$

Blue terms are new! Y. Bu, M. Lublinsky and A. Sharon, JHEP 1604, 136 (2016)

$\sigma_{\mathrm{m}}^0 > 0$ in a pure QED plasma with one Dirac fermion at one loop level

B. B. Brandt, A. Francis, and H. B. Meyer, (2014)

$\sigma_{\mathrm{m}}^0 = 0$ based on Boltzmann equations J. Hong and D. Teaney, (2010)



Re[$\mathcal{D}$]
0.4
0.2
0.0
0
2
4
ω
0
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q^2
Im[$\mathcal{D}$]
0.4
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0.2
0.1
0.0
0
2
4
ω
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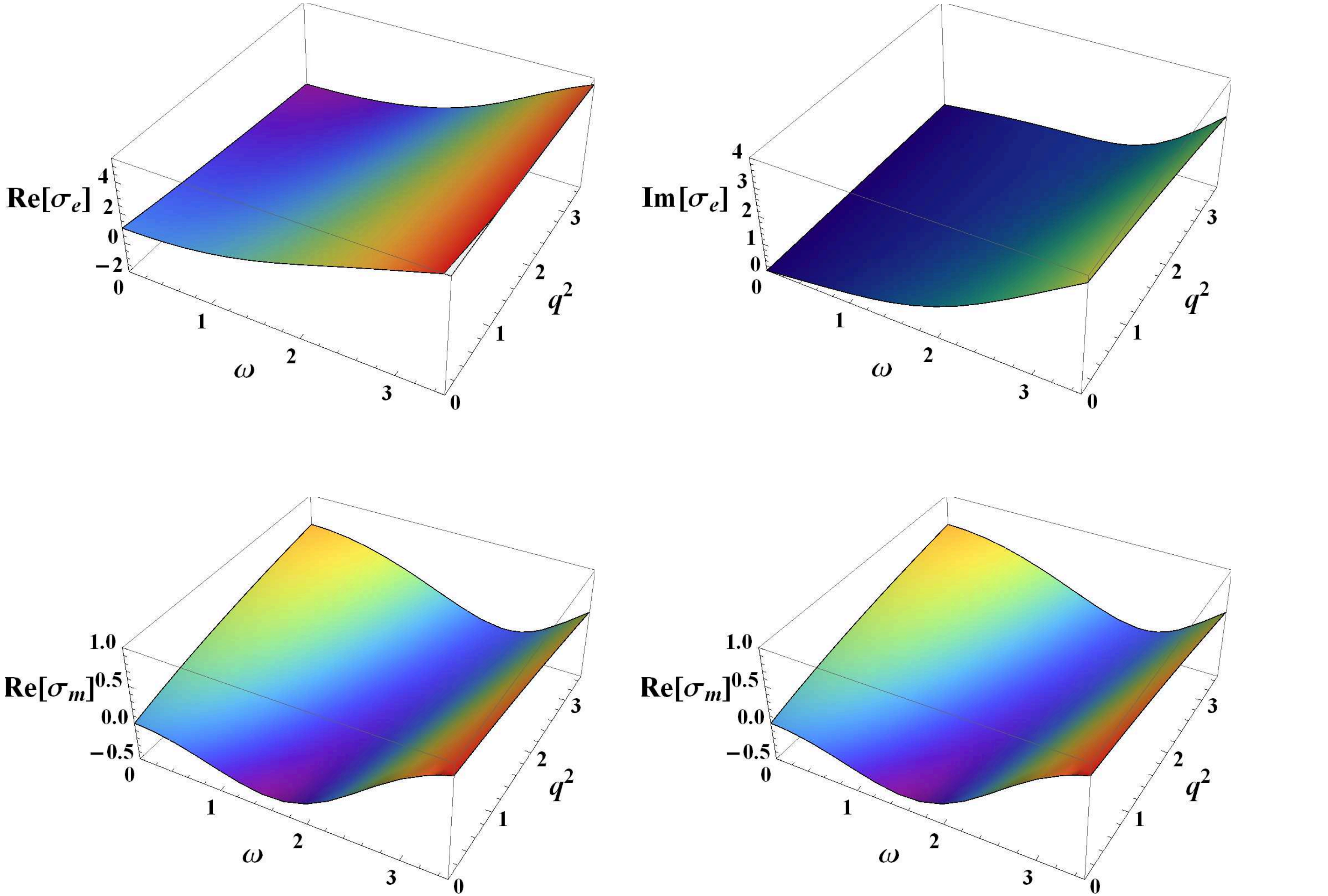
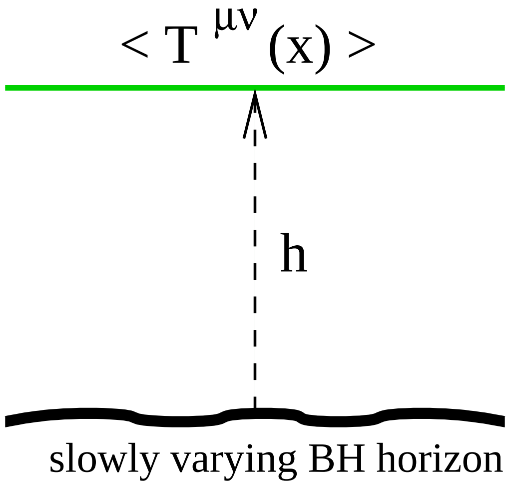
2
4
q^2



