Charmonium decays at BESIII

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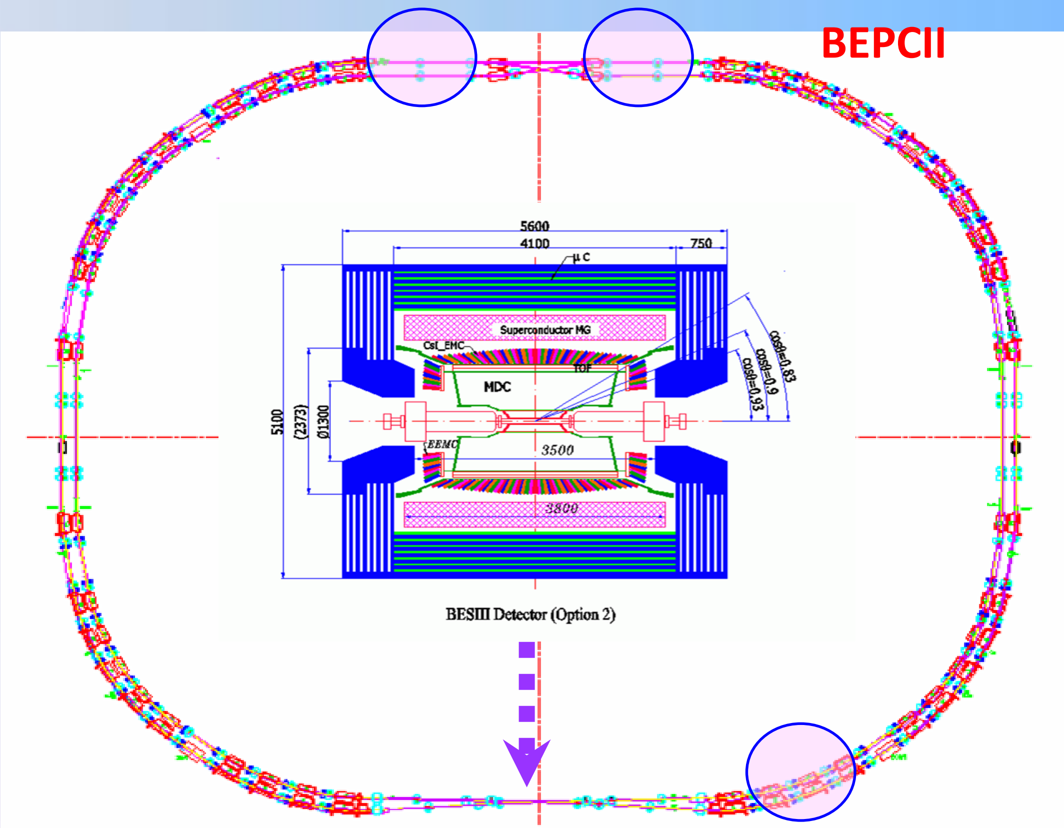
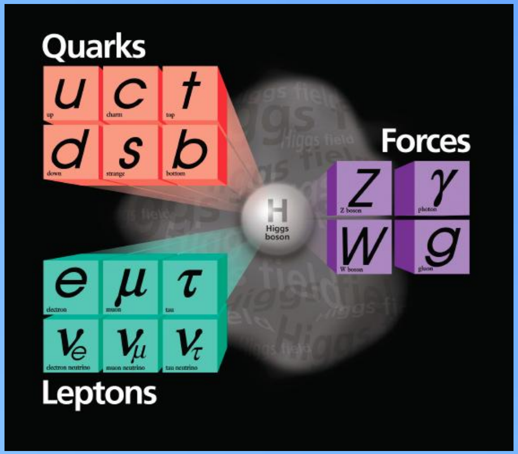
BEPCII/BESIII





The BEPCII has achieved the designed luminosity 1×10^{33} $cm^{-2}s^{-1}$ at Apr. 2016.

