Initial state perspective on the ridge

(weak-coupling)

Odd azimuthal anisotropy in CGC

Vladimir Skokov (RBRC BNL)

September 18, 2017

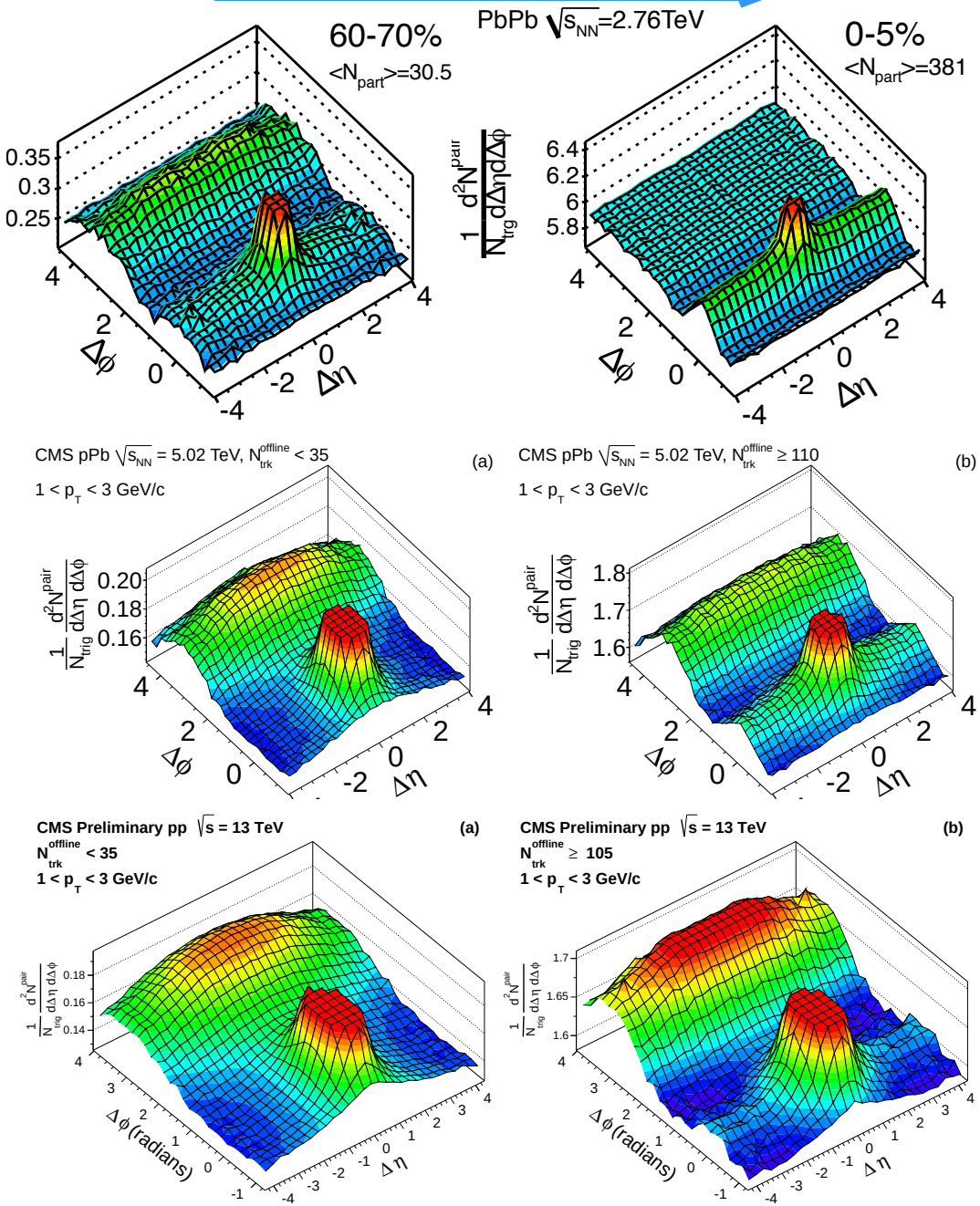
Main Focus: L. McLerran and V. S., arXiv:1611.09870
Outlook: A. Kovner, M. Lublinsky and V. S., arXiv:1612.07790
γ_{SS} in CGC: A. Kovner, M. Lublinsky and V. S., arXiv:1706.02330

Striking resemblance

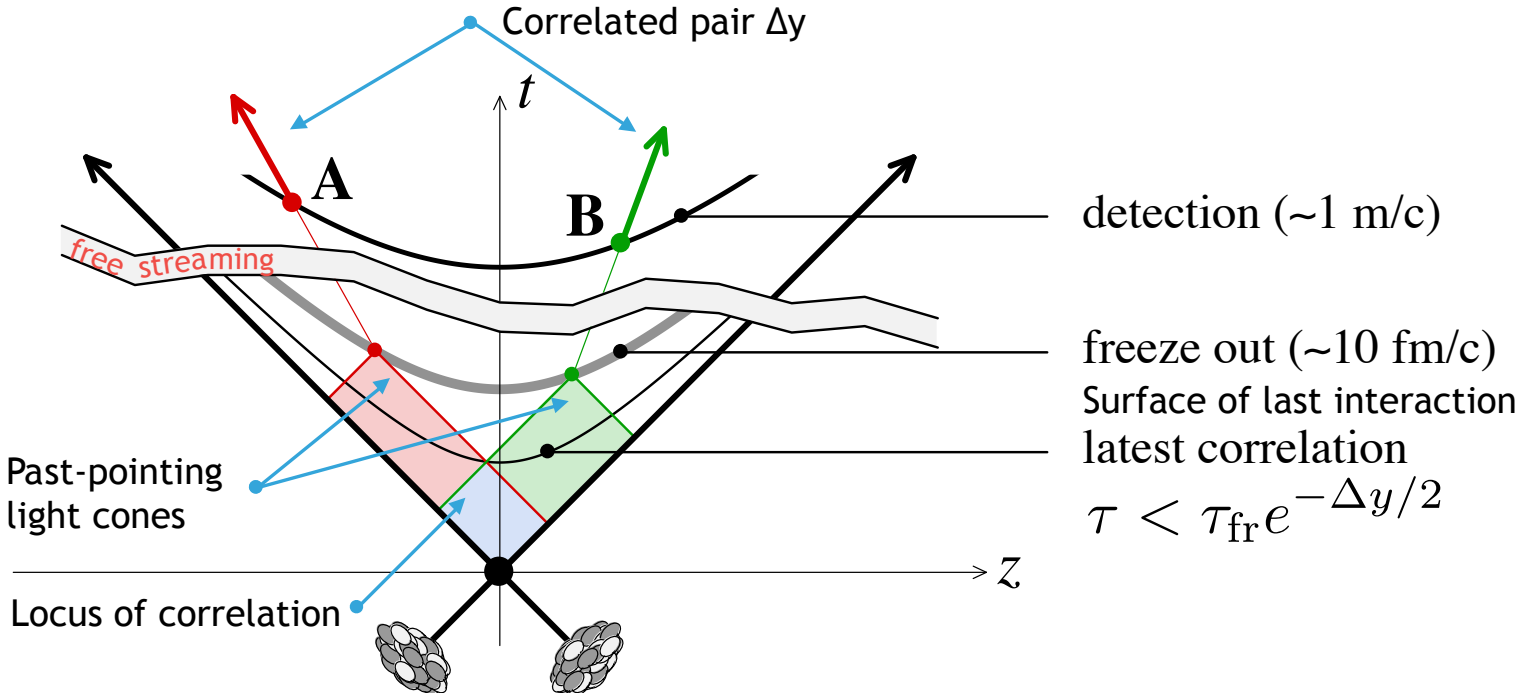
- Present also in multi-particle correlations
- Collectivity in systems of drastically different sizes
- Does it imply the same underlying mechanism?

System Size

Multiplicity



Introduction: the Ridge



Suppress final state effects in order to probe initial state physics

- Regardless of nature of the ridge
  - long-range rapidity correlations either pre-exist in initial wave function or develop very early after collision
  - understanding initial stage – creation of quarks and gluons before possible thermalization and formation of QGP – is of paramount importance for understanding A-A, p-A and p-p.

figure adapted from A. Dumitru, F. Gelis, L. McLerran and R. Venugopalan, arXiv:0804.3858

Collectivity: possible origin

Final state

- Driven by initial state geometric correlations
- Develops gradually during (hydro) evolution
- Requires large multiplicities to facilitate final state interaction
- Requires non-trivial initial state geometry (proton shape fluctuations)

VS

Initial state

- Driven by initial momentum correlations
- Pre-exist before collisions or develops very soon after
- High-multiplicities are not required, but allowed
- Momentum correlations are present (suppressed by $1/N_c^2$) for "round" p

Collectivity: possible origin

Final state

- Driven by initial state geometric correlations
- Develops gradually during (hydro) evolution
- Requires large multiplicities to facilitate final state interaction
- Requires non-trivial initial state geometry (proton shape fluctuations)

~~vs~~

and

Initial state

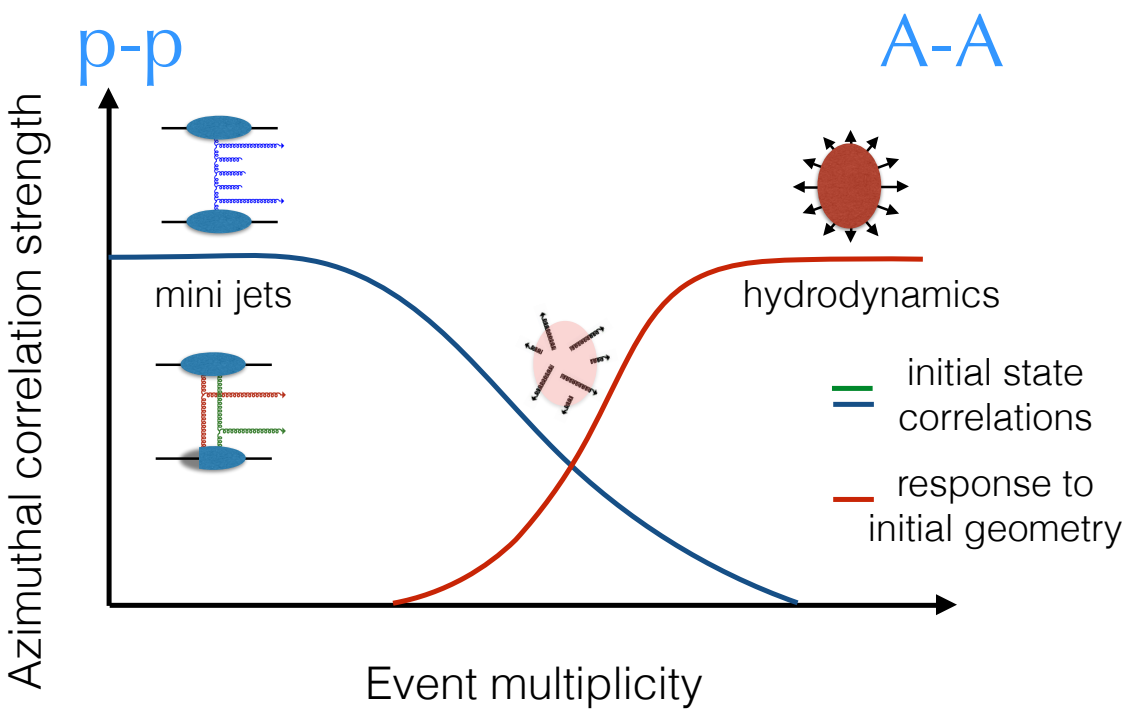
- Driven by initial momentum correlations
- Pre-exist before collisions or develops very soon after
- High-multiplicities are not required, but allowed
- Momentum correlations are present (suppressed by $1/N_c^2$) for "round" p

Collectivity: tuning in

Final state — Initial state

control parameters

multiplicity, transverse momentum, system size

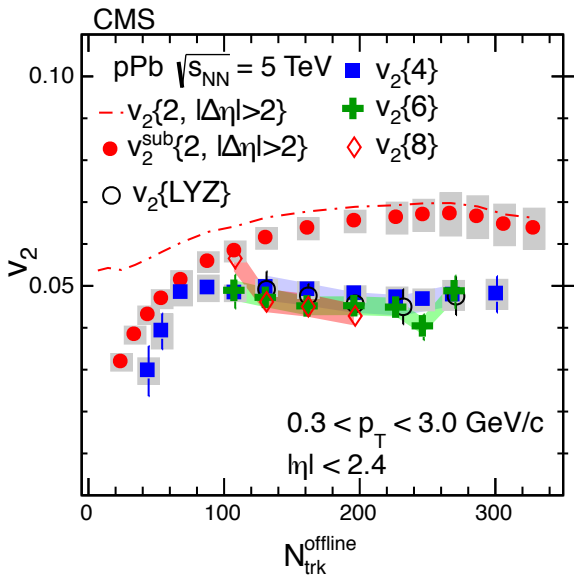


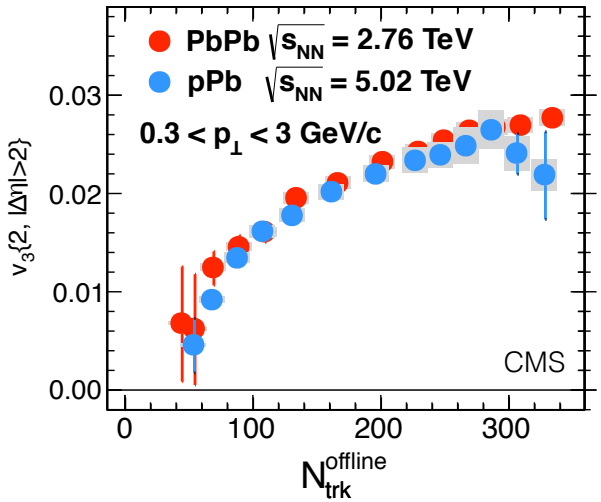
Adapted from S. Schlichting and P. Tribedy, arXiv:1611.00329

Challenges and approaches

- Higher order cumulants and hierarchy
  - — Dipole models (talks by M. Mace and A. Kovner)
  - — Finite number of sources / interference: $1/N_s$ and $1/N_c$ effects (talks by U. Wiedemann, A. Rezaeian)
  - — Domains/Pomeron loops (A. Kovner, M. Lublinsky, A. Dumitru, V. S. etc)
- Odd harmonics
  - — Ad-hoc models based on quark scattering
  - — Numerical classical Yang-Mills
  - — "First saturation" correction to momentum-odd double inclusive production (this talk)
- Hierarchy in p-A, d-A, He^3-A collisions at RHIC
  - — ?
- Multiplicity bias
  - — Cranking up Q_s might not be enough; (. . . , T. Liou, A. H. Mueller and S. Munier, arXiv:1608.00852; A. Dumitru, V. S. 1704.05917)

  high multiplicity events $\Longleftrightarrow$ rare configurations of hadronic Ψ
  - — CGC density matrix?! (N. Armesto, A. Kovner, M. Lublinsky, V. S. work in progress)
- Thermalization/isotropization (many talks during IS'17)
  - — Final state interactions are important





Challenges and approaches

- Higher order cumulants and hierarchy
— Dipole models (talks by M. Mace and A. Kovner)
— Finite number of sources / interference: $1/N_s$ and $1/N_c$ effects (talks by U. Wiedemann, A. Rezaeian)
— Domains/Pomeron loops (A. Kovner, M. Lublinsky, A. Dumitru, V. S. etc)

- Odd harmonics
— Ad-hoc models based on quark scattering
— Numerical classical Yang-Mills
— "First saturation" correction to momentum-odd double inclusive production (this talk)

- Hierarchy in p-A, d-A, He^3-A collisions at RHIC
— ?

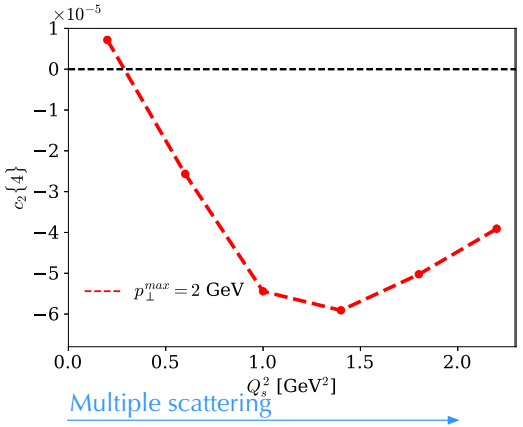
- Multiplicity bias
— Cranking up Q_s might not be enough; (. . . , T. Liou, A. H. Mueller and S. Munier, arXiv:1608.00852; A. Dumitru, V. S. 1704.05917)
high multiplicity events $\Longleftrightarrow$ rare configurations of hadronic Ψ
— CGC density matrix?! (N. Armesto, A. Kovner, M. Lublinsky, V. S. work in progress)

- Thermalization/isotropization (many talks during IS'17)
— Final state interactions are important

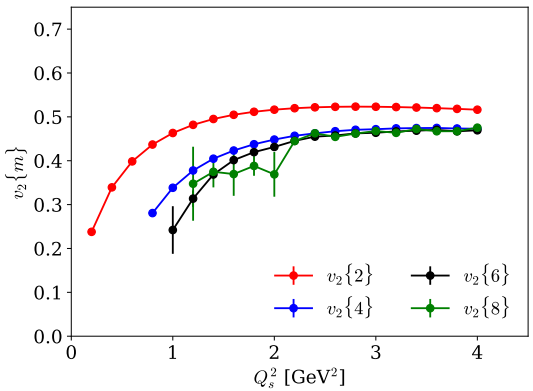
Previous result in Glasma graph approximation

A. Dumitru, L. McLerran, V. S., arXiv:1410.4844

$$c_2\{4\} = \frac{1}{N_D^3}\frac{1}{4(N_c^2-1)^3} > 0$$



Multiple scattering



K. Dusling, M. Mace and R. Venugopalan, arXiv:1705.00745

Challenges and approaches

- Higher order cumulants and hierarchy

— Dipole models (talks by M. Mace and A. Kovner)

— Finite number of sources / interference: $1/N_s$ and $1/N_c$ effects (talks by U. Wiedemann, A. Rezaeian)

— Domains/Pomeron loops (A. Kovner, M. Lublinsky, A. Dumitru, V. S. etc)

- Odd harmonics

— Ad-hoc models based on quark scattering

— Numerical classical Yang-Mills

— "First saturation" correction to momentum-odd double inclusive production (this talk)

- Hierarchy in p-A, d-A, He^3-A collisions at RHIC

— ?