Re[σ_e]
Im[σ_e]
Re[σ_m]
Re[σ_m]
ω
q^2

Hydrodynamics from Fluid-Gravity correspondence

5d GR with negative cosmological constant:



$$\mathbf{S} = \frac{1}{16\pi G_N} \int d^5x \sqrt{-g} \,(R + 12)\,,$$

Einstein Equations

$$E_{MN} \equiv R_{MN} - \frac{1}{2} g_{MN} R - 6\, g_{MN} = 0.$$

Solution: Boosted Black Brane in asymptotic AdS$_5$

$$ds^2 = -2u_\mu dx^\mu dr - r^2 f\,(br)\, u_\mu u_\nu dx^\mu dx^\nu + r^2 \mathcal{P}_{\mu\nu} dx^\mu dx^\nu,$$

$$f(r) = 1 - 1/r^4 \quad \text{and} \quad \mathcal{P}_{\mu\nu} = \eta_{\mu\nu} + u_\mu u_\nu$$

Hawking temperature

$$T = \frac{1}{\pi b},$$

S. Bhattacharyya, V. E Hubeny, S. Minwalla, M. Rangamani, JHEP 0802:045, (2008):

Promote u_i and b into a slowly varying functions of boundary coordinates x^α

$$\mathbf{ds^2 = -2u_\mu(x^\alpha)dx^\mu dr - r^2 f\,(b(x^\alpha)r)\,u_\mu(x^\alpha)u_\nu(x^\alpha)dx^\mu dx^\nu + r^2\mathcal{P}_{\mu\nu}(x^\alpha)dx^\mu dx^\nu},$$

Use gradient expansion of the fields $\mathbf{u(x) = u_0 + \delta x\,\nabla u}$ and $\mathbf{b(x) = b + \delta x \nabla b}$ to set up a perturbative procedure