MDC: σ_{xy}=130 μm, dE/dx=6%
σ_p/p = 0.5% at 1 GeV

TOF:
Plastic scintillator: σ_T(barrel): 80 ps
MRPC: σ_T(endcap): 70 ps

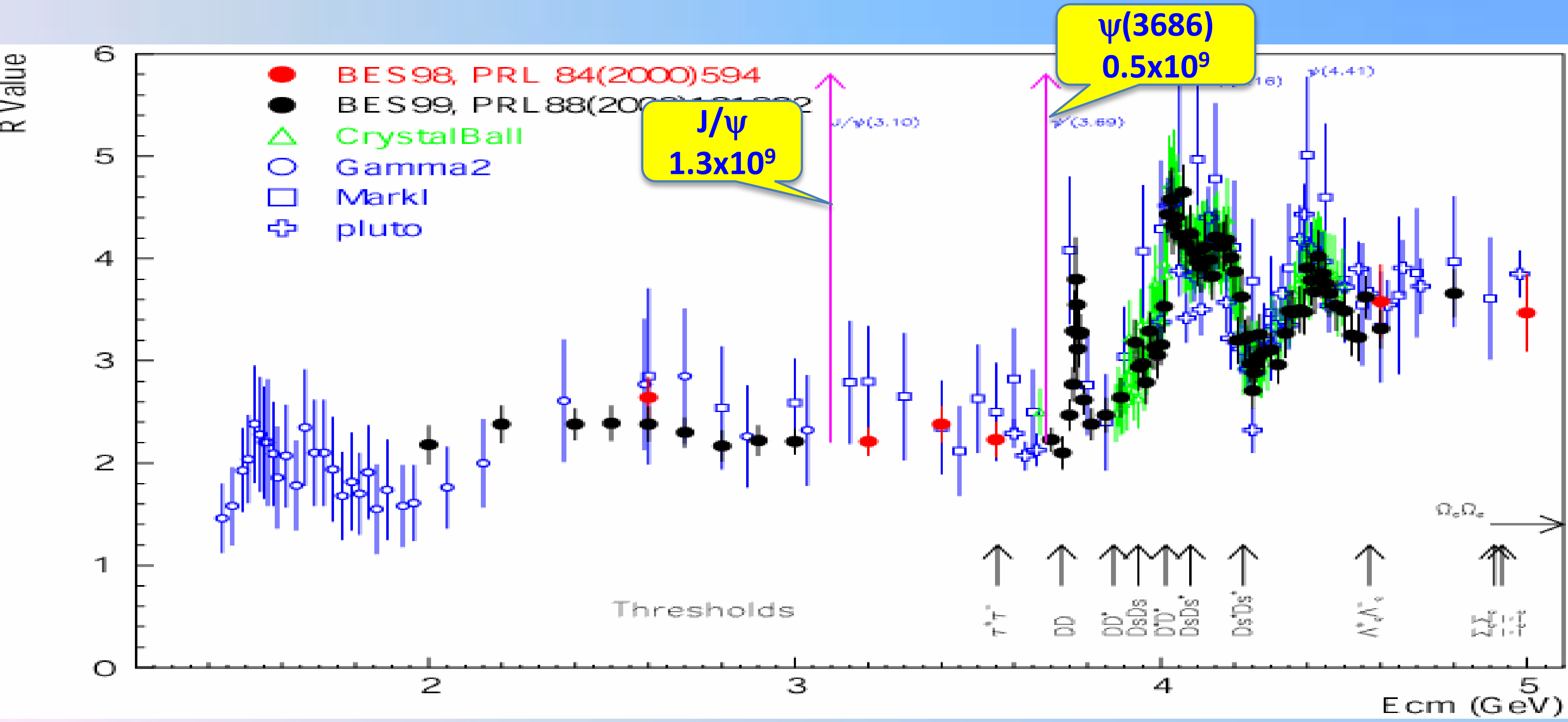
EMC: CsI(Tl)

At 1 GeV	σ_E (%)	σ_l(mm)
Barrel:	2.5	6.1
Endcap:	5	9

MUC: $\sigma_{spatial}$: 1.48 cm

Data samples at BESIII

World larges charmonium data sets directly produced from e^+e^- collision on J/ψ and ψ(3686) resonance, and many other data sets from 2-4.6 GeV



Recent charmonium results

Study of $\psi(nS) \rightarrow \Lambda\bar{\Lambda}$, $\Sigma^0\bar{\Sigma}^0$, $\Sigma(1385)^0\bar{\Sigma}(1385)^0$, $\Xi^0\bar{\Xi}^0$