- Multiplicity bias

— Cranking up Q_s might not be enough; (. . . , T. Liou, A. H. Mueller and S. Munier, arXiv:1608.00852; A. Dumitru, V. S. 1704.05917)

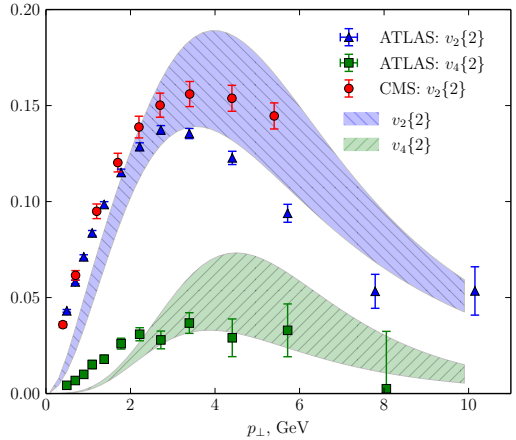
high multiplicity events $\Longleftrightarrow$ rare configurations of hadronic Ψ

— CGC density matrix?! (N. Armesto, A. Kovner, M. Lublinsky, V. S. work in progress)

- Thermalization/isotropization (many talks during IS'17)

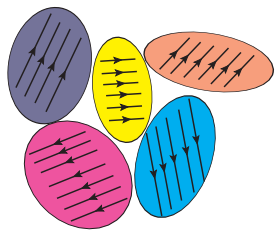
— Final state interactions are important

$N_c \to \infty$; $1/N_s$ suppressed contributions



L. McLerran, V. S., arXiv:1510.08072

$$c_2\{4\} = \frac{1}{N_D^3}\left(-\underbrace{\mathcal{A}^4}_{\text{finite } N_s} + \underbrace{\frac{1}{4(N_c^2-1)^3}}_{\text{glasma graph}}\right)$$



Role of statistics: talks by A. Kovner

Challenges and approaches

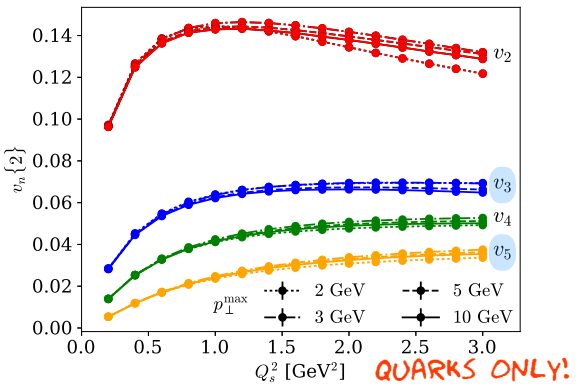
- Higher order cumulants and hierarchy
  — Dipole models (talks by M. Mace and A. Kovner)
  — Finite number of sources / interference: $1/N_s$ and $1/N_c$ effects (talks by U. Wiedemann, A. Rezaeian)
  — Domains/Pomeron loops (A. Kovner, M. Lublinsky, A. Dumitru, V. S. etc)

- Odd harmonics
  — Ad-hoc models based on quark scattering
  — Numerical classical Yang-Mills
  — "First saturation" correction to momentum-odd double inclusive production (this talk)

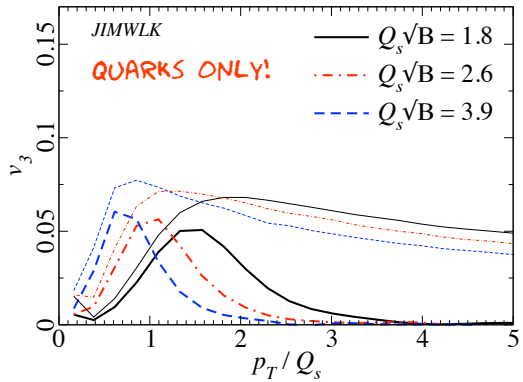
- Hierarchy in p-A, d-A, He^3-A collisions at RHIC
  — ?

- Multiplicity bias
  — Cranking up Q_s might not be enough; (. . . , T. Liou, A. H. Mueller and S. Munier, arXiv:1608.00852; A. Dumitru, V. S. 1704.05917)
  high multiplicity events $\Longleftrightarrow$ rare configurations of hadronic Ψ
  — CGC density matrix?! (N. Armesto, A. Kovner, M. Lublinsky, V. S. work in progress)

- Thermalization/isotropization (many talks during IS'17)
  — Final state interactions are important



K. Dusling, M. Mace and R. Venugopalan, arXiv:1705.00745



T. Lappi, arXiv:1501.05505

$v_{\text{odd}} = 0$ if anti-quarks are added

Challenges and approaches

- Higher order cumulants and hierarchy

— Dipole models (talks by M. Mace and A. Kovner)

— Finite number of sources / interference: $1/N_s$ and $1/N_c$ effects (talks by U. Wiedemann, A. Rezaeian)

— Domains/Pomeron loops (A. Kovner, M. Lublinsky, A. Dumitru, V. S. etc)

- Odd harmonics

— Ad-hoc models based on quark scattering

— Numerical classical Yang-Mills

— "First saturation" correction to momentum-odd double inclusive production (this talk)

- Hierarchy in p-A, d-A, He^3-A collisions at RHIC

— ?

- Multiplicity bias

— Cranking up Q_s might not be enough; (. . . , T. Liou, A. H. Mueller and S. Munier, arXiv:1608.00852; A. Dumitru, V. S. 1704.05917)

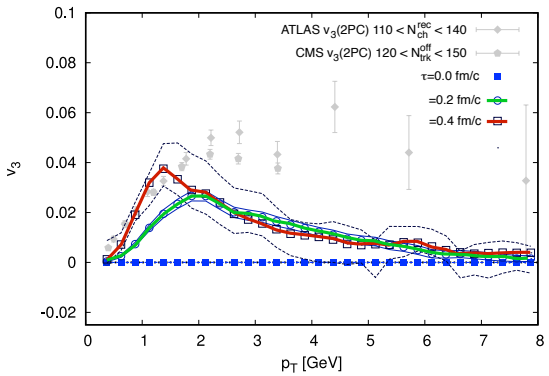
high multiplicity events $\Longleftrightarrow$ rare configurations of hadronic Ψ

— CGC density matrix?! (N. Armesto, A. Kovner, M. Lublinsky, V. S. work in progress)

- Thermalization/isotropization (many talks during IS'17)

— Final state interactions are important

Beyond dilute approximation for p



B. Schenke, S. Schlichting and R. Venugopalan, arXiv:1502.01331

Challenges and approaches

- Higher order cumulants and hierarchy

— Dipole models (talks by M. Mace and A. Kovner)

— Finite number of sources / interference: $1/N_s$ and $1/N_c$ effects (talks by U. Wiedemann, A. Rezaeian)

— Domains/Pomeron loops (A. Kovner, M. Lublinsky, A. Dumitru, V. S. etc)

- Odd harmonics

— Ad-hoc models based on quark scattering

— Numerical classical Yang-Mills

— "First saturation" correction to momentum-odd double inclusive production (this talk)

- Hierarchy in p-A, d-A, He^3-A collisions at RHIC

— ?

- Multiplicity bias

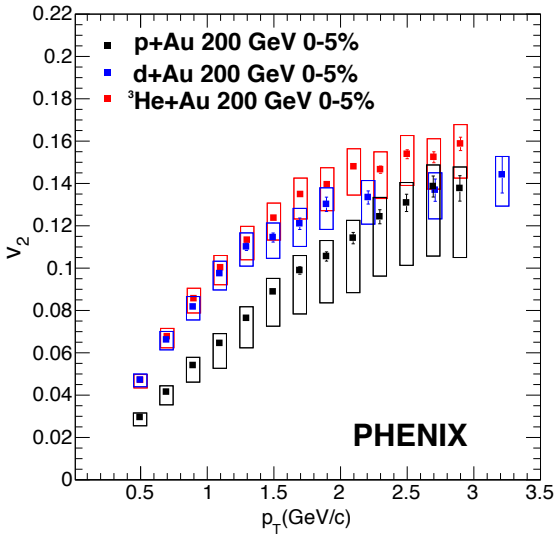
— Cranking up Q_s might not be enough; (. . . , T. Liou, A. H. Mueller and S. Munier, arXiv:1608.00852; A. Dumitru, V. S. 1704.05917)

high multiplicity events $\Longleftrightarrow$ rare configurations of hadronic Ψ

— CGC density matrix?! (N. Armesto, A. Kovner, M. Lublinsky, V. S. work in progress)

- Thermalization/isotropization (many talks during IS'17)

— Final state interactions are important



Is difference significant at high $p_\perp$?

Challenges and approaches

- Higher order cumulants and hierarchy

— Dipole models (talks by M. Mace and A. Kovner)

— Finite number of sources / interference: $1/N_s$ and $1/N_c$ effects (talks by U. Wiedemann, A. Rezaeian)

— Domains/Pomeron loops (A. Kovner, M. Lublinsky, A. Dumitru, V. S. etc)

- Odd harmonics

— Ad-hoc models based on quark scattering

— Numerical classical Yang-Mills

— "First saturation" correction to momentum-odd double inclusive production (this talk)

- Hierarchy in p-A, d-A, He^3-A collisions at RHIC

— ?

- Multiplicity bias

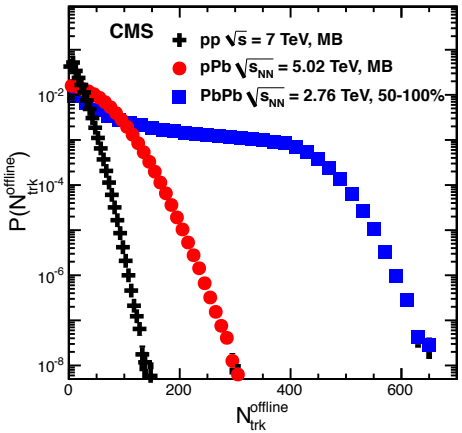
— Cranking up Q_s might not be enough; (. . . , T. Liou, A. H. Mueller and S. Munier, arXiv:1608.00852; A. Dumitru, V. S. 1704.05917)

high multiplicity events $\Longleftrightarrow$ rare configurations of hadronic Ψ

— CGC density matrix?! (N. Armesto, A. Kovner, M. Lublinsky, V. S. work in progress)

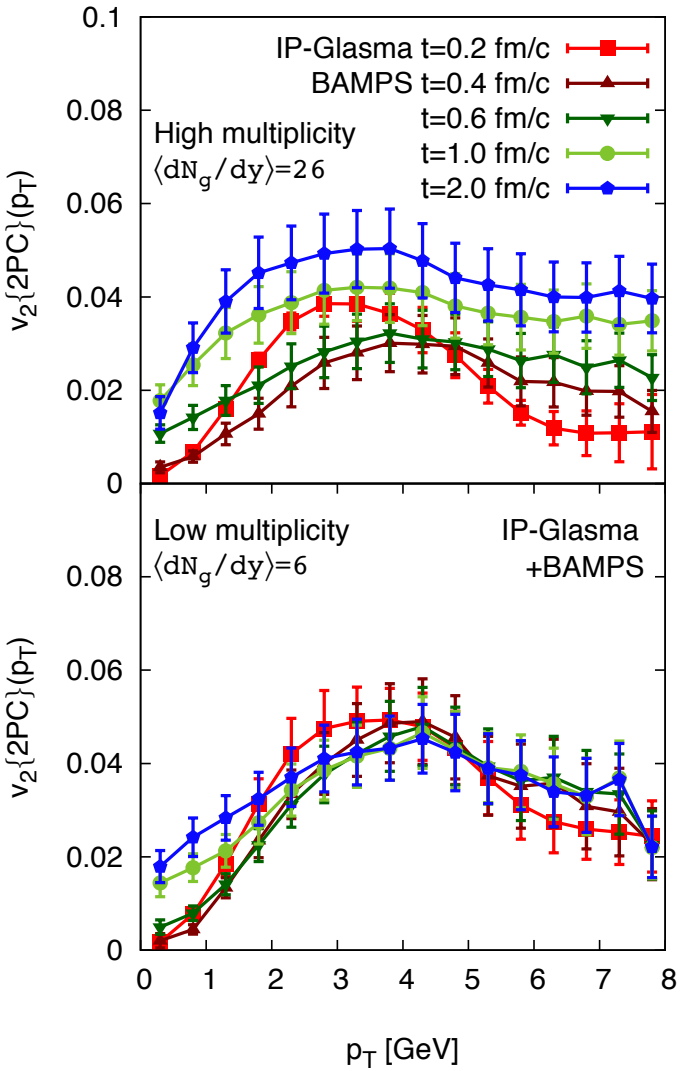
- Thermalization/isotropization (many talks during IS'17)

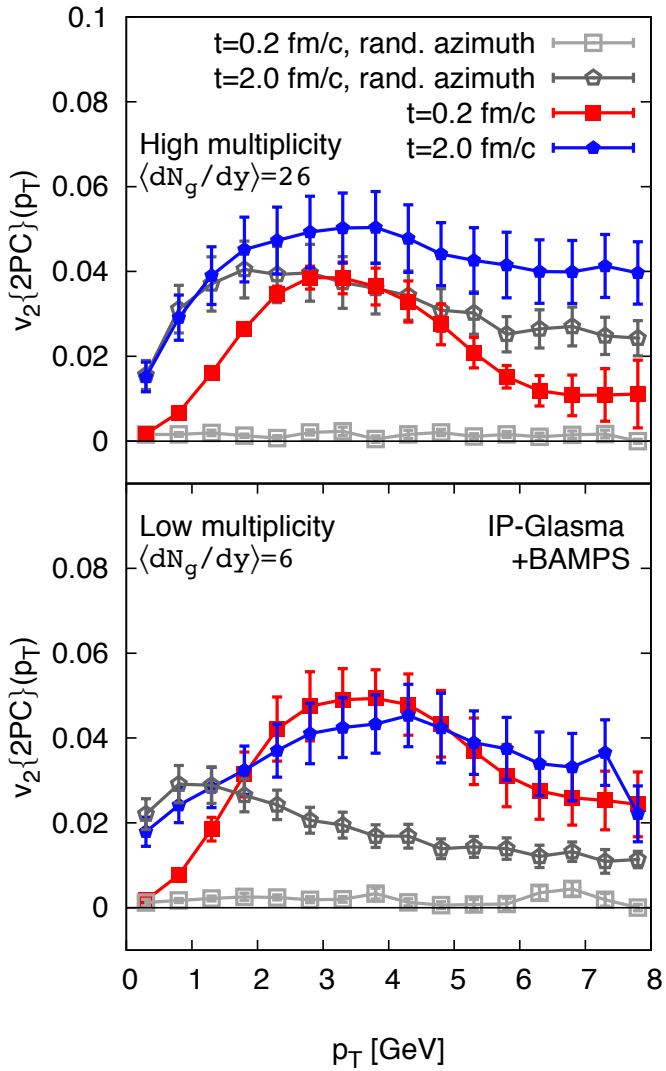
— Final state interactions are important