The stress tensor for the dual fluid

$$\mathbf{T}^\mu_\nu = \lim_{\mathbf{r}\to\infty} \tilde{\mathbf{T}}^\mu_\nu(\mathbf{r})\,; \qquad \tilde{\mathbf{T}}^\mu_\nu(\mathbf{r}) \equiv \mathbf{r}^4\left(\mathcal{K}^\mu_\nu - \mathcal{K}\gamma^\mu_\nu + \mathbf{3}\gamma^\mu_\nu - \frac{\mathbf{1}}{\mathbf{2}}\mathbf{G}^\mu_\nu\right),$$

$$\mathbf{T}^{\mu\nu} = \mathbf{T}^{\mu\nu}_{\mathbf{ideal}} + \mathbf{\Pi}^{\mu\nu}_{\mathbf{NS}} + \tau_{\mathbf{R}}\,(\mathbf{u}\,\nabla)\,\mathbf{\Pi}^{\mu\nu}_{\mathbf{NS}} + \mathbf{O}[\,(\nabla\,\mathbf{u})^2\,]$$

P. Kovtun, G. Policastro, D. Son, A. Starinets (2001-2004)

$$\frac{\eta_0}{\mathbf{s}} = \frac{\mathbf{1}}{\mathbf{4}\,\pi}, \qquad\qquad \tau_{\mathbf{R}} = \mathbf{2} - \log(\mathbf{2})$$

R. Baier, P. Romatschke, D. T. Son, A. O. Starinets, M. A. Stephanov

JHEP 0804, 100 (2008)

M. P. Heller and R. A. Janik, Phys. Rev. D 76, 025027 (2007)

We do it somewhat differently, linearizing in the velocity amplitude

$$\mathbf{u}_\mu(\mathbf{x}^\alpha) = (-\mathbf{1}, \epsilon\beta_\mathbf{i}(\mathbf{x}^\alpha)) + \mathcal{O}(\epsilon^2), \qquad \mathbf{b}(\mathbf{x}^\alpha) = \mathbf{b}_0 + \epsilon\mathbf{b}_1(\mathbf{x}^\alpha) + \mathcal{O}(\epsilon^2),$$

"seed" metric, i.e., a linearized version of the BH metric

$$\mathbf{ds}^2_{\rm seed} = \mathbf{2drdv} - \mathbf{r}^2\mathbf{f}(\mathbf{r})\mathbf{dv}^2 + \mathbf{r}^2\mathbf{d}\vec{\mathbf{x}}^2 - \epsilon\left[\mathbf{2}\beta_\mathbf{i}(\mathbf{x}^\alpha)\mathbf{drdx}^\mathbf{i} + \frac{\mathbf{2}}{\mathbf{r}^2}\beta_\mathbf{i}(\mathbf{x}^\alpha)\mathbf{dvdx}^\mathbf{i} + \frac{\mathbf{4}}{\mathbf{r}^2}\mathbf{b}_1(\mathbf{x}^\alpha)\mathbf{dv}^2\right] + \mathcal{O}(\epsilon^2),$$

$$\mathbf{ds}^2 = \mathbf{ds}^2_{\rm seed} + \mathbf{ds}^2_{\rm corr}[\beta] \qquad \text{gauge fix} \quad \mathbf{g}_{\mathbf{rr}} = \mathbf{0}, \quad \mathbf{g}_{\mathbf{r}\mu} \propto \mathbf{u}_\mu$$

$$\mathbf{ds}^2_{\rm corr} = \epsilon\left(-\mathbf{3h}\,\mathbf{drdv} + \frac{\mathbf{k}}{\mathbf{r}^2}\mathbf{dv}^2 + \mathbf{r}^2\,\mathbf{h}\,\mathbf{d}\vec{\mathbf{x}}^2 + \frac{\mathbf{2}}{\mathbf{r}^2}\mathbf{j}_\mathbf{i}\,\mathbf{dvdx}^\mathbf{i} + \mathbf{r}^2\,\alpha_{\mathbf{ij}}\,\mathbf{dx}^\mathbf{i}\mathbf{dx}^\mathbf{j}\right)$$

$\mathbf{h}[\beta]$, $\mathbf{k}[\beta]$, $\mathbf{j}[\beta]$, $\alpha[\beta]$ are to be found by solving the Einstein equations.