Measurement of $\psi(3686) \rightarrow \gamma\pi^0, \gamma\eta, \gamma\eta'$

Observation of $\psi(3686) \rightarrow e^+e^-\chi_{cJ}$ and $\chi_{cJ} \rightarrow e^+e^-J/\psi$

Higher-order multipole amplitudes in $\psi(3686) \rightarrow \gamma\chi_{c1,2}$

Observation of $\chi_{c0,2} \rightarrow \eta\eta', \eta'\eta'$

Observation of $\chi_{c2} \rightarrow K(892)^*K$ and study of $\chi_{c2} \rightarrow \rho^{\pm}\pi$

Observation of $\chi_{cJ} \rightarrow \Sigma^+\bar{\Sigma}^-$, $\Sigma^0\bar{\Sigma}^0$

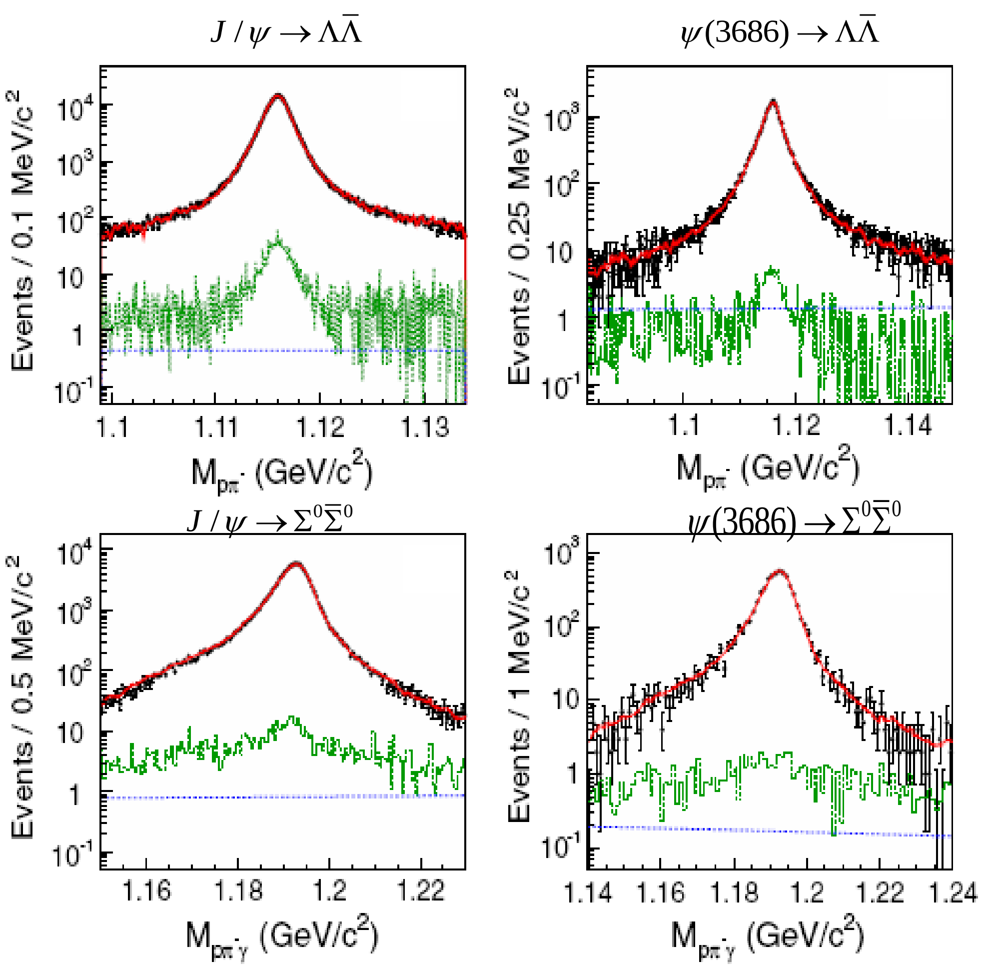
Measurement of $\eta_c \rightarrow \phi\phi, \omega\phi$