Classical Yang-Mills + Parton cascade for p-Pb

M. Greif, C. Greiner, B. Schenke, S. Schlichting and Z. Xu, arXiv:1708.02076; talk by B. Schenke





Low multiplicity: dominance of initial state

High multiplicity: dominance of final state

Low $p_\perp$: dominance of final state

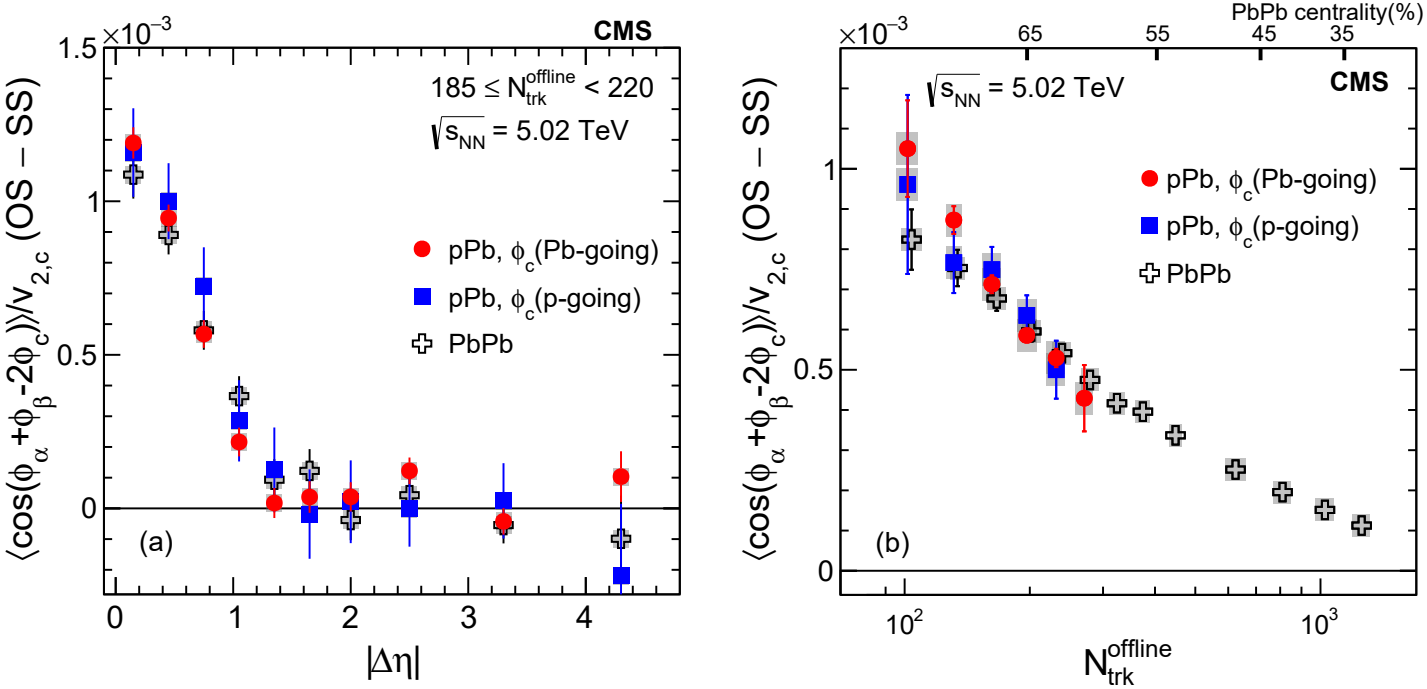
High $p_\perp$: both effects have similar importance

For p-p, p-Pb and (peripheral) Pb-Pb: understanding of initial state effects is as important as final state effects

Initial state effects in collisions of small systems (p-p/p-Pb) $\rightsquigarrow$ input for future EIC.

More challenges

- Observable for Chiral Magnetic Effect $\gamma_{\alpha\beta} \propto \langle \cos(\phi_\alpha + \phi_\beta - 2\phi_c) \rangle$



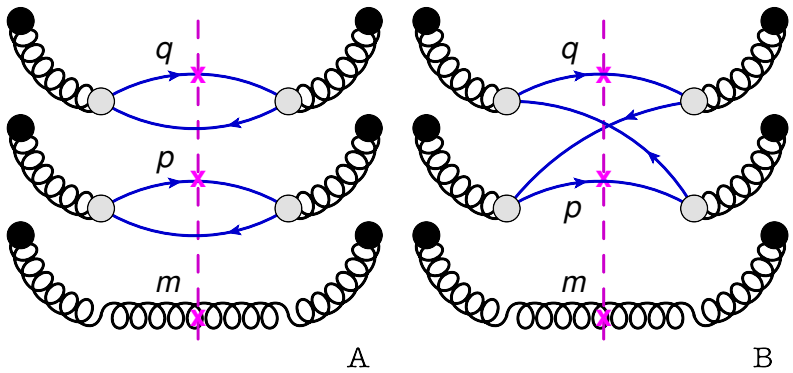
- $\gamma_{\alpha\beta}$ measures correlation between charged particles with respect to ridge orientation
- Challenges **conventional v_2 driven background**, as v_2 is greater in Pb-Pb than in p-Pb by 40%!
- Initial state to rescue?

More challenges

- Observable for Chiral Magnetic Effect $\gamma_{\alpha\beta} \propto \langle \cos(\phi_\alpha + \phi_\beta - 2\phi_c) \rangle$
- $\gamma_{\alpha\beta}$ measures correlation between charged particles with respect to ridge orientation
- Challenges **conventional v_2 driven background**, as v_2 is greater in Pb-Pb than in p-Pb by 40%!
- Initial state to rescue? Pre-existing initial correlation in projectile?

- Same-charge correlations, two contributions of the same order $g^4(g^6\rho^4)$
- Rapidity independent pedestal, A, $\rightsquigarrow \gamma_S < 0$
- Pauly blocking, B, $\rightsquigarrow \gamma_S(\Delta\eta \to 0) > 0$

  $\rightsquigarrow \gamma_S(\Delta\eta \to \infty) < 0$
- Oposite charge correlations: leading order is rapidity independent and positive

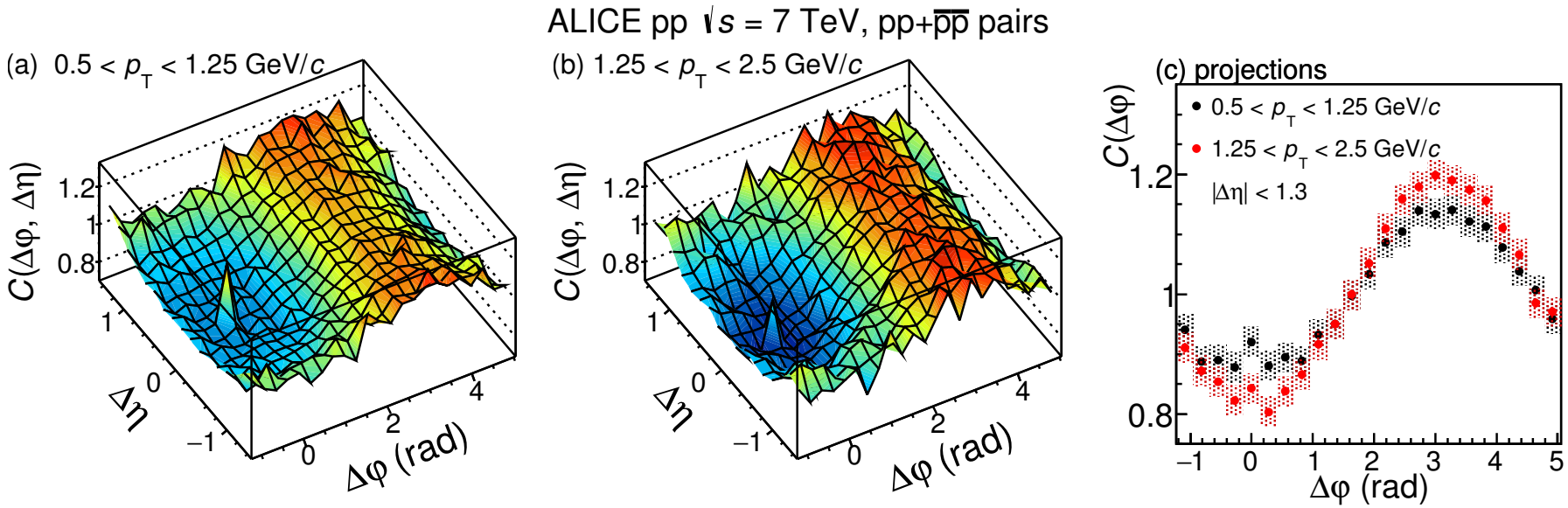


A. Kovner, M. Lublinsky, V. S., 1706.0233

- Quark Fermi statistics effect could be responsible for proton-proton anti-correlation seen by ALICE?

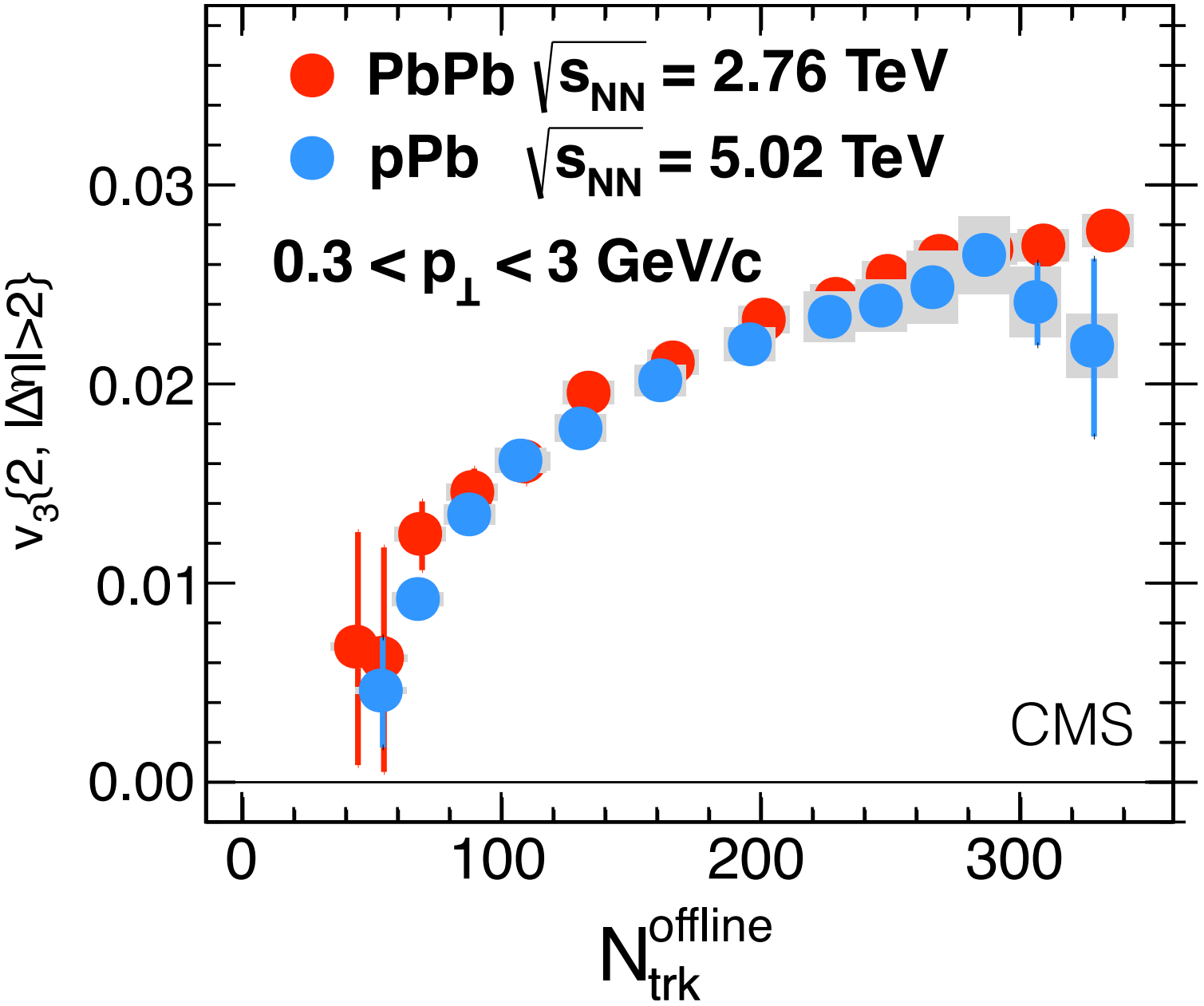
Talks by A. Kovner and T. Altinoluk

Fermi statistics from initial state?

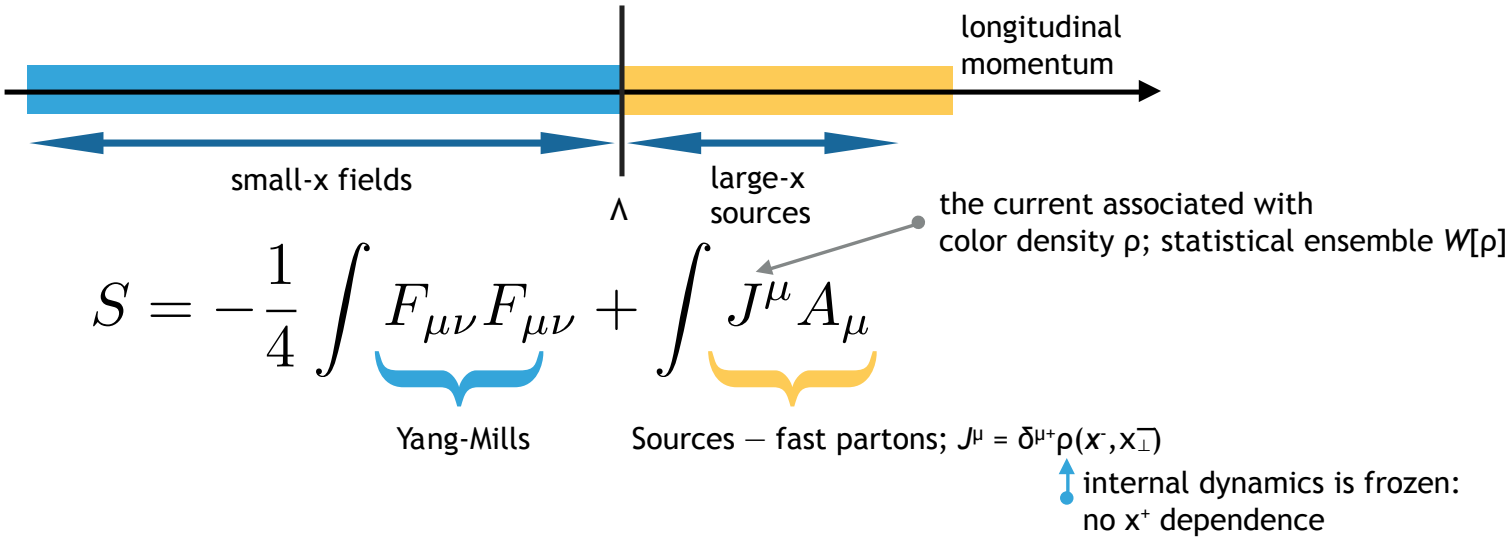


- Almost identical for $p\Lambda + \bar{p}\bar{\Lambda}$

Odd harmonics



Talk by F. Gelis



- Large x: color sources from different overlapping in longitudinal direction nucleons
  Sources are approximately Gaussian. Correlations are local in the transverse plane because they originate from uncorrelated nucleons $\rightsquigarrow$ McLerran-Venugopalan model for color charge distribution $W[\rho]$
- Small x: high gluon densities at leading order, classical field description
- To organize calculations use an EFT of QCD: the Color Glass Condensate

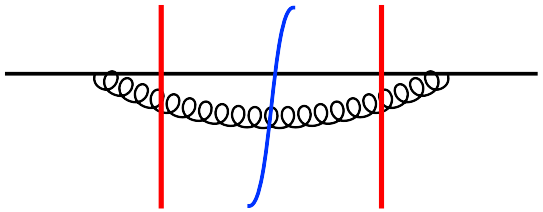
Single-inclusive gluon production

- In general, single-inclusive gluon production (classical approximation)

$$\left\langle \frac{d^2N}{dk_\perp^2} \right\rangle_{\rho_1,\,\rho_2} \propto \frac{1}{\alpha_s} \sum_{n=1,\,m=1}^{n\le N,\,m\le M} c_{n,m} \left(\frac{Q_{s1}^2}{k_\perp^2}\right)^n \left(\frac{Q_{s2}^2}{k_\perp^2}\right)^m$$

  1 (2) stands for Projectile (Target)

- Lowest order $c_{1,1}$ was computed by A. Kovner, L. McLerran, H. Weigert, arXiv:9506320
  reproducing results of E. Kuraev, L. Lipatov, V. Fadin, '77

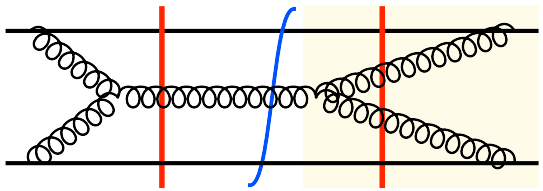


- Analytic results for dilute-dense limit $N = 1, M = \infty$
  Yu. Kovchegov, A. Mueller, arXiv:9802440; A. Dumitru, L. McLerran, arXiv:0105268;
  J.-P. Blaizot, F. Gelis, R. Venugopalan, arXiv:0402256

- Numerical results for dense-dense limit $N = \infty, M = \infty$
  Pioneering work by A. Krasnitz, R. Venugopalan, arXiv:9809433

- First saturation correction in projectile $N = 2, M = \infty$ – Incomplete

  (diagrams where one of the nucleons is
  a spectator in the amplitude of order g are not included)



  G. Chirilli, Yu. Kovchegov, D. Wertepny, arXiv:1501.03106

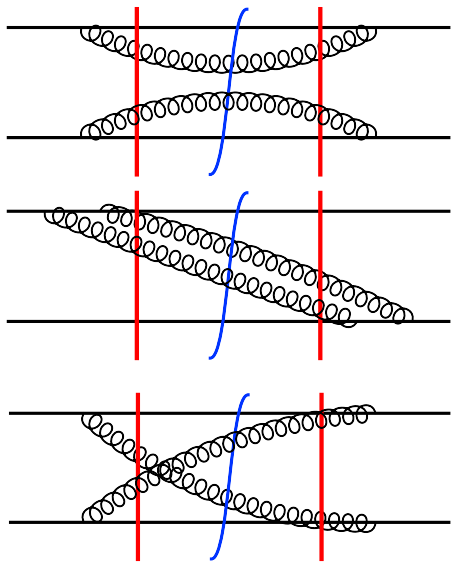
- To describe physics of the ridge: double-inclusive gluon production...