Boundary cond: no singularities

$$\mathbf{h} < \mathcal{O}(\mathbf{r}^0), \quad \mathbf{k} < \mathcal{O}(\mathbf{r}^4), \quad \mathbf{j}_\mathbf{i} < \mathcal{O}(\mathbf{r}^4), \quad \alpha_{\mathbf{ij}} < \mathcal{O}(\mathbf{r}^0).$$

Einstein equations for the metric corrections

Dynamical equations:

$$\mathbf{E_{rr}} = 0 : \qquad 5\,\partial_r h + r\partial_r^2 h = 0\,.$$

$$\mathbf{E_{rv}} = 0 : \qquad 3\,r^2\partial_r k = 6\,r^4\,\partial\beta + r^3\,\partial_v\partial\beta - 2\,\partial j - r\,\partial_r\partial j - r^3\,\partial_i\partial_j\alpha_{ij}$$

$$\mathbf{E_{ri}} = 0 : \qquad -\partial_r^2 j_i = \left(\partial^2\beta_i - \partial_i\partial\beta\right) + 3r\,\partial_v\beta_i - \tfrac{3}{r}\partial_r j_i + r^2\partial_r\partial_j\alpha_{ij}\,.$$

$$\begin{aligned}\mathbf{E_{ij}} = 0 : \quad & (r^7 - r^3)\partial_r^2\alpha_{ij} + (5r^6 - r^2)\partial_r\alpha_{ij} + 2r^5\partial_v\partial_r\alpha_{ij} + 3r^4\partial_v\alpha_{ij} \\ & + r^3\left\{\partial^2\alpha_{ij} - \left(\partial_i\partial_k\alpha_{jk} + \partial_j\partial_k\alpha_{ik} - \frac{2}{3}\delta_{ij}\partial_k\partial_l\alpha_{kl}\right)\right\} \\ & + \left(\partial_i j_j + \partial_j j_i - \frac{2}{3}\delta_{ij}\partial j\right) - r\partial_r\left(\partial_i j_j + \partial_j j_i - \frac{2}{3}\delta_{ij}\partial j\right) \\ & + 3r^4\left(\partial_i\beta_j + \partial_j\beta_i - \frac{2}{3}\delta_{ij}\partial\beta\right) + r^3\partial_v\left(\partial_i\beta_j + \partial_j\beta_i - \frac{2}{3}\delta_{ij}\partial\beta\right) = 0\,.\end{aligned}$$

Constraint equations

$$\mathbf{E_{vv}} = 0 \ \text{ and } \ \mathbf{E_{vi}} = 0 \quad \longrightarrow \quad \partial_\mu T^{\mu\nu} = 0$$

Results from the Fluid/Gravity correspondence

$$\mathbf{\Pi}_{\mu\nu} = -\eta\nabla_\mu \mathbf{u}_\nu - \zeta\nabla_\mu\nabla_\nu\nabla\mathbf{u} + \kappa\mathbf{u}^\alpha\mathbf{u}^\beta\mathbf{C}_{\mu\alpha\nu\beta} + \rho\mathbf{u}^\alpha\nabla^\beta\mathbf{C}_{\mu\alpha\nu\beta} + \xi\nabla^\alpha\nabla^\beta\mathbf{C}_{\mu\alpha\nu\beta} - \theta\mathbf{u}^\alpha\nabla_\alpha\mathbf{R}_{\mu\nu},$$

Analytical results in the hydrodynamic regime $\omega, \mathbf{q} \ll \mathbf{1}$ **($\pi\mathbf{T} = \mathbf{1}$):**

$$\eta(\omega, \mathbf{q}^2) = \mathbf{2} + (\mathbf{2} - \ln\mathbf{2})\mathbf{i}\omega - \frac{\mathbf{1}}{\mathbf{4}}\mathbf{q}^2 - \frac{\mathbf{1}}{\mathbf{24}}\left[\mathbf{6}\pi - \pi^2 + \mathbf{12}\left(\mathbf{2} - \mathbf{3}\ln\mathbf{2} + \ln^2\mathbf{2}\right)\right]\omega^2 + \cdots$$

$$\zeta(\omega, \mathbf{q}^2) = \frac{\mathbf{1}}{\mathbf{12}}(\mathbf{5} - \pi - \mathbf{2}\ln\mathbf{2}) + \cdots$$