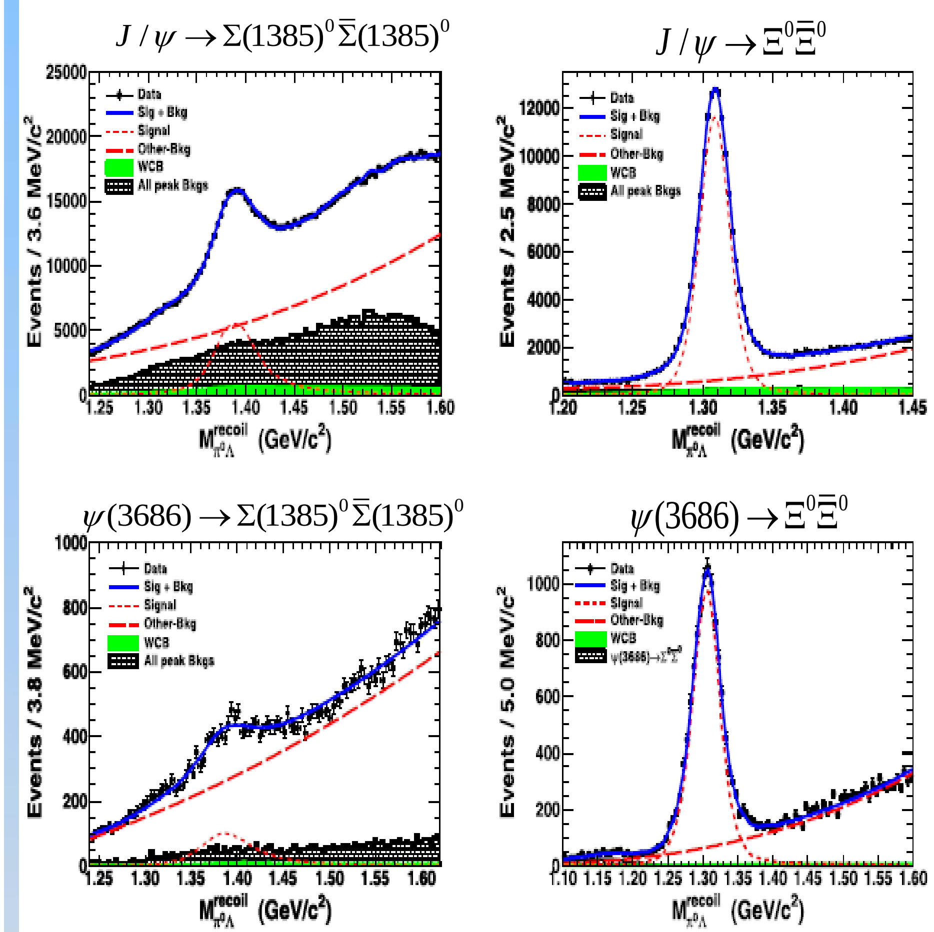
Search for $\eta_c/\eta(1405) \rightarrow \pi^+\pi^-\pi^0$

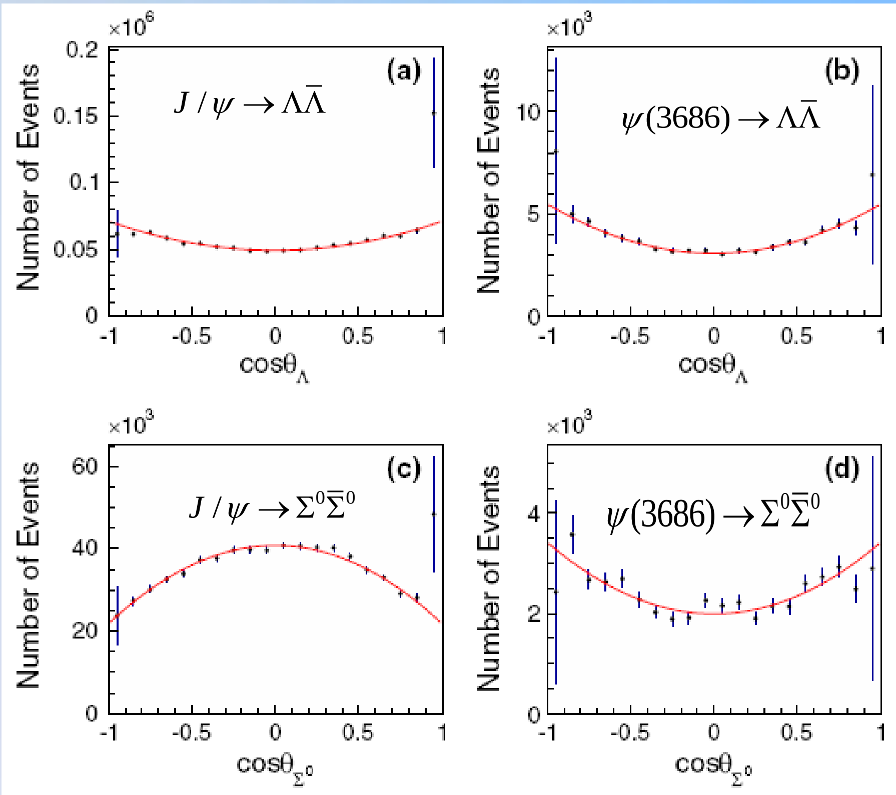
Study of $\psi(nS) \to \Lambda\bar{\Lambda}$, $\Sigma^0\bar{\Sigma}^0$, $\Sigma(1385)^0\bar{\Sigma}(1385)^0$, $\Xi^0\bar{\Xi}^0$