Double-inclusive gluon production

- "Glasma" graph – the correlated leading order contribution

  A. Dumitru, F. Gelis, L. McLerran, R. Venugopalan, arXiv:0804.3858

$$\left\langle \frac{d^2N(\vec{k}_\perp,\vec{p}_\perp)}{dk_\perp^2\,dp_\perp^2} \right\rangle_{\rho_1,\,\rho_2} \propto \frac{1}{\alpha_s^2}\left(\frac{Q_s^2}{k_\perp^2}\right)^2\left(\frac{Q_s^2}{p_\perp^2}\right)^2$$



- Analytical calculations to any order in the target density

  Yu. Kovchegov, D. Wertepny, arXiv:1212.1195

- The nature of the "glasma" graph correlations: pre-existing gluon correlation in projectile

  T. Altinoluk, N. Armesto, G. Beuf, A. Kovner and M. Lublinsky, arXiv:1503.07126
  "Bose enhancement of gluons in the projectile leads to azimuthal collimation of . . . ";

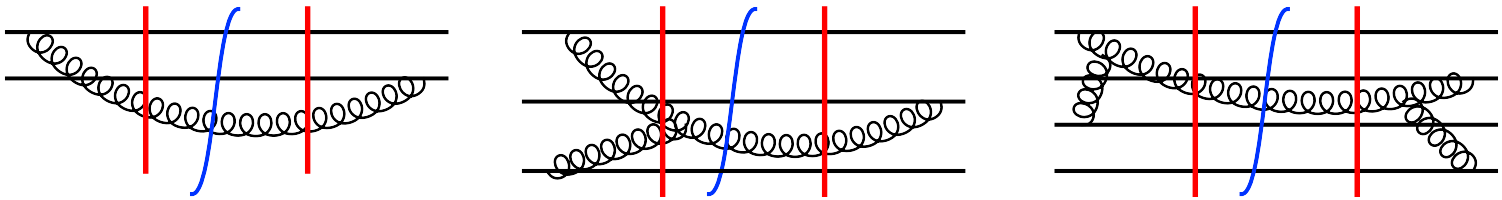
- Dense-dense limit: numerical simulations of Classical Yang-Mills

  T. Lappi, S. Srednyak and R. Venugopalan, arXiv:0911.2068; B. Schenke, S. Schlichting and R. Venugopalan, arXiv:1502.01331

Key properties of single- & double-inclusive production I

- Dilute-dense power-counting on configuration-by-configuration basis (for a given ρ_1):

$$\frac{dN}{dk_\perp^2} \propto g^2\rho_1^2 + g^4\rho_1^3 + g^6\rho_1^4 + \cdots$$

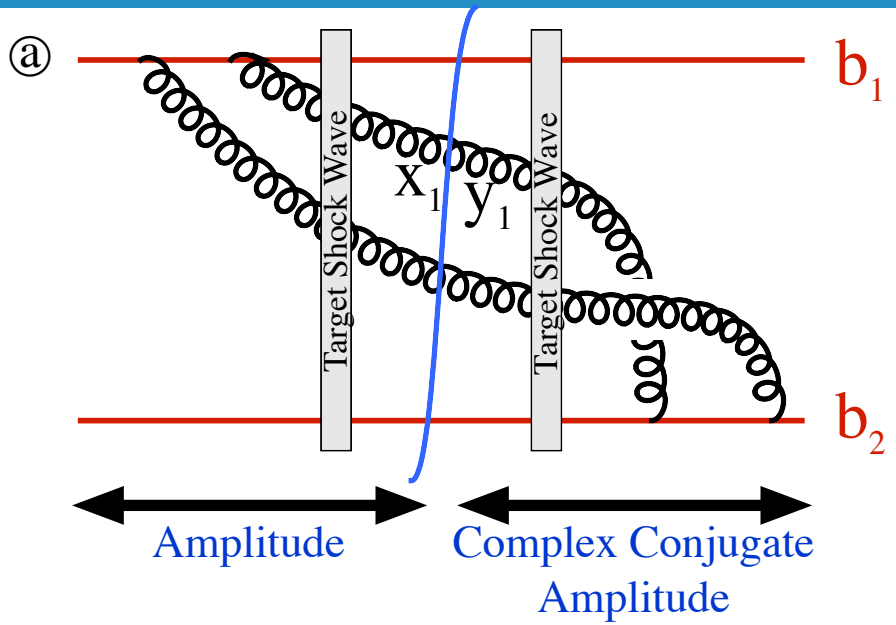


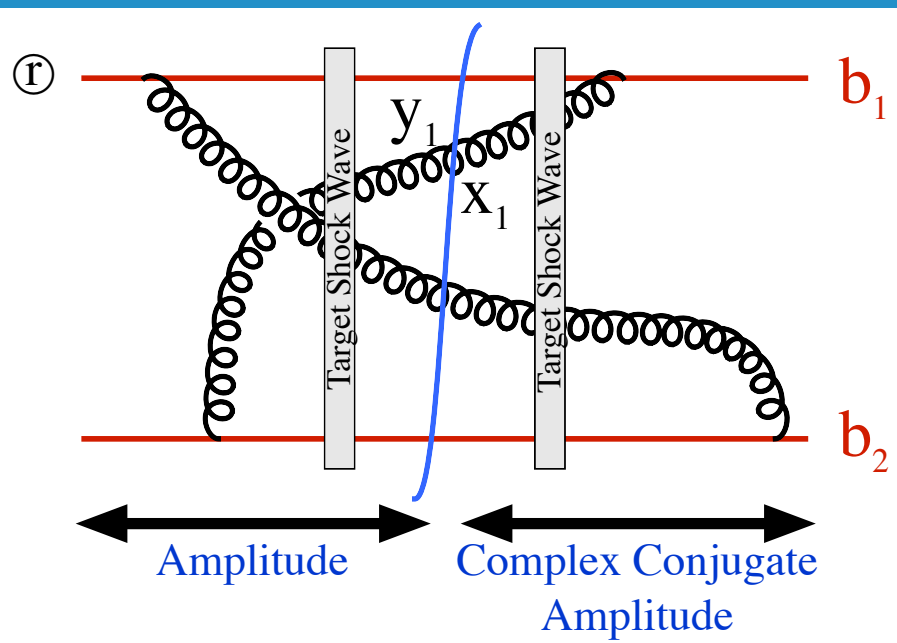
- The leading contribution $g^2\rho_1^2$ (dilute-dense limit) is symmetric under $\vec{k}_\perp \to -\vec{k}_\perp$!
  Corollary: the leading log double-inclusive particle production
  $\left\langle \frac{d^2N(\vec{k}_\perp,\vec{p}_\perp)}{dk_\perp^2 dp_\perp^2} \right\rangle_{\rho_1,\,\rho_2} = \left\langle \frac{dN}{dk_\perp^2}\frac{dN}{dp_\perp^2} \right\rangle_{\rho_1,\,\rho_2}$ shares this symmetry at the leading order. No $v_3\{2\}$!
  Yu. Kovchegov, D. Wertepny, arXiv:1212.1195

- The first saturation correction contribution $g^6\rho_1^4$ is also symmetric under $\vec{k}_\perp \to -\vec{k}_\perp$
  see argument in L. McLerran, V. S. arXiv:1611.09870;

- The contribution $g^4\rho_1^3$ averages out (i.e. it is zero) in a Gaussian ensemble.
  My goal is to derive this contribution and show that it is not symmetric under $\vec{k}_\perp \to -\vec{k}_\perp$.
  I will argue that it gives a non-vanishing $g^8\rho_1^6$ contribution to double inclusive production.

Why are there no odd harmonics in Glasma graph?





$$\sigma_{ⓐ}(k,p) = \int_{x_1,y_1,x_2,y_2} e^{-ik\cdot(x_1-y_1)} e^{-ip\cdot(x_2-y_2)} \underbrace{A(x_1,b_1)A(x_2,b_1)}_{\text{ampl.}} \underbrace{A^*(y_1,b_2)A^*(y_2,b_2)}_{\text{c.c.ampl.}}$$

$$\sigma_{ⓡ}(k,p) = \int_{x_1,y_1,x_2,y_2} e^{-ik\cdot(y_1-x_1)} e^{-ip\cdot(x_2-y_2)} \underbrace{A(y_1,b_2)A(x_2,b_1)}_{\text{ampl.}} \underbrace{A^*(x_1,b_1)A^*(y_2,b_2)}_{\text{c.c.ampl.}}$$

But $A(x,b)$ is real: $A(x,b) = A^*(x,b) \rightsquigarrow$

$$\sigma_{ⓐ}(k,p) + \sigma_{ⓡ}(k,p) = \sigma_{ⓐ}(k,p) + \sigma_{ⓐ}(-k,p)$$

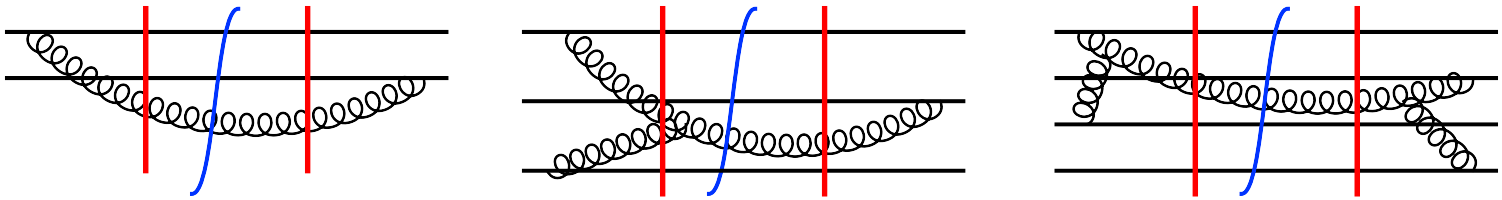
$$\rightsquigarrow v_3 = 0$$

A. Kovner and M. Lublinsky, arXiv:1012.3398,
Yu. Kovchegov, D. Wertepny, arXiv:1212.1195,
Y. Kovchegov, V.S., work in progress

Key properties of single- & double-inclusive production I

- Dilute-dense power-counting on configuration-by-configuration basis (for a given ρ_1):

$$\frac{dN}{dk_\perp^2} \propto g^2\rho_1^2 + g^4\rho_1^3 + g^6\rho_1^4 + \cdots$$



- The leading contribution $g^2\rho_1^2$ (dilute-dense limit) is symmetric under $\vec{k}_\perp \to -\vec{k}_\perp$!
  Corollary: leading log double-inclusive particle production
  $\left\langle \frac{d^2N(\vec{k}_\perp,\vec{p}_\perp)}{dk_\perp^2\, dp_\perp^2} \right\rangle_{\rho_1,\,\rho_2} = \left\langle \frac{dN}{dk_\perp^2}\frac{dN}{dp_\perp^2} \right\rangle_{\rho_1,\,\rho_2}$ shares this symmetry at leading order. No $v_3\{2\}$!
  Yu. Kovchegov, D. Wertepny, arXiv:1212.1195

- The first saturation correction contribution $g^6\rho_1^4$ is also symmetric under $\vec{k}_\perp \to -\vec{k}_\perp$
  see argument in L. McLerran, V. S. arXiv:1611.09870;

- The contribution $g^4\rho_1^3$ averages out (i.e. it is zero) in a Gaussian ensemble.
  My goal is to derive this contribution and show that it is not symmetric under $\vec{k}_\perp \to -\vec{k}_\perp$.
  I will argue that it gives a non-vanishing $g^8\rho_1^6$ contribution to double inclusive production.

Key properties of single- & double-inclusive production II

- Dense-dense limit: analysis of 0911.2068 (section "3.1 Numerical approach") showed that

$$\frac{dN}{dk_\perp^2} \propto f_\gamma(\vec{k}_\perp) f_\gamma(-\vec{k}_\perp) + i\left[E_\gamma(\vec{k}_\perp)A_\gamma(-\vec{k}_\perp) - A_\gamma(\vec{k}_\perp)E_\gamma(-\vec{k}_\perp)\right]$$

T. Lappi, S. Srednyak and R. Venugopalan, arXiv:0911.2068; F. Gelis, ...

E_γ and A_γ are solutions of CYM at asymptotic infinity, $\tau \to \infty$.

- $\rightsquigarrow$ non-vanishing odd azimuthal anisotropy in double-inclusive production!

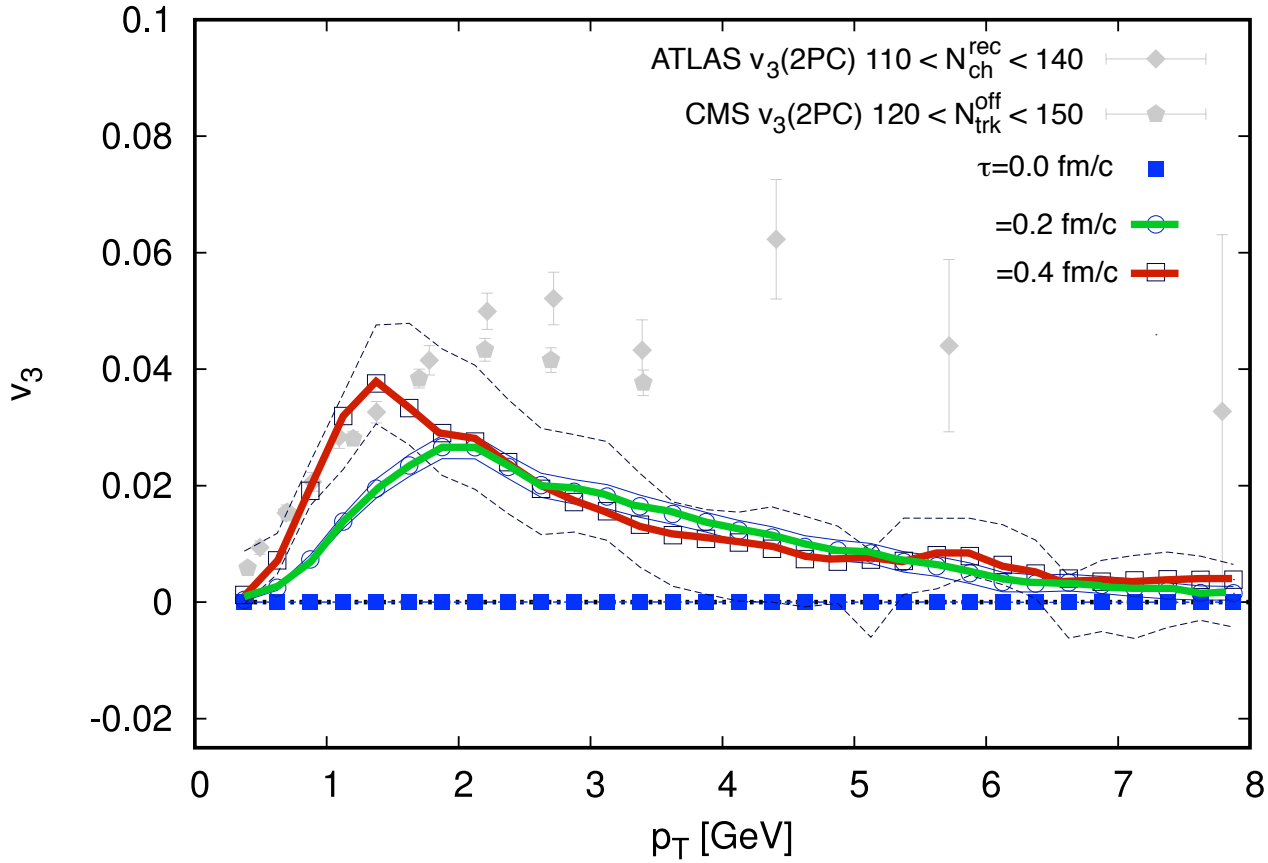
T. Lappi, S. Srednyak and R. Venugopalan, arXiv:0911.2068; B. Schenke, S. Schlichting and R. Venugopalan, arXiv:1502.01331

- Apparent contradiction between dilute-dense limit and dense-dense argument...
- Challenges
  a) to build a bridge between dilute-dense and dense-dense regimes;
  b) to provide semi-analytic expression for odd asymmetry

L. McLerran, V. S. arXiv:1611.09870;

- Why do we care about an analytical result? — no need to solve time evolution conf.-by-conf. numerically; — account for low-x evolution; — incorporate running-coupling corrections