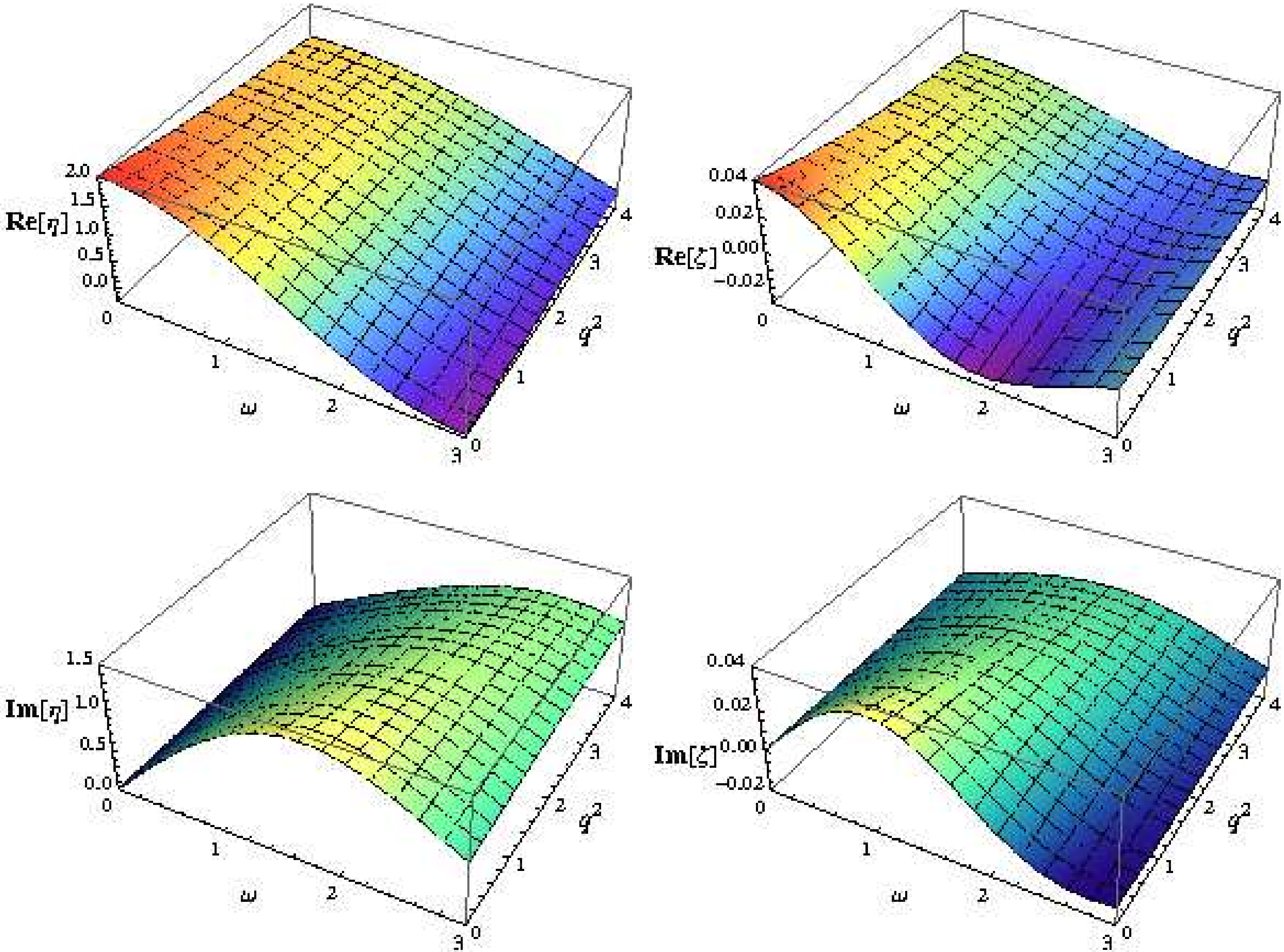
Blue terms are new!