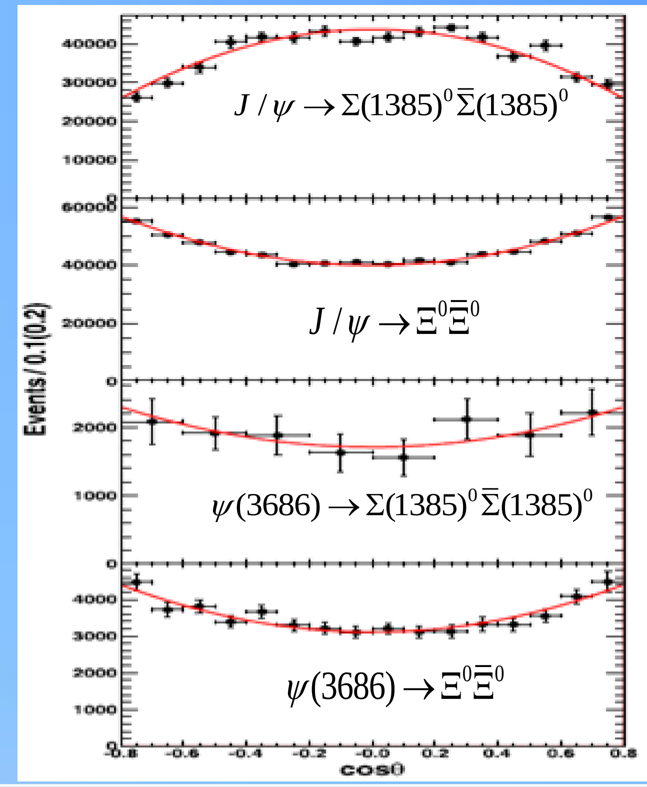
Test of 12% rule
Test of the helicity conservation rule
Test of isospin symmetry
First measurement or with best precision

PRD 95, 052003 (2017)
PLB 770, 217 (2017)









Channel	BF ($\times 10^{-4}$)		α	
	J/ψ→	ψ(3686) →	J/ψ→	ψ(3686) →
$\Lambda\bar{\Lambda}$	19.43±0.03±0.33	3.97±0.02±0.12	0.469±0.026±0.008	0.82±0.08±0.02
$\Sigma^0\bar{\Sigma}^0$	11.64±0.04±0.23	2.44±0.03±0.11	-0.449±0.020±0.008	0.71±0.11±0.04
$\Sigma(1385)^0\bar{\Sigma}(1385)^0$	10.71±0.09±0.82	0.69±0.05±0.05	-0.64±0.03±0.10	0.59±0.25±0.25
$\Xi^0\bar{\Xi}^0$	11.65±0.04±0.43	2.73±0.03±0.13	0.66±0.03±0.05	0.65±0.09±0.14

More than 3σ deviation to predictions: Int.J.Mod.Phys. 2 (1987) 249, Phys. Rev.D 25 (1982) 1345

Test of 12% rule:

$$\frac{\mathcal{B}(\psi(3686)\to\Lambda\bar{\Lambda})}{\mathcal{B}(J/\psi\to\Lambda\bar{\Lambda})}=(20.43\pm0.11\pm0.58)\%$$

$$\frac{\mathcal{B}(\psi(3686)\to\Sigma^0\bar{\Sigma}^0)}{\mathcal{B}(J/\psi\to\Sigma^0\bar{\Sigma}^0)}=(20.96\pm0.27\pm0.92)\%$$

$$\frac{\mathcal{B}(\psi(3686)\to\Sigma(1385)^0\bar{\Sigma}(1385)^0)}{\mathcal{B}(J/\psi\to\Sigma(1385)^0\bar{\Sigma}(1385)^0)}=(6.44\pm0.47\pm0.64)\%$$