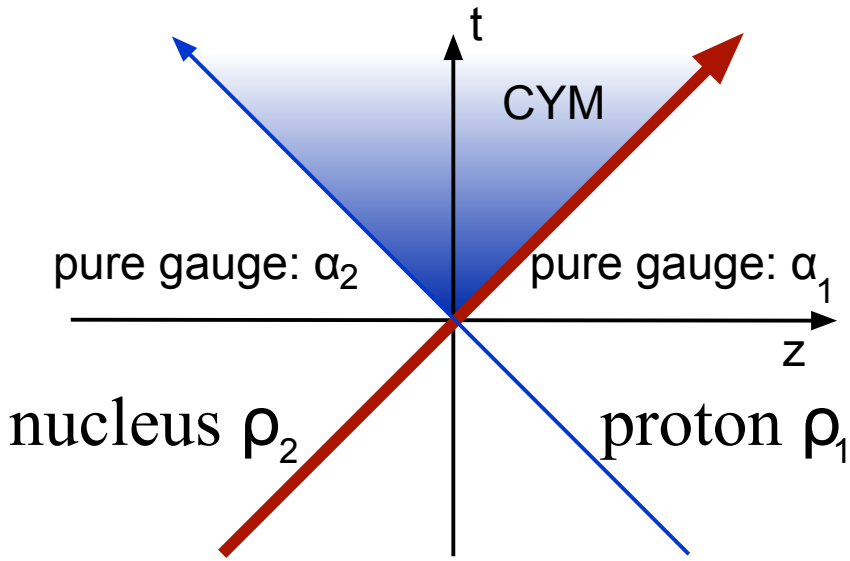
Odd harmonics in dense-dense scattering



B. Schenke, S. Schlichting, R. Venugopalan, 1502.01331

- Zero v_3 from initial state; expected in classical limit, as initial conditions for classical field are real functions.
- v_3 develops quickly with time

Classical Yang-Mills



- Before collision, pure gauge soft fields created by "valence" currents

$$\partial_i \alpha^i_{1,2}(\vec{x}_\perp) = g\,\rho_{1,2}(\vec{x}_\perp)$$
$$\alpha^i_{1,2}(\vec{x}_\perp) = -\tfrac{1}{ig} U_{1,2}(\vec{x}_\perp)\partial^i U^\dagger_{1,2}(\vec{x}_\perp)$$

- Just after collision, $\tau \to 0+$, (Fock-Schwinger gauge $A_\tau = 0$)

A. Kovner, L. McLerran, H. Weigert, arXiv:9506320

$$\alpha^i(\tau \to 0, \vec{x}_\perp) = \alpha^i_1(\vec{x}_\perp) + \alpha^i_2(\vec{x}_\perp)$$

$$A_\eta(\tau \to 0, \vec{x}_\perp) = \tau^2 \alpha(\tau \to 0, \vec{x}_\perp); \quad \alpha(\tau \to 0, \vec{x}_\perp) = \frac{ig}{2}[\alpha^i_1(\vec{x}_\perp), \alpha^i_2(\vec{x}_\perp)]$$

- In forward light-cone $[D_\mu, F^{\mu\nu}] = 0$

Classical Yang-Mills: expansion in $g\rho_1$

- In order to perform expansion, it is convenient to rotate out **nucleus field** from initial conditions:

$$\alpha(\tau, \vec{x}_\perp) = U_2(\vec{x}_\perp)\beta(\tau, \vec{x}_\perp)U_2^\dagger(\vec{x}_\perp)$$

$$\alpha^i(\tau, \vec{x}_\perp) = U_2(\vec{x}_\perp)\left(\beta^i(\tau, \vec{x}_\perp) - \frac{1}{ig}\partial_i\right)U_2^\dagger(\vec{x}_\perp)$$

$$\beta(\tau \to 0, \vec{x}_\perp) = U_2^\dagger(\vec{x}_\perp)\alpha(\tau \to 0, \vec{x}_\perp)U_2^\dagger(\vec{x}_\perp)$$

$$\beta^i(\tau \to 0, \vec{x}_\perp) = U_2^\dagger(\vec{x}_\perp)\alpha_1^i(\vec{x}_\perp)U_2^\dagger(\vec{x}_\perp)$$

- Perform expansion in powers of ρ_1: $\beta_\gamma = \beta_\gamma^{(1)} + \beta_\gamma^{(2)} + \cdots$

First order, $\beta^{(1)}$

- At leading order, CYM equations are (in Milne coordinates)

$$\left[\partial_\tau^2 + \frac{3}{\tau}\partial_\tau - \partial_\perp^2\right]\beta^{(1)}(\tau, \vec{x}_\perp) = 0,$$
$$\partial_\tau \partial_i \beta_i^{(1)}(\tau, \vec{x}_\perp) = 0,$$
$$\left[\delta^{ij}\left(\partial_\tau^2 + \frac{1}{\tau}\partial_\tau - \partial_\perp^2\right) + \partial_i\partial_j\right]\beta_j^{(1)}(\tau, \vec{x}_\perp) = 0$$

No non-linear terms $\rightsquigarrow$ solution can be found trivially. Analogous to $\Box\phi = 0$

- Solutions are

in momentum space:

$$\beta^{(1)}(\tau, \vec{k}_\perp) = b_1(\vec{k}_\perp)\frac{J_1(k_\perp\tau)}{k_\perp\tau},$$
$$\beta_i^{(1)}(\tau, \vec{k}_\perp) = i\frac{\varepsilon^{ij}k_j}{k_\perp^2}b_2(\vec{k}_\perp)J_0(k_\perp\tau) + ik_i\Lambda(\vec{k}_\perp)$$

$[U(\vec{x}_\perp) \equiv U_2(\vec{x}_\perp)]$

$$b_1(\vec{x}_\perp) = \delta^{ij}\Omega_{ij}(\vec{x}_\perp),$$
$$b_2(\vec{x}_\perp) = \epsilon^{ij}\Omega_{ij}(\vec{x}_\perp),$$
$$\Omega_{ij}(\vec{x}_\perp) = g\left[\frac{\partial_i}{\partial_\perp^2}\rho_1^a(\vec{x}_\perp)\right]\partial^j\left(U^\dagger(\vec{x}_\perp)t_a U(\vec{x}_\perp)\right),$$
$$\Lambda(\vec{x}_\perp) = g\frac{\partial^i}{\partial_\perp^2}\left(U^\dagger(\vec{x}_\perp)\left[\frac{\partial_i}{\partial_\perp^2}\rho_1^a(\vec{x}_\perp)\right]U(\vec{x}_\perp)\right)$$

Second order, $\beta^{(2)}$

- At next-to-leading order:

$$\left[\partial_\tau^2 + \frac{3}{\tau}\partial_\tau - \partial_\perp^2\right]\beta^{(2)}(\tau,\vec{x}_\perp) = -ig\left(\partial_i[\beta^{(1)}{}_i(\tau,\vec{x}_\perp),\beta^{(1)}(\tau,\vec{x}_\perp)] + [\beta^{(1)}{}_i(\tau,\vec{x}_\perp),\partial_i\beta^{(1)}(\tau,\vec{x}_\perp)]\right)$$

$$\left[\delta^{ij}\left(\partial_\tau^2 + \frac{1}{\tau}\partial_\tau - \partial_\perp^2\right) + \partial_i\partial_j\right]\beta^{(2)}{}_j(\tau,\vec{x}_\perp) = -ig\left(\partial_j[\beta^{(1)}{}_j(\tau,\vec{x}_\perp),\beta^{(1)}{}_i(\tau,\vec{x}_\perp)]\right.$$
$$\left.+[\beta^{(1)}{}_j(\tau,\vec{x}_\perp),\partial_j\beta^{(1)}{}_i(\tau,\vec{x}_\perp) - \partial_i\beta^{(1)}{}_j(\tau,\vec{x}_\perp)] - \tau^2[\beta^{(1)}(\tau,\vec{x}_\perp),\partial_i\beta^{(1)}(\tau,\vec{x}_\perp)]\right)$$

  First non-linear corrections!

- This looks very discouraging as (in momentum space) β^1 are Bessel functions
- Goal is to compute $g^6\rho_1^3$ correction to particle production.
  Do we have to solve these equations?

Particle production: Lehmann-Symanzik-Zimmermann I

- For simplicity – Minkowski-space and semi-classical scalar field $\phi(x)$.
  The creation operator

$$a^+(\vec{k},t) = \frac{1}{i}\int d^3x\ \exp(-ik\cdot x)\overset{\leftrightarrow}{\partial_0}\phi(x) \qquad k\cdot x = k_\mu x^\mu$$

- The difference

$$a^+(\vec{k},t\to\infty) - a^+(\vec{k},t\to t_0) = \frac{1}{i}\int_{t_0}^{\infty} dt\,\partial_0\left(\int d^3x\ \exp(-ik\cdot x)\overset{\leftrightarrow}{\partial_0}\phi(x)\right)$$
$$= \frac{1}{i}\int_{t_0}^{\infty} dt\,d^3x\ \exp(-ik\cdot x)\underbrace{\left(\Box + m^2\right)\phi\,(x)}_{\text{interaction}}$$

  Instead of a usual choice $t_0\to-\infty$, in order to mimic initial conditions on light cone, $t_0\to 0$.

- Thus for creation operator at out-state, there are two contributions

$$a^+(\vec{k},\infty) = \underbrace{\frac{1}{i}\int_{t=0} d^3x\exp(-ik\cdot x)\overset{\leftrightarrow}{\partial_0}\phi(x)}_{\text{initial flux through t=0 hypersurface}} + \underbrace{\frac{1}{i}\int_0^\infty dt\int d^3x\exp(-ik\cdot x)\left(\Box+m^2\right)\phi(x)}_{\text{interaction; evolution in the forward light cone}}$$

- Single-inclusive gluon production $E_k\frac{dN}{d^3k} = \frac{1}{2(2\pi)^3}a^+(\vec{k},\infty)\,a(\vec{k},\infty)$

Particle production: Lehmann-Symanzik-Zimmermann II

- Single-inclusive gluon production

$$E_k \frac{dN}{d^3k} = \frac{1}{2(2\pi)^3} \left| \overbrace{\frac{1}{i}\int_{t=0} d^3x \exp(-ik\cdot x)\, \overleftrightarrow{\partial_0}\, \phi(x)}^{\text{"surface"}} + \overbrace{\frac{1}{i}\int_0^\infty dt \int d^3x \exp(-ik\cdot x)\, \left(\Box + m^2\right) \phi(x)}^{\text{"bulk"}} \right|^2$$

initial flux

interaction; evolution in the forward light cone

- Leading order: no "bulk" contribution
- "Surface": Initial conditions (to any order) are real functions of $\vec{x}_\perp \rightsquigarrow$ symmetric under $\vec{k}_\perp \to -\vec{k}_\perp \rightsquigarrow$ absence of odd azimuthal anisotropy
- Both contributions schematically:

$$E_k \frac{dN}{d^3k} = \left(\underbrace{a^{(1)}(\vec{k}_\perp)}_{\text{surface only}} + a^{(2)}(\vec{k}_\perp) + \ldots \right) \left(a^{(1)}(\vec{k}_\perp) + \underbrace{a^{(2)}(\vec{k}_\perp)}_{\text{surface and bulk}} + \ldots \right)^*$$

$$\approx \underbrace{a^{(1)}(\vec{k}_\perp) a^{(1)}(-\vec{k}_\perp)}_{\text{symmetric}} + \underbrace{a^{(1)}(\vec{k}_\perp) \left(a^{(2)}(\vec{k}_\perp)\right)^* + a^{(1)}(-\vec{k}_\perp) a^{(2)}(\vec{k}_\perp)}_{\text{odd asymmetry is possible}}$$

- Realistic calculations: Milne coordinates (Hankel functions instead of exp), gluon polarization, non-trivial time integrals involving up to three Bessel functions

Single-inclusive production: leading order

$$E_k \frac{dN^{\rm LO}}{d^3k} = \frac{1}{16\pi}\left[\left|\mathcal{S}_\perp(\vec{k}_\perp)\right|^2 + \left|\mathcal{S}_\eta(\vec{k}_\perp)\right|^2\right]$$

with

$$\mathcal{S}_\perp^{(1)}(\vec{k}_\perp) = -\lim_{\tau\to 0+}\left(\tau\partial_\tau H_0^{(1)}(k_\perp\tau)\beta_\perp^{(1)}(\tau,\vec{k}_\perp)\right) = -\frac{2}{\pi}i\,\beta_\perp^{(1)}(\tau=0,\vec{k}_\perp) = \frac{2i}{\pi k_\perp}\underbrace{b_2(\vec{k}_\perp)}_{\text{init. cond.}}$$

$$\mathcal{S}_\eta^{(1)}(\tau,\vec{k}_\perp) = -\frac{4}{\pi}i\,\frac{\beta(\tau=0,\vec{k}_\perp)}{k_\perp} = -\frac{2i}{\pi k_\perp}\underbrace{b_1(\vec{k}_\perp)}_{\text{init. cond.}}$$

- Substituting initial condition

  reproducing result of J.-P. Blaizot, F. Gelis, R. Venugopalan, arXiv:0402256

$$\begin{aligned} E_k \frac{dN^{\rm LO}}{d^3k} &= \frac{1}{8\pi^3 k_\perp^2}(\delta_{ij}\delta_{lm}+\epsilon_{ij}\epsilon_{lm})\,\Omega_{ij}^b(\vec{k}_\perp)\left[\Omega_{lm}^b(\vec{k}_\perp)\right]^* \\ &= \frac{g^2}{8\pi^3 k_\perp^2}(\delta_{ij}\delta_{lm}+\epsilon_{ij}\epsilon_{lm})\int\frac{d^2p_\perp}{(2\pi)^2}\frac{d^2q_\perp}{(2\pi)^2}\frac{p_i(k-p)_j}{p_\perp^2}\frac{q_l(k-q)_m}{q_\perp^2} \\ &\quad\times\rho_a^*(\vec{q}_\perp)\left[W^\dagger(\vec{k}_\perp-\vec{q}_\perp)\underbrace{W(\vec{k}_\perp-\vec{p}_\perp)}_{\text{adjoint Wilson line}}\right]_{ab}\rho_b(\vec{p}_\perp) \end{aligned}$$

- Manifestly symmetric under $\vec{k}_\perp \to -\vec{k}_\perp$

Next-to-leading order I

$$E_k \frac{dN^{\rm NLO}}{d^3k} = \frac{1}{16\pi} \sum_{\gamma=\perp,\eta} \Big| \underbrace{\mathfrak{S}^{(1)}_\gamma(\vec{k}_\perp) + \mathfrak{S}^{(2)}_\gamma(\vec{k}_\perp)}_{\text{init. cond.}} + \underbrace{\mathfrak{B}^{(2)}_\gamma(\vec{k}_\perp)}_{\text{evolution}} \Big|^2$$

$$= \frac{1}{16\pi} \sum_{\gamma=\perp,\eta} \Bigg[\Big|\mathfrak{S}^{(1)}_\gamma(\vec{k}_\perp)\Big|^2 + \underbrace{\mathfrak{S}^{(1)}_\gamma(\vec{k}_\perp)\left(\mathfrak{S}^{(2)}_\gamma(\vec{k}_\perp) + \mathfrak{B}^{(2)}_\gamma(\vec{k}_\perp)\right)^*}_{!}$$

$$+ \underbrace{\left(\mathfrak{S}^{(1)}_\gamma(\vec{k}_\perp)\right)^*\left(\mathfrak{S}^{(2)}_\gamma(\vec{k}_\perp) + \mathfrak{B}^{(2)}_\gamma(\vec{k}_\perp)\right)}_{!} \Bigg]$$

- Surface contributions are defined by initial condition on light cone and can be readily found. At any order n, $\mathfrak{S}^{(n)}_\gamma(\vec{k}_\perp) = -(\mathfrak{S}^{(n)}_\gamma(-\vec{k}_\perp))^*$
- Anti-symmetric combination:

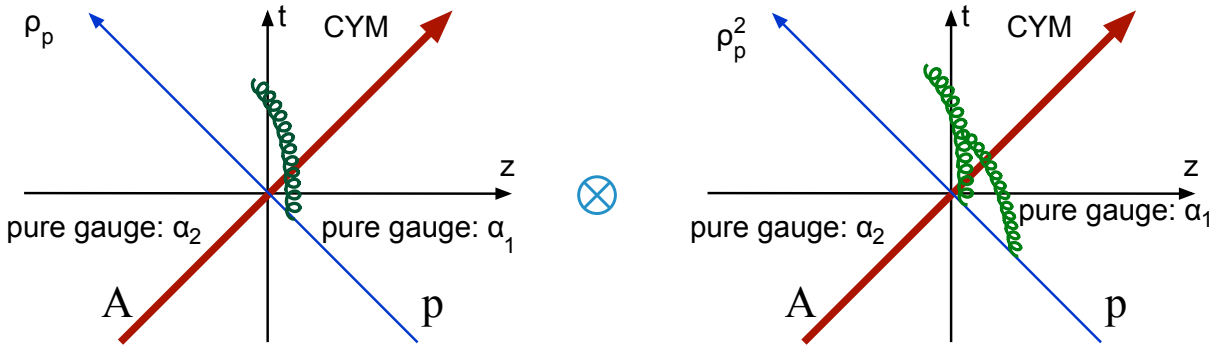
$$\frac{E_k}{2}\left(\frac{dN^{\rm NLO}}{d^3k}(\vec{k}_\perp) - \frac{dN^{\rm NLO}}{d^3k}(-\vec{k}_\perp)\right) = \frac{1}{16\pi}\mathrm{Re}\sum_{\gamma=\perp,\eta}\left(\mathfrak{S}^{(1)}_\gamma(\vec{k}_\perp)\right)^*\underbrace{\left[\mathfrak{B}^{(2)}_\gamma(\vec{k}_\perp) + \left(\mathfrak{B}^{(2)}_\gamma(-\vec{k}_\perp)\right)^*\right]}_{!}$$

- This combination is defined by NLO solution.