$$\kappa = \mathbf{2} + \frac{\mathbf{1}}{\mathbf{4}}(\mathbf{5} + \pi - \mathbf{6}\log\mathbf{2})\,\mathbf{i}\omega + \cdots, \quad \rho = \mathbf{2} + \cdots, \quad \xi = \ln\mathbf{2} - \frac{\mathbf{1}}{\mathbf{2}} + \cdots, \quad \theta = \frac{\mathbf{3}}{\mathbf{2}}\zeta.$$

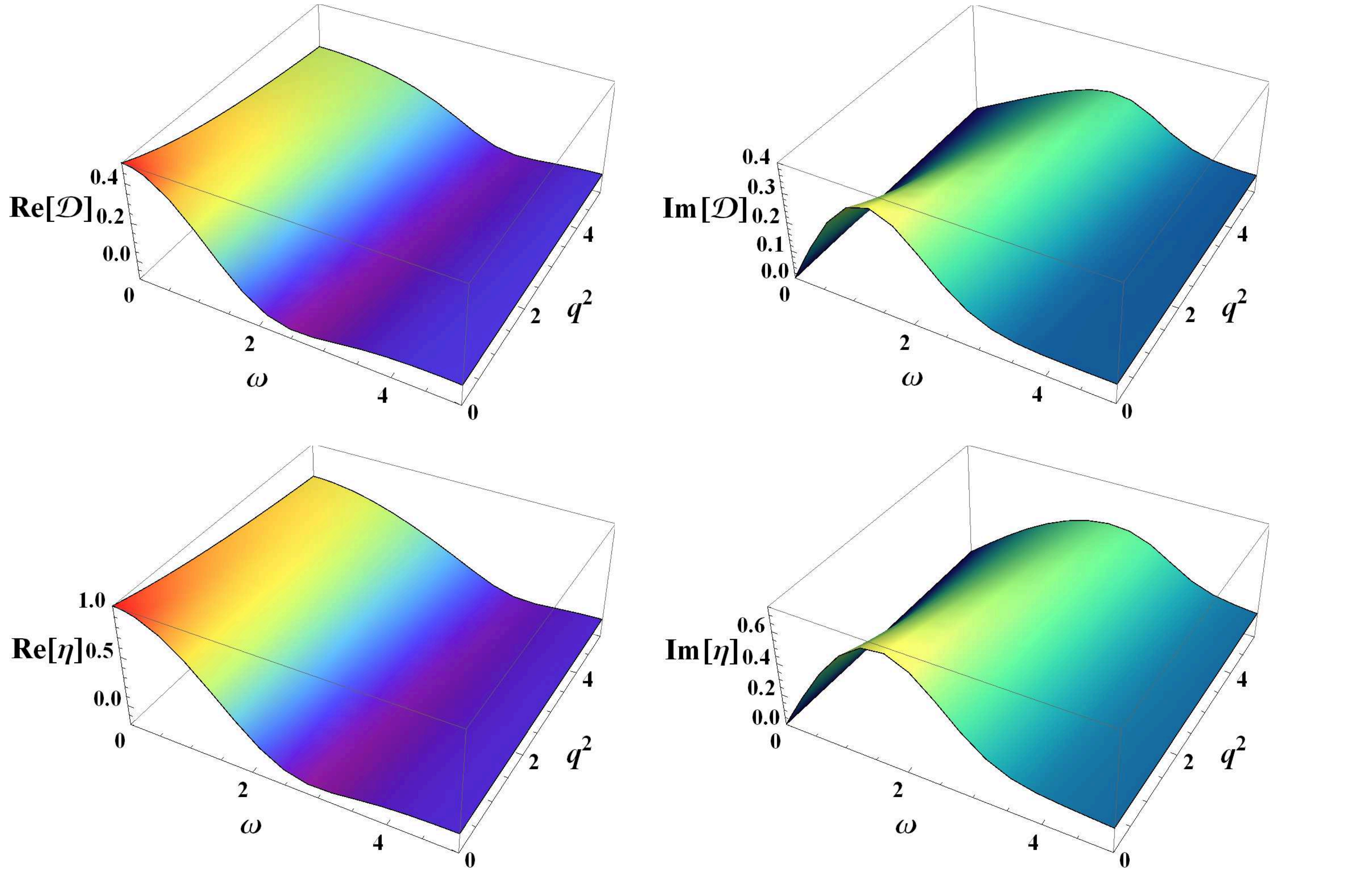
R. Baier, P. Romatschke, D. T. Son, A. O. Starinets and M. A. Stephanov, JHEP 0804, 100 (2008)