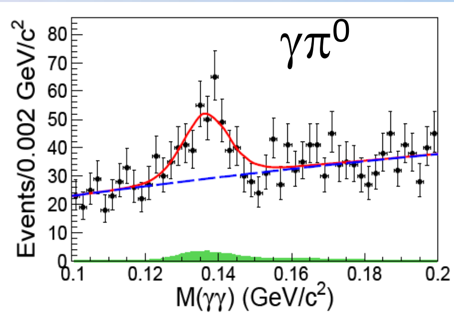
$$\frac{\mathcal{B}(\psi(3686)\to\Xi^0\bar{\Xi}^0)}{\mathcal{B}(J/\psi\to\Xi^0\bar{\Xi}^0)}=(23.43\pm0.26\pm1.09)\%$$

Test of isospin symmetry:

Mode	$\frac{\mathcal{B}(\psi\to\Xi^0\bar{\Xi}^0)}{\mathcal{B}(\psi\to\Xi^-\bar{\Xi}^+)}$	$\frac{\mathcal{B}(\psi\to\Sigma(1385)^0\bar{\Sigma}(1385)^0)}{\mathcal{B}(\psi\to\Sigma(1385)^-\bar{\Sigma}(1385)^+)}$	$\frac{\mathcal{B}(\psi\to\Sigma(1385)^0\bar{\Sigma}(1385)^0)}{\mathcal{B}(\psi\to\Sigma(1385)^+\bar{\Sigma}(1385)^-)}$
J/ψ	$1.12\pm0.01\pm0.07$	$0.98\pm0.01\pm0.08$	$0.85\pm0.02\pm0.09$
$\psi(3686)$	$0.98\pm0.02\pm0.07$	$0.81\pm0.12\pm0.12$	$0.82\pm0.11\pm0.11$

Measurement of $\psi(3686)\to\gamma\pi^0,\gamma\eta,\gamma\eta'$

hep-ex:1708.03103

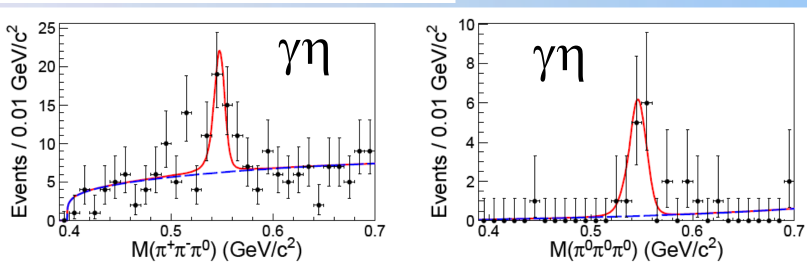


$$R_{J/\psi} \equiv \frac{\mathcal{B}(J/\psi\to\gamma\eta)}{\mathcal{B}(J/\psi\to\gamma\eta')}$$

$$R_{\psi(3686)} \equiv \frac{\mathcal{B}(\psi(3686)\to\gamma\eta)}{\mathcal{B}(\psi(3686)\to\gamma\eta')}$$

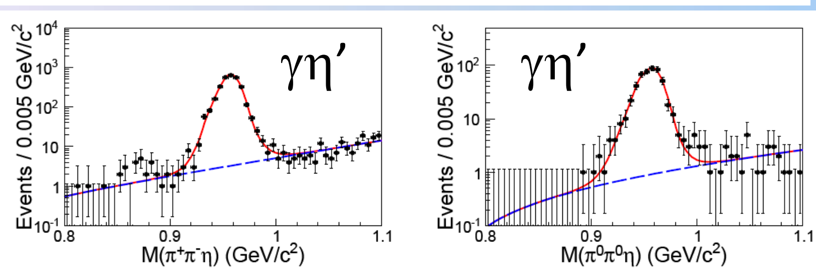
$$R_{\psi(3686)} \approx R_{J/\psi}$$

Phys. Rev. 112