Next-to-leading order II

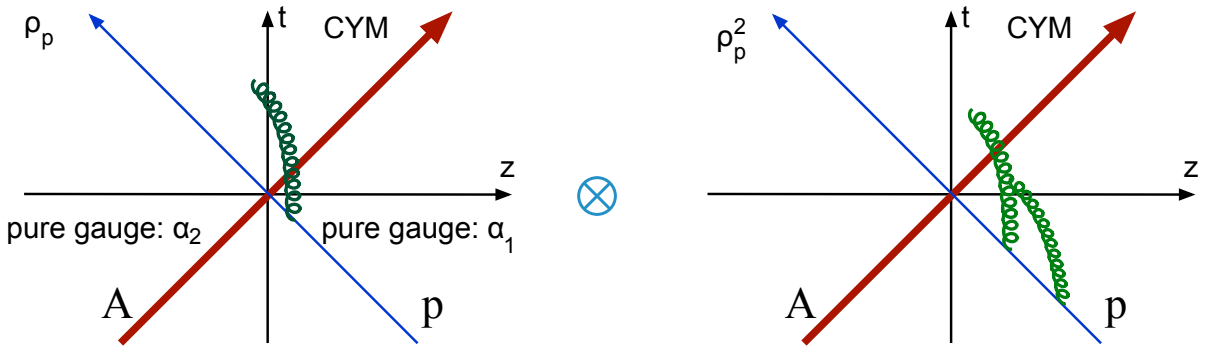
Graphically

$\left(\mathfrak{S}_\gamma^{(1)}(\vec{k}_\perp)\right)^*$ $\mathfrak{B}_\gamma^{(2)}(\vec{k}_\perp)$



This diagram does not contribute:

$\left(\mathfrak{S}_\gamma^{(1)}(\vec{k}_\perp)\right)^*$ $\mathfrak{S}_\gamma^{(2)}(\vec{k}_\perp)$



Next-to-leading order III

- Definition of bulk contributions (η-component only):

$$\mathfrak{B}_\eta^{(2)}(\vec{k}_\perp) = \int_0^\infty d\tau \tau^2 H_1^{(1)}(k_\perp \tau) \underbrace{\left\{ \frac{1}{\tau^3} \partial_\tau \tau^3 \partial_\tau \beta^{(2)}(\tau, \vec{k}_\perp) - \partial_\perp^2 \beta^{(2)}(\tau, \vec{k}_\perp) \right\}}_{\Box}$$

- There is no need to solve for $\beta^{(2)}$!

$$\underbrace{\left[\partial_\tau^2 + \frac{3}{\tau} \partial_\tau - \partial_\perp^2 \right]}_{\Box} \beta^{(2)}(\tau, \vec{x}_\perp) = -ig \left(\partial_i [\beta^{(1)}{}_i(\tau, \vec{x}_\perp), \beta^{(1)}(\tau, \vec{x}_\perp)] + [\beta^{(1)}{}_i(\tau, \vec{x}_\perp), \partial_i \beta^{(1)}(\tau, \vec{x}_\perp)] \right)$$

- $\rightsquigarrow$

$$\mathfrak{B}_\eta^{(2)}(\vec{k}_\perp) = \int_0^\infty d\tau \tau^2 H_1^{(1)}(k_\perp \tau) \left(g \int \frac{d^2 q}{(2\pi)^2} [(2\vec{k}_\perp - \vec{q}_\perp) \cdot \underbrace{\vec{\beta}_\perp^{(1)}(\tau, \vec{q}_\perp)}_{\propto J_0(\tau q_\perp)}, \underbrace{\beta^{(1)}(\tau, \vec{k}_\perp - \vec{q}_\perp)}_{\propto J_1(\tau |\vec{k}_\perp - \vec{q}_\perp|)/\tau}] \right)$$

τ-integrals involving three Bessel functions and one power of τ

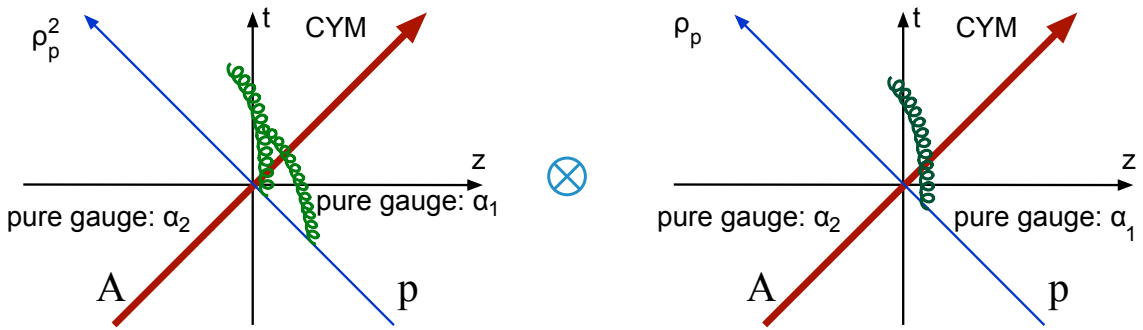
Odd azimuthal anisotropy at NLO

- Performing τ-integration in

$$\frac{E_k}{2}\left(\frac{dN^{\rm NLO}}{d^3k}(\vec{k}_\perp)-\frac{dN^{\rm NLO}}{d^3k}(-\vec{k}_\perp)\right)=\frac{1}{16\pi}{\rm Re}\sum_{\gamma=\perp,\eta}\left(\mathfrak{S}^{(1)}_\gamma(\vec{k}_\perp)\right)^*\left[\mathfrak{B}^{(2)}_\gamma(\vec{k}_\perp)+\left(\mathfrak{B}^{(2)}_\gamma(-\vec{k}_\perp)\right)^*\right]$$

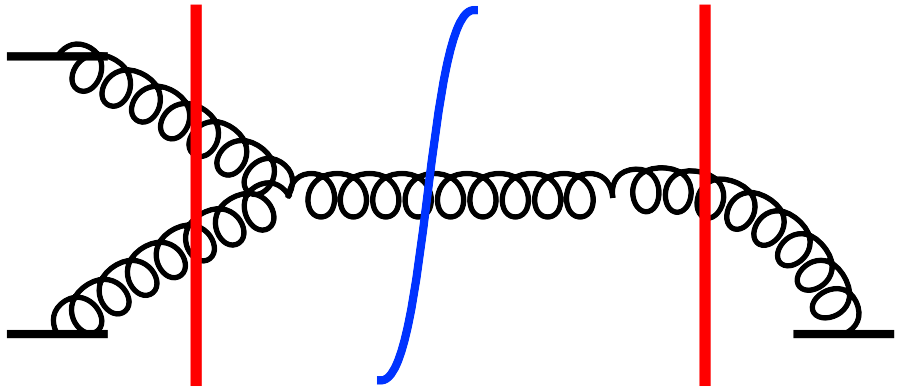
$$=\frac{1}{16\pi}{\rm Im}\left\{\frac{2g}{\pi^2k_\perp^2}\int\frac{d^2q}{(2\pi)^2}\frac{\vec{k}_\perp\times\vec{q}_\perp}{|\vec{k}_\perp\times\vec{q}_\perp|}\frac{1}{q_\perp^2|\vec{k}_\perp-\vec{q}_\perp|^2}\,f^{abc}\,\Omega^a_{ij}(\vec{q}_\perp)\,\Omega^b_{mn}(\vec{k}_\perp-\vec{q}_\perp)\,\Omega^{c\star}_{rp}(\vec{k}_\perp)\times\right.$$
$$\left.\left[\left(k_\perp^2\epsilon^{ij}\epsilon^{mn}-\vec{q}_\perp\cdot(\vec{k}_\perp-\vec{q}_\perp)(\epsilon^{ij}\epsilon^{mn}+\delta^{ij}\delta^{mn})\right)\epsilon^{rp}+2\vec{k}_\perp\cdot(\vec{k}_\perp-\vec{q}_\perp)\epsilon^{ij}\delta^{mn}\delta^{rp}\right]\right\}$$

Ω represent sources rotated by target: $\Omega^b_{ij}(\vec{k}_\perp)=g\int\frac{d^2p_\perp}{(2\pi)^2}\frac{p_i(k-p)_j}{p_\perp^2}\rho_a(\vec{p}_\perp)\underbrace{W_{ba}(\vec{k}_\perp-\vec{p}_\perp)}_{\text{ad. Wilson line}}$

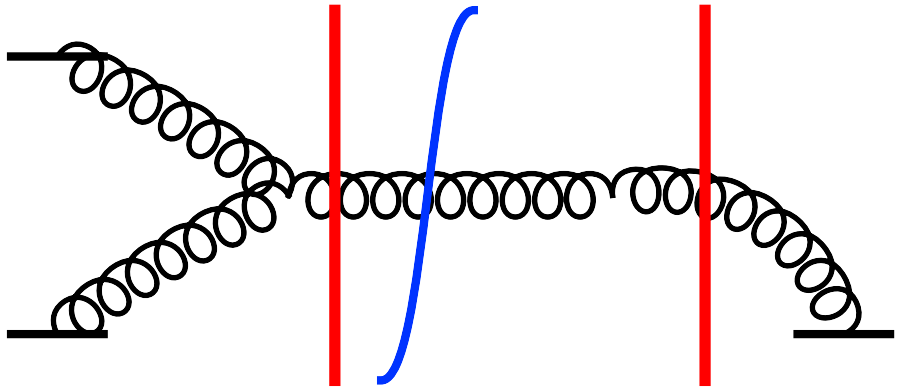


Odd azimuthal anisotropy at NLO

- An example of a digram that contributes:



- Vanishing odd azimuthal anisotropy (an example):

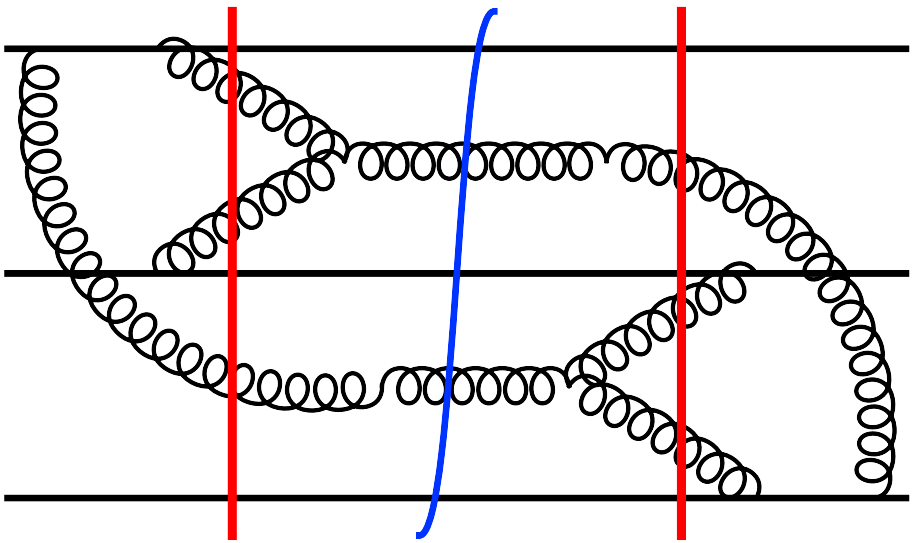


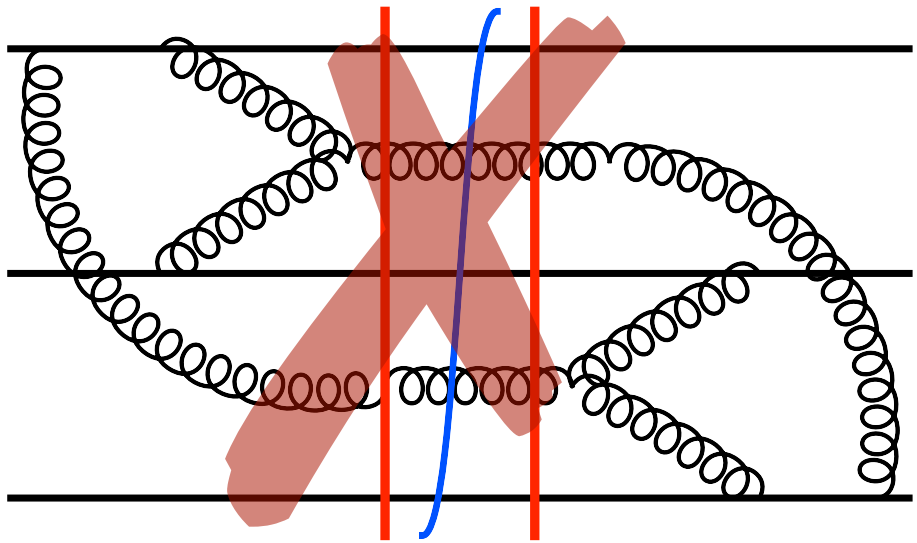
Odd azimuthal asymmetry in double inclusive production

- Odd contribution to single inclusive production is $\propto \rho^3$
- In a Gaussian MV, $\rightsquigarrow \left\langle \frac{E_k}{2} \left(\frac{dN^{\rm NLO}}{d^3k}(\vec{k}_\perp) - \frac{dN^{\rm NLO}}{d^3k}(-\vec{k}_\perp) \right) \right\rangle_\rho = 0$
- However, asymmetry survives in double inclusive production!

$$\overbrace{\left\langle \frac{E_k E_p}{4} \left(\frac{dN^{\rm NLO}}{d^3k}(\vec{k}_\perp) - \frac{dN^{\rm NLO}}{d^3k}(-\vec{k}_\perp) \right) \frac{dN^{\rm NLO}}{d^3p}(\vec{p}_\perp) \right\rangle_\rho}^{\propto \rho^6} \neq 0$$

- One of diagrams





Final result for antisymmetrized part

$$\langle\cdots\rangle_{\rho_1} = \left(\frac{2g}{\pi^2}\right)^2 \frac{1}{p_\perp^2 k_\perp^2} \int \frac{d^2q}{(2\pi)^2} \int \frac{d^2q'}{(2\pi)^2} \frac{\vec{p}_\perp \times \vec{q}_\perp}{|\vec{p}_\perp \times \vec{q}_\perp|} \frac{\vec{k}_\perp \times \vec{q}'_\perp}{|\vec{k}_\perp \times \vec{q}'_\perp|} \frac{f^{abc}}{q_\perp^2 |\vec{p}_\perp - \vec{q}_\perp|^2} \frac{f^{a'b'c'}}{{q'_\perp}^2 |\vec{k}_\perp - \vec{q}'_\perp|^2} \Big[\Big(p_\perp^2 \epsilon^{ij}\epsilon^{mn} - \vec{q}_\perp \cdot (\vec{p}_\perp - \vec{q}_\perp)(\epsilon^{ij}\epsilon^{mn} + \delta^{ij}\delta^{mn})\Big)\epsilon^{rp}$$

$$+2\vec{p}_\perp \cdot (\vec{p}_\perp - \vec{q}_\perp)\delta^{ij}\epsilon^{mn}\delta^{rp}\Big]\Big[\Big(k_\perp^2\, \epsilon^{i'j'}\epsilon^{m'n'} - \vec{q}'_\perp \cdot (\vec{k}_\perp - \vec{q}'_\perp)(\epsilon^{i'j'}\epsilon^{m'n'} + \delta^{i'j'}\delta^{m'n'})\Big)\epsilon^{r'p'} + 2\vec{k}_\perp \cdot (\vec{k}_\perp - \vec{q}'_\perp)\delta^{i'j'}\epsilon^{m'n'}\delta^{r'p'}\Big] \times$$

$$\Big[\Omega^{a,b}_{ij,mn}(\vec{q}_\perp, \vec{k}_\perp - \vec{q}_\perp)\Big(\Omega^{c,a'}_{rp,i'j'}(-\vec{p}_\perp, \vec{q}'_\perp)\Omega^{b',c'}_{m'n',r'p'}(\vec{k}_\perp - \vec{q}'_\perp, -\vec{k}_\perp) + \Omega^{c,b'}_{rp,m'n'}(-\vec{p}_\perp, \vec{k}_\perp - \vec{q}'_\perp)\Omega^{a',c'}_{i'j',r'p'}(\vec{q}'_\perp, -\vec{k}_\perp) + \Omega^{c,c'}_{rp,r'p'}(-\vec{p}_\perp, -\vec{k}_\perp)\Omega^{a',b'}_{i'j',m'n'}(\vec{q}'_\perp, \vec{k}_\perp - \vec{q}'_\perp)\Big)$$

$$\Omega^{a,c}_{ij,rp}(\vec{q}_\perp, -\vec{p}_\perp)\Big(\Omega^{b,a'}_{mn,i'j'}(\vec{p}_\perp - \vec{p}_\perp, \vec{q}'_\perp)\Omega^{b',c'}_{m'n',r'p'}(\vec{k}_\perp - \vec{q}'_\perp, -\vec{k}_\perp) + \Omega^{b,b'}_{mn,m'n'}(\vec{p}_\perp - \vec{q}_\perp, \vec{k}_\perp - \vec{q}'_\perp)\Omega^{a',c'}_{i'j',r'p'}(\vec{q}'_\perp, -\vec{k}_\perp) +$$

$$\Omega^{b,c'}_{mn,r'p'}(\vec{p}_\perp - \vec{q}_\perp, -\vec{k}_\perp)\Omega^{a',b'}_{i'j',m'n'}(\vec{q}'_\perp, \vec{k}_\perp - \vec{q}'_\perp)\Big) + \Omega^{a,a'}_{ij,i'j'}(\vec{q}_\perp, \vec{q}'_\perp)\Big(\Omega^{b,c}_{mn,rp}(\vec{p}_\perp - \vec{p}_\perp, -\vec{p}_\perp)\Omega^{b',c'}_{m'n',r'p'}(\vec{k}_\perp - \vec{q}'_\perp, -\vec{k}_\perp) +$$

$$\Omega^{b,b'}_{mn,m'n'}(\vec{p}_\perp - \vec{q}_\perp, \vec{k}_\perp - \vec{q}'_\perp)\Omega^{c,c'}_{rp,r'p'}(-\vec{p}_\perp, -\vec{k}_\perp) + \Omega^{b,c'}_{mn,r'p'}(\vec{p}_\perp - \vec{q}_\perp, -\vec{k}_\perp)\Omega^{c,b'}_{rp,m'n'}(-\vec{p}'_\perp, \vec{k}_\perp - \vec{q}'_\perp)\Big) +$$

$$\Omega^{a,b'}_{ij,m'n'}(\vec{q}_\perp, \vec{k}_\perp - \vec{q}'_\perp)\Big(\Omega^{b,c}_{mn,rp}(\vec{p}_\perp - \vec{p}_\perp, -\vec{p}_\perp)\Omega^{a',c'}_{i'j',r'p'}(\vec{q}'_\perp, -\vec{k}_\perp) + \Omega^{b,a'}_{mn,i'j'}(\vec{p}_\perp - \vec{q}_\perp, \vec{q}'_\perp)\Omega^{c,c'}_{rp,r'p'}(-\vec{p}_\perp, -\vec{k}_\perp) +$$