Modified sound dispersion:

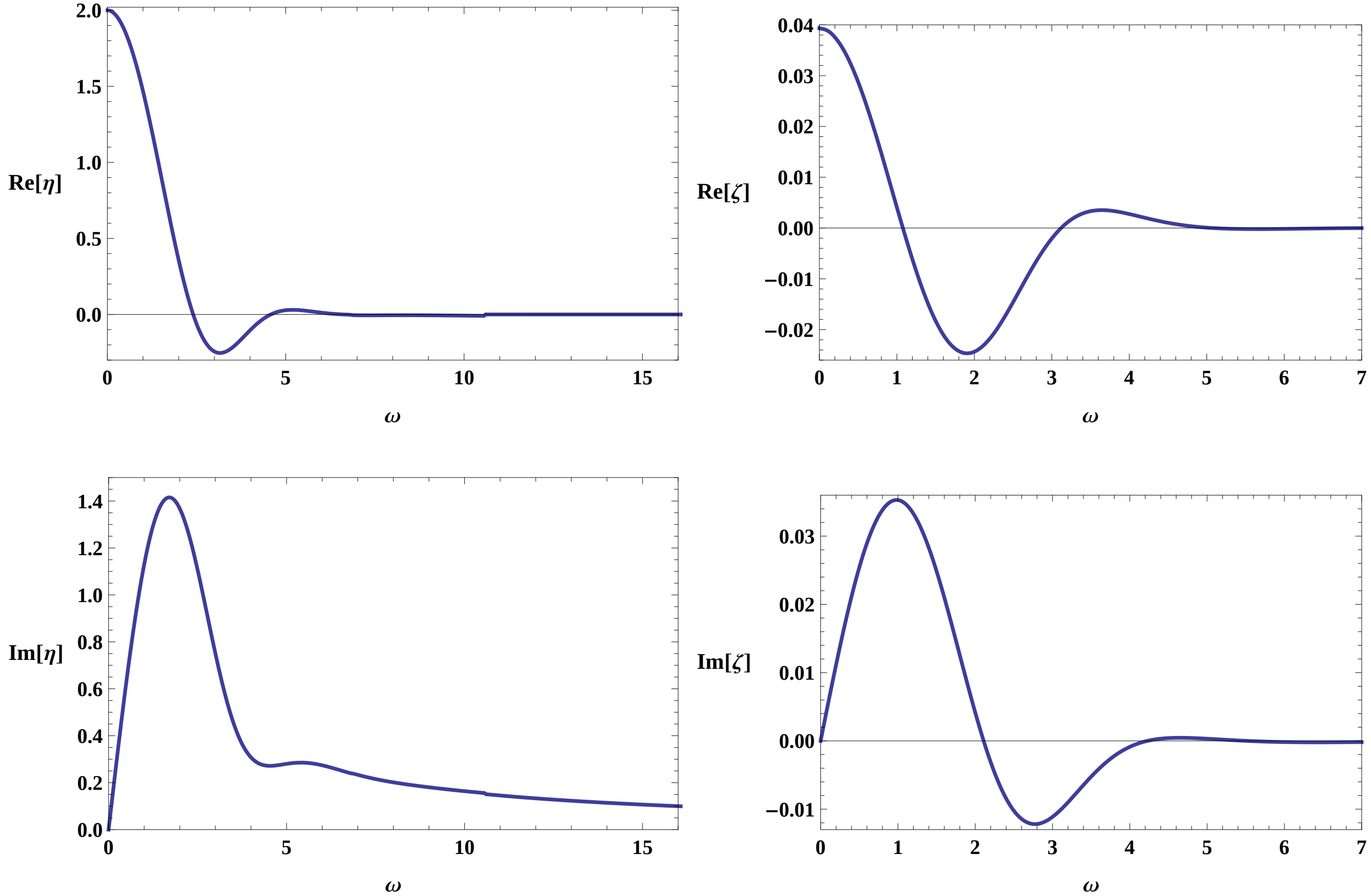
$$\omega = \pm\frac{\mathbf{1}}{\sqrt{\mathbf{3}}}\mathbf{q} - \frac{\mathbf{i}}{\mathbf{6}}\mathbf{q}^2 \pm \frac{\mathbf{1}}{\mathbf{24}\sqrt{\mathbf{3}}}(\mathbf{3} - \mathbf{2}\ln\mathbf{2})\,\mathbf{q}^3 + \frac{\mathbf{i}}{\mathbf{288}}\left(\mathbf{8} - \frac{\pi^2}{\mathbf{3}} + \mathbf{4}\ln^2\mathbf{2} - \mathbf{4}\ln\mathbf{2}\right)\mathbf{q}^4 +$$