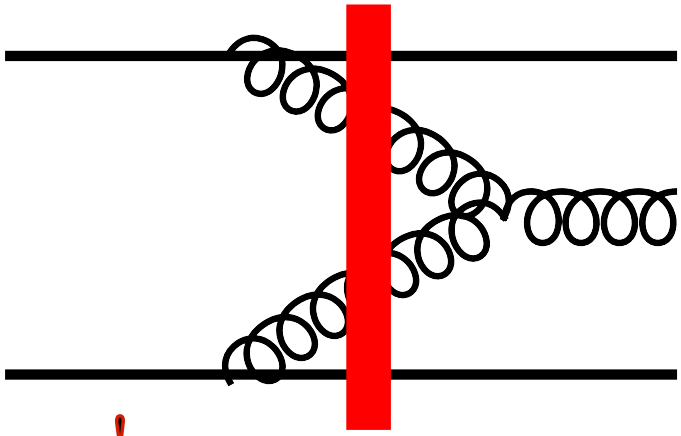
$$\Omega^{b,c'}_{mn,r'p'}(\vec{p}_\perp - \vec{q}_\perp, -\vec{k}_\perp)\Omega^{c,a'}_{rp,i'j'}(-\vec{p}'_\perp, \vec{q}'_\perp)\Big) + \Omega^{a,c'}_{ij,r'p'}(\vec{q}_\perp, -\vec{k}_\perp)\Big(\Omega^{b,c}_{mn,rp}(\vec{p}_\perp - \vec{p}_\perp, -\vec{p}_\perp)\Omega^{a',b'}_{i'j',m'n'}(\vec{q}'_\perp, \vec{k}_\perp - \vec{q}'_\perp) +$$

$$\Omega^{b,a'}_{mn,i'j'}(\vec{p}_\perp - \vec{q}_\perp, \vec{q}'_\perp)\Omega^{c,b'}_{rp,m'n'}(-\vec{p}_\perp, \vec{k}_\perp - \vec{q}'_\perp) + \Omega^{b,b'}_{mn,m'n'}(\vec{p}_\perp - \vec{q}_\perp, \vec{k}_\perp - \vec{q}'_\perp)\Omega^{c,a'}_{rp,i'j'}(-\vec{p}'_\perp, \vec{q}'_\perp)\Big)\Big].$$

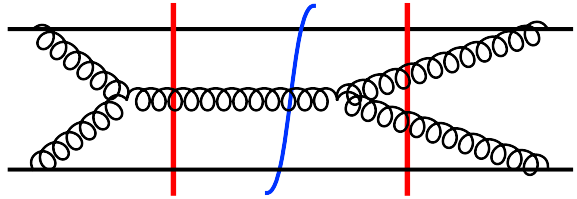
with

$$\Omega^{a,b}_{ij,lm}(\vec{p}_\perp, \vec{q}_\perp) \equiv \langle\Omega^a_{ij}(\vec{p}_\perp)\Omega^b_{lm}(\vec{q}_\perp)\rangle_{\rho_1} = g^2 \int \frac{d^2u}{(2\pi)^2} \mu^2(\vec{u}_\perp) \frac{u_i(u+p)_j u_l (u-q)_m}{u_\perp^4} \Big[W(\vec{u}_\perp + \vec{p}_\perp)W^\dagger(\vec{u}_\perp - \vec{q}_\perp)\Big]_{ab},$$

Cross check?!



G. Chirilli, Yu. Kovchegov, D. Wertepny, 1501.03106



!

$$= -2i g^3 \int d^2x_1\, d^2x_2 \int \frac{d^2k}{(2\pi)^2}\frac{d^2l}{(2\pi)^2}\frac{d^2q_1}{(2\pi)^2}\frac{d^2q_2}{(2\pi)^2} e^{i\vec{q}_{1\perp}\cdot(\vec{x}_{1\perp}-\vec{b}_{1\perp})+i\vec{q}_{2\perp}\cdot(\vec{x}_{2\perp}-\vec{b}_{2\perp})+i\vec{l}_\perp\cdot(\vec{x}_{2\perp}-\vec{x}_{1\perp})+i\vec{k}_\perp\cdot(\vec{z}_\perp-\vec{x}_{2\perp})}$$

$$\times \frac{1}{q_{1\perp}^2\, q_{2\perp}^2}\left(-\vec{q}_{1\perp}\cdot\vec{q}_{2\perp}\,\frac{\vec{\epsilon}_\perp^{\lambda *}\times\vec{k}_\perp}{k_\perp^2} + \vec{\epsilon}_\perp^{\lambda *}\cdot\vec{q}_{1\perp}\,\frac{\vec{q}_{2\perp}\times(\vec{k}_\perp-\vec{l}_\perp)}{(\vec{k}_\perp-\vec{l}_\perp)^2} + \vec{\epsilon}_\perp^{\lambda *}\cdot\vec{q}_{2\perp}\,\frac{\vec{q}_{1\perp}\times\vec{l}_\perp}{\vec{l}_\perp^2}\right)\mathrm{Sign}(\vec{k}_\perp\times\vec{l}_\perp)$$

$$\times f^{abc}\left[U^{bd}_{\vec{x}_{1\perp}} - U^{bd}_{\vec{b}_{1\perp}}\right]\left[U^{ce}_{\vec{x}_{2\perp}} - U^{ce}_{\vec{b}_{2\perp}}\right]\left(V_{\vec{b}_{1\perp}}t^d\right)_1\left(V_{\vec{b}_{2\perp}}t^e\right)_2$$

$$-2g^3\int d^2x\int\frac{d^2k}{(2\pi)^2}\frac{d^2l}{(2\pi)^2}\frac{d^2q}{(2\pi)^2} e^{i\vec{q}_\perp\cdot(\vec{x}_\perp-\vec{b}_{2\perp})+i\vec{l}_\perp\cdot(\vec{x}_\perp-\vec{b}_{1\perp})+i\vec{k}_\perp\cdot(\vec{z}_\perp-\vec{x}_\perp)} f^{abc}\left(V_{\vec{b}_{1\perp}}t^d\right)_1\left(V_{\vec{b}_{2\perp}}t^e\right)_2$$

$$\times\left[U^{bd}_{\vec{b}_{1\perp}}\left(U^{ce}_{\vec{x}_\perp}-U^{ce}_{\vec{b}_{2\perp}}\right)\left(\frac{\vec{\epsilon}_\perp^{\lambda *}\cdot\vec{k}_\perp}{k_\perp^2}\frac{\vec{l}_\perp\cdot\vec{q}_\perp}{l_\perp^2\, q_\perp^2} - \frac{\vec{\epsilon}_\perp^{\lambda *}\cdot\vec{l}_\perp}{l_\perp^2}\frac{(\vec{k}_\perp-\vec{l}_\perp)\cdot\vec{q}_\perp}{(\vec{k}_\perp-\vec{l}_\perp)^2\, q_\perp^2} - \frac{\vec{\epsilon}_\perp^{\lambda *}\cdot\vec{q}_\perp}{2\, l_\perp^2\, q_\perp^2} + \frac{\vec{\epsilon}_\perp^{\lambda *}\cdot\vec{k}_\perp}{k_\perp^2}\frac{(\vec{k}_\perp-\vec{l}_\perp)\cdot\vec{q}_\perp}{(\vec{k}_\perp-\vec{l}_\perp)^2\, q_\perp^2}\right)\right.$$

$$\left.-\left(U^{bd}_{\vec{x}_\perp}-U^{bd}_{\vec{b}_{1\perp}}\right)U^{ce}_{\vec{b}_{2\perp}}\left(\frac{\vec{\epsilon}_\perp^{\lambda *}\cdot\vec{k}_\perp}{k_\perp^2}\frac{\vec{l}_\perp\cdot\vec{q}_\perp}{l_\perp^2\, q_\perp^2} - \frac{\vec{\epsilon}_\perp^{\lambda *}\cdot\vec{q}_\perp}{q_\perp^2}\frac{(\vec{k}_\perp-\vec{q}_\perp)\cdot\vec{l}_\perp}{(\vec{k}_\perp-\vec{q}_\perp)^2\, l_\perp^2} - \frac{\vec{\epsilon}_\perp^{\lambda *}\cdot\vec{l}_\perp}{2\, l_\perp^2\, q_\perp^2} + \frac{\vec{\epsilon}_\perp^{\lambda *}\cdot\vec{k}_\perp}{k_\perp^2}\frac{(\vec{k}_\perp-\vec{q}_\perp)\cdot\vec{l}_\perp}{(\vec{k}_\perp-\vec{q}_\perp)^2\, l_\perp^2}\right)\right]$$

$$+2g^3\int d^2x\int\frac{d^2k}{(2\pi)^2}\frac{d^2q_1}{(2\pi)^2}\frac{d^2q_2}{(2\pi)^2} e^{i\vec{q}_{1\perp}\cdot(\vec{x}_\perp-\vec{b}_{1\perp})+i\vec{q}_{2\perp}\cdot(\vec{x}_\perp-\vec{b}_{2\perp})+i\vec{k}_\perp\cdot(\vec{z}_\perp-\vec{x}_\perp)}\frac{\vec{q}_{1\perp}\cdot\vec{q}_{2\perp}}{q_{1\perp}^2\, q_{2\perp}^2}\frac{\vec{\epsilon}_\perp^{\lambda *}\cdot\vec{k}_\perp}{k_\perp^2}\mathrm{Sign}(b_2^- - b_1^-)$$

$$\times\left[U^{ab}_{x_\perp}-U^{ab}_{z_\perp}\right] f^{bde}\left(V_{\vec{b}_{1\perp}}t^d\right)_1\left(V_{\vec{b}_{2\perp}}t^e\right)_2$$

IMAGINARY REAL

G. Chirilli, Yu. Kovchegov, D. Wertepny, 1501.03106
Yu. Kovchegov, V.S., relation to v_3 – work in progress

Double inclusive production

- The obtained result can only be used to extract antisymmetrized part, or leading contribution to $v_3\{2\}$.
- **It is not full NLO for double inclusive production!**
- $\sigma(\vec{k}_\perp)$ as a shorthand notation for $E_k\, dN/dk^3$:

$$\sigma(\vec{k}_\perp) = \underbrace{\sigma_2(\vec{k}_\perp)}_{g^2\rho^2}\ (\text{LO}) + \underbrace{\sigma_3(\vec{k}_\perp)}_{g^4\rho^3}\ (\text{NLO}) + \underbrace{\sigma_4(\vec{k}_\perp)}_{g^6\rho^4}\ (\text{FSC}) + \cdots$$

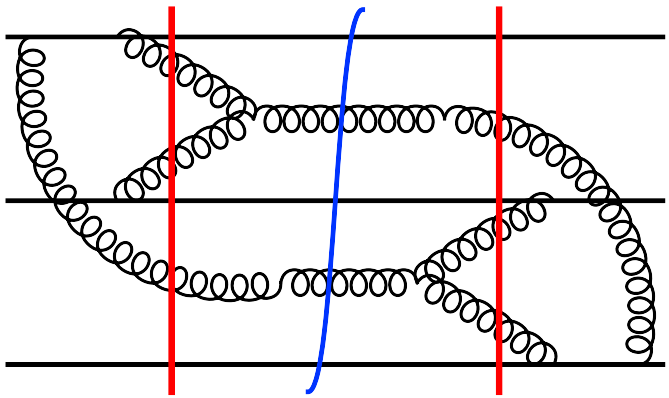
Double inclusive:

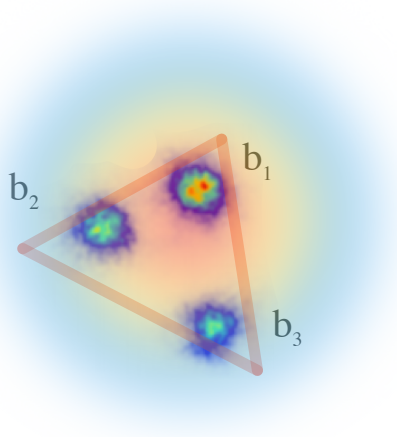
$$\langle\sigma(\vec{k}_\perp)\sigma(\vec{p}_\perp)\rangle = \underbrace{\langle\sigma_2(\vec{k}_\perp)\sigma_2(\vec{p}_\perp)\rangle}_{\text{LO}} + \overbrace{\underbrace{\langle\sigma_3(\vec{k}_\perp)\sigma_3(\vec{p}_\perp)\rangle}_{\text{computed here}} + \underbrace{\langle\sigma_4(\vec{k}_\perp)\sigma_2(\vec{p}_\perp) + \sigma_4(\vec{p}_\perp)\sigma_2(\vec{k}_\perp)\rangle}_{\text{even anisotropy only}}}^{\text{complete NLO}} + \cdots$$

- Complete NLO requires computing first saturation correction, $\sigma_4(\vec{k}_\perp)$.
- Nevertheless, this represents a first explicit analytic result for odd azimuthal anisotropy.

Properties of odd azimuthal anisotropy

- The surface contribution on light cone gives zero odd azimuthal anisotropy to all orders.
  It is T-even and can be written in a local form.
- The odd harmonics originate from time evolution in forward light cone.
  They are non-local and not T-even.
- The argument that double inclusive cross section is even under inversion of one of gluon momenta is valid only for surface contribution (at any order in classical approximation).
  The bulk contribution to single inclusive production is not symmetric under $\vec{k}_\perp \to -\vec{k}_\perp$ configuration-by-configuration.
- The result can be used to calculate leading order $v_3\{2\}$ without solving Classical Yang-Mills numerically.
- Is it related to initial state geometric asymmetry ε_3?





Towards numerical result

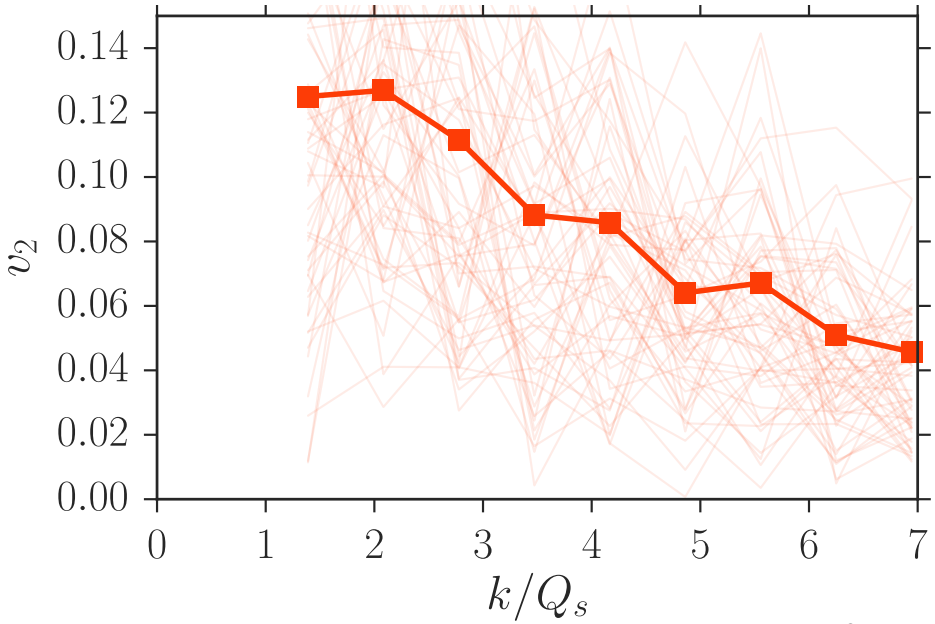
- Generate (target and projectile) sources according to MV on a latice 2024x2024
- Compute Wilson lines for target
- Rotate projectile sources by target Wilson lines $\rightsquigarrow \Omega$
- Compute single inclusive asymmetry on conf.-by-conf. basis

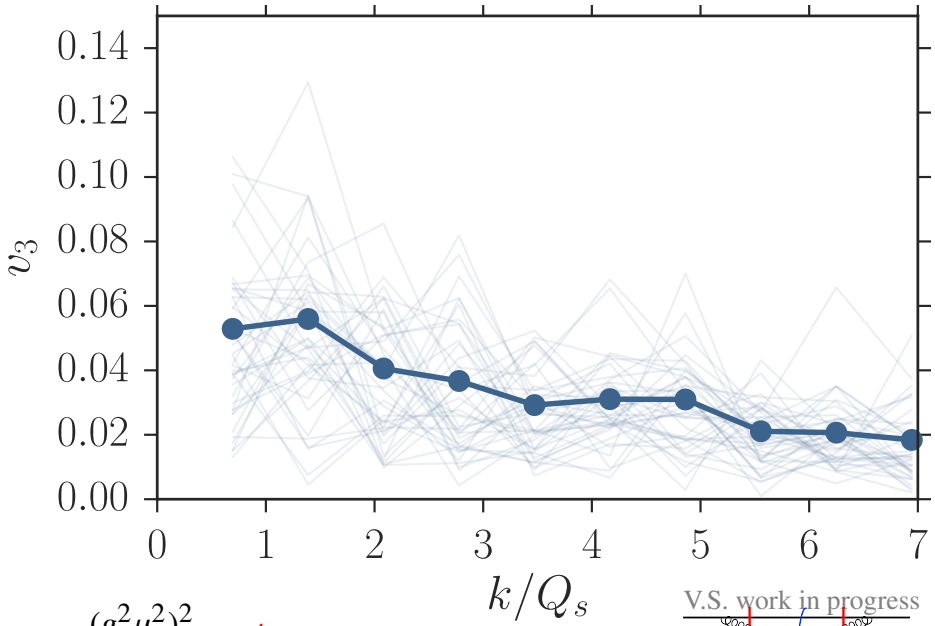
$$\frac{E_k}{2}\left(\frac{dN^{\mathrm{NLO}}}{d^3k}(\vec{k}_\perp)-\frac{dN^{\mathrm{NLO}}}{d^3k}(-\vec{k}_\perp)\right)=\frac{1}{16\pi}\mathrm{Im}\left\{\frac{2g}{\pi^2k_\perp^2}\int\frac{d^2q}{(2\pi)^2}\frac{\vec{k}_\perp\times\vec{q}_\perp}{|\vec{k}_\perp\times\vec{q}_\perp|}\frac{1}{q_\perp^2|\vec{k}_\perp-\vec{q}_\perp|^2}\ f^{abc}\ \Omega^a_{ij}(\vec{q}_\perp)\,\Omega^b_{mn}(\vec{k}_\perp-\vec{q}_\perp)\,\Omega^{c\star}_{rp}(\vec{k}_\perp)\times\right.$$
$$\left.\left[\left(k_\perp^2\epsilon^{ij}\epsilon^{mn}-\vec{q}_\perp\cdot(\vec{k}_\perp-\vec{q}_\perp)(\epsilon^{ij}\epsilon^{mn}+\delta^{ij}\delta^{mn})\right)\epsilon^{rp}+2\vec{k}_\perp\cdot(\vec{k}_\perp-\vec{q}_\perp)\epsilon^{ij}\delta^{mn}\delta^{rp}\right]\right\}$$

- Compute double inclusive asymmetry on conf.-by-conf. basis
- Average w.r.t. $W[\rho_1]$ and $W[\rho_2]$

V. S. work in progress

Crunching numbers



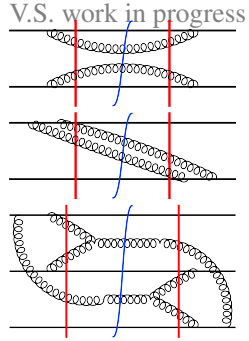


— LO in DI uncorrelated production $dN/d^2p_\perp d^2k_\perp \propto \frac{(g^2\mu_p^2)^2}{p_\perp^2 k_\perp^2} N_c^4$

— Even LO in DI correlated production $dN/d^2p_\perp d^2k_\perp \propto \frac{(g^2\mu_p^2)^2}{p_\perp^2 k_\perp^2} N_c^2$

— Odd LO in DI correlated production $dN/d^2p_\perp d^2k_\perp \propto g^2 \frac{g^2\mu_p^2}{g^2\mu_A^2} \frac{(g^2\mu_p^2)^2}{p_\perp^2 k_\perp^2} N_c^3$

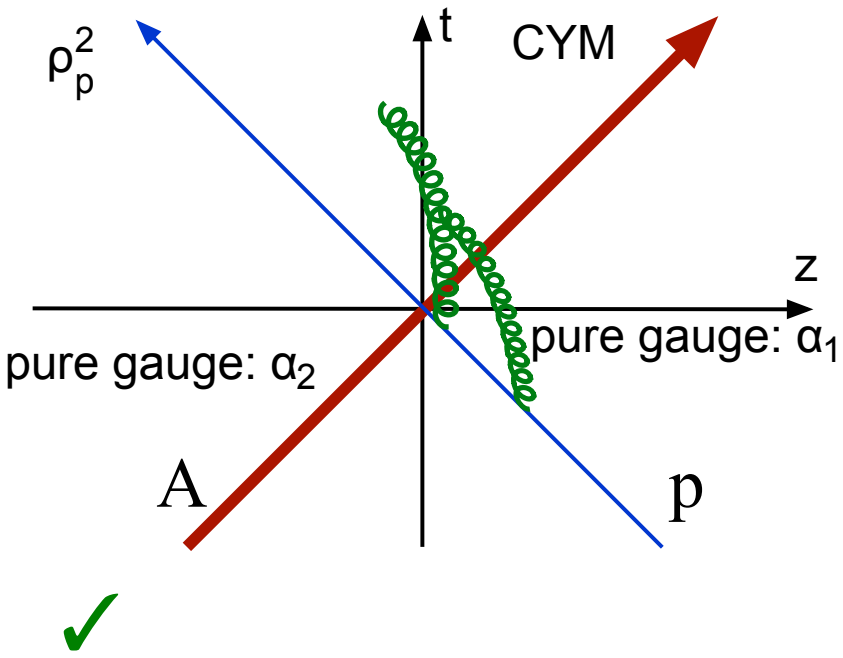
V.S. work in progress

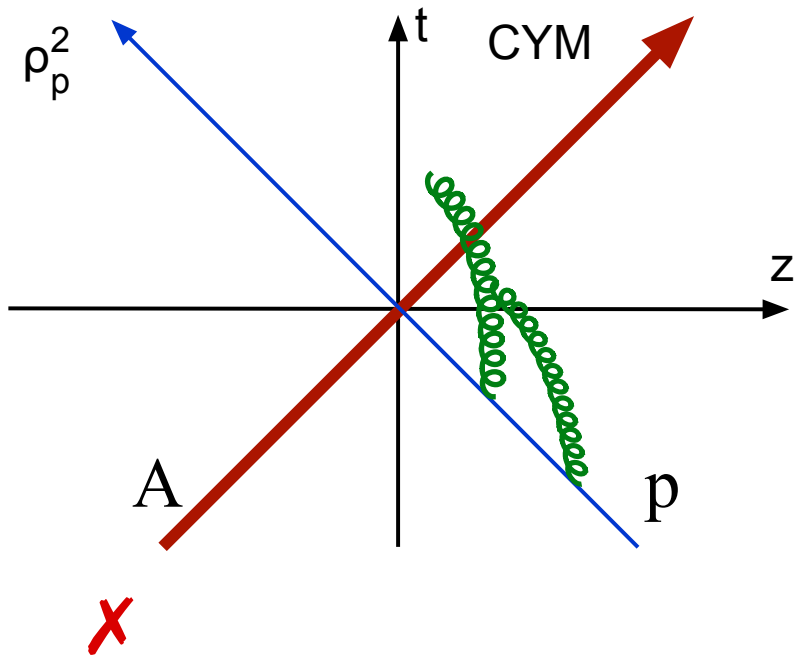


— Second harmonic $v_2^2 \propto \frac{1}{N_c^2}$ — Third harmonic $v_3^2 \propto g^2 \frac{g^2\mu_p^2}{g^2\mu_A^2} \frac{1}{N_c}$ or $\frac{v_3^2}{v_2^2} \propto g^2 N_c \frac{Q_{sp}^2}{Q_{sA}^2}$

- Proton Q_s & coupling constant can be used to adjust relative amplitude between v_2 and v_3
- SC(2,3) is possible to compute with higher statistics: preliminarily $\approx -1.6 \cdot 10^{-7} < 0$
- Applications to phenomenology?!

Outlook: Beyond classical limit





Some degree of final state interaction is needed?!

- Is there odd azimuthal anisotropy from genuine initial wave-function beyond classical limit?
- Using projectile wave function describing quantum high energy evolution (JIMWLK), we were able to show that there is!

A. Kovner, M. Lublinsky and V. S., 1612.07790

Summary and Conclusions

- $v_3\{2\} \neq 0$ from initial state calculations!
- Explicit expression for odd azimuthal anisotropy;
  diagrammatic interpretation.
- First correction is sufficient to have some $v_3\{2\}$.
  It is a quantitative question whether it is comparable to dense-dense limit.

Back up

LSZ: Milne coordinates

- Single-inclusive gluon production

$$E_k \frac{dN}{d^3k} = \frac{1}{16\pi}\left[\left|\underbrace{\mathfrak{S}_\perp(\vec{k}_\perp)}_{\text{surface}} + \underbrace{\mathfrak{B}_\perp(\vec{k}_\perp)}_{\text{bulk}}\right|^2 + \left|\mathfrak{S}_\eta(\vec{k}_\perp) + \mathfrak{B}_\eta(\vec{k}_\perp)\right|^2\right]$$

- Surface contributions:

$$\mathfrak{S}_\perp(\vec{k}_\perp) = \lim_{\tau\to 0+}\left(\tau \underbrace{H_0^{(1)}(k_\perp\tau)}_{H_n^{(1,2)}=J_n\pm iY_n} \overset{\leftrightarrow}{\partial}_0 \beta_\perp(\tau,\vec{k}_\perp)\right)$$

$$\beta_\perp(\tau,\vec{k}_\perp) = i\frac{\epsilon^{ij}k_j}{k_\perp}\beta_i(\tau,\vec{k}_\perp)$$

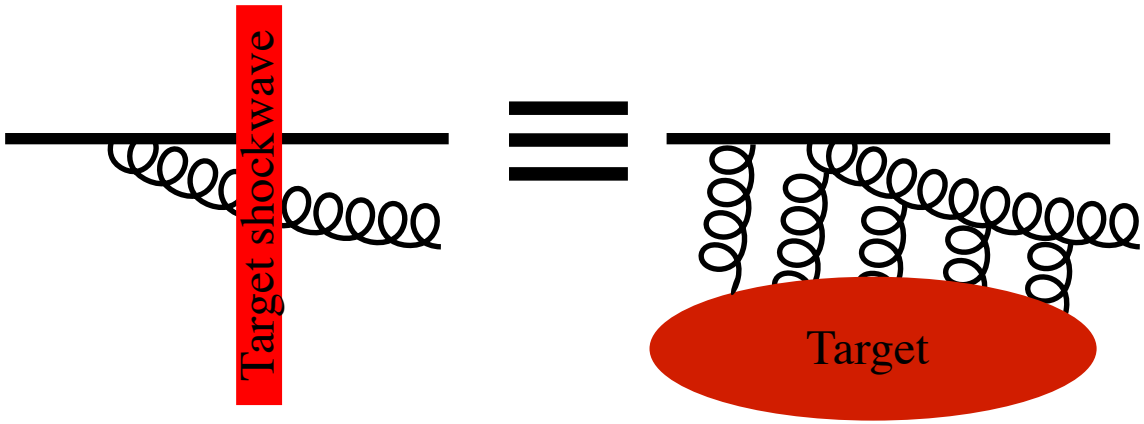
$$\mathfrak{S}_\eta(\vec{k}_\perp) = \lim_{\tau\to 0+}\left(\tau^3\left\{\frac{H_1^{(1)}(k_\perp\tau)}{\tau} \overset{\leftrightarrow}{\partial}_0 \beta(\tau,\vec{k}_\perp)\right\}\right)$$

- Bulk contributions:

$$\mathfrak{B}_\perp(\vec{k}_\perp) = \int_0^\infty d\tau\tau H_0^{(1)}(k_\perp\tau)\left\{\frac{1}{\tau}\partial_\tau\tau\partial_\tau\beta_\perp(\tau,\vec{k}_\perp) - \partial_\perp^2\,\beta_\perp(\tau,\vec{k}_\perp)\right\}$$

$$\mathfrak{B}_\eta(\vec{k}_\perp) = \int_0^\infty d\tau\tau^2 H_1^{(1)}(k_\perp\tau)\left\{\frac{1}{\tau^3}\partial_\tau\tau^3\partial_\tau\beta(\tau,\vec{k}_\perp) - \partial_\perp^2\,\beta(\tau,\vec{k}_\perp)\right\}$$

Multiple scattering



- Multiple rescattering in higher energy limit is described by eikonal phase

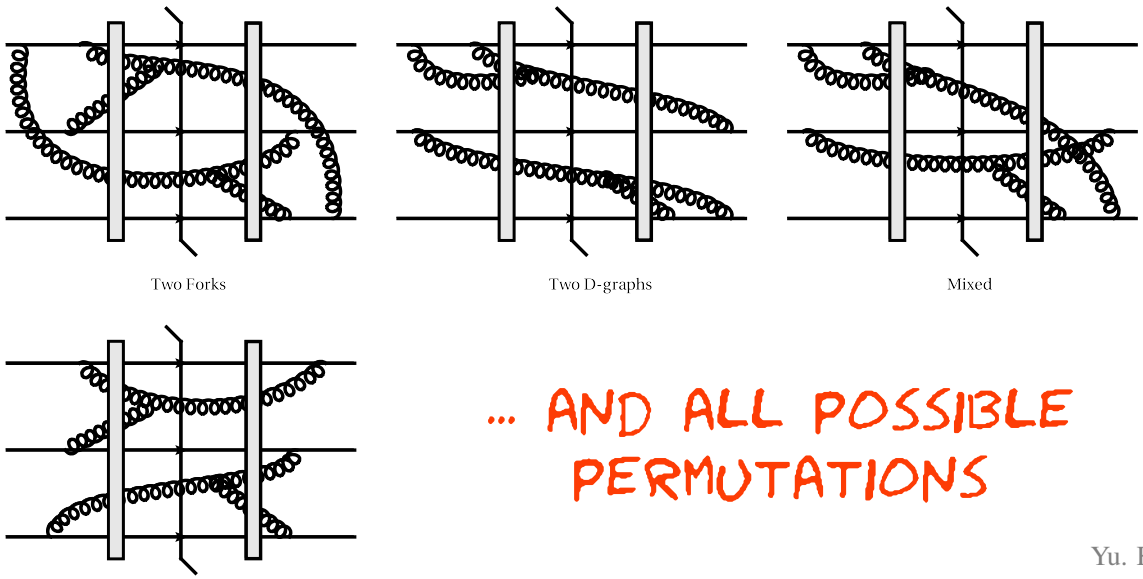
$$U(x_\perp) = \mathcal{P}\exp\left(ig\int dx^- A_+(x^-, x_\perp)\right)$$

- Only important property I need for further narrative: classical gluon field is real
  (for rigorous discussion see L. McLerran and V. S., arXiv:1611.09870)
- $A(x)$ is real function; therefore $A^*(k) = A(-k)$
- $$\sigma(k) \propto \underbrace{\left(\int d^2x A(x)\exp(-ikx)\right)}_{\text{amplitude}}\underbrace{\left(\int d^2y A(y)\exp(-iky)\right)^*}_{\text{c.c.amplitude}} = A(k)A(-k)$$
- Thus $\sigma(k) = \sigma(-k)$

"Why did not you use diagrammatic technique?"

The presented results could have been obtained using diagrammatic approach, but

- amount of diagrams is enormous; ensuing expressions are bulky and cumbersome
- Diagrams do not provide any intuition about properties of different contributions, e.g. that surface contribution gives no odd azimuthal anisotropy
- key point was to compute single inclusive production on configuration-by-configuration basis with explicit ρ.



Yu. Kovchegov, V. S. work in progress