Re[η]
2.0
1.5
1.0
0.5
0.0
0
1
2
3
ω
0
1
2
3
4
q^2
Re[ζ]
0.04
0.02
0.00
−0.02
Im[η]
1.5
1.0
0.5
0.0
Im[ζ]
0.04
0.02
0.00
−0.02



Re[𝒟]
0.4
0.2
0.0
0
2
4
ω
q²
Im[𝒟]
0.4
0.3
0.2
0.1
0.0
Re[η]
1.0
0.5
0.0
Im[η]
0.6
0.4
0.2
0.0



$q^2 = 0$

- **Real parts of the viscosities are decreasing functions of momenta. Oscillations are consistent with the expectations about the viscosities have infinitely many complex poles.**
- **Imaginary parts have a clear maximum near $\omega \sim 2$, introducing a (new?) transition scale.**
- **Viscosity vanish at large momenta, which is what is required to restore causality.**
- **ζ is always subleading vs η.**

Conclusions

- **Memory function is an important ingredient of causal relativistic hydrodynamics. Fluid-gravity correspondence provides a calculational framework to rigorously address transports In QFT, including all order resummation. Unfortunately not in QCD ...**

- **All order dissipative terms of a weakly perturbed conformal fluid are fully accounted for by two shear viscosity functions $\eta(\omega, q^2)$ and $\zeta(\omega, q^2)$. We propose to use all-order viscosity functions for hydro simulations as an improvement beyond the Israel-Stewart formalism.**

- **For a weakly curved background space, there are additional four transport functions called Gravitational Susceptibilities of the Fluid.**

- **An off-shell constitutive relation for $U(1)$ current consists of a momenta-dependent diffusion term and two conductivities. Certain universality between dissipative transport coefficients η and $\mathcal{D}$ is observed.**

- **At large momenta, the effective viscosity (diffusion constant) is a decreasing function of both frequency and momentum. The corresponding memory functions have support in the past only, the behaviour consistent with causality restoration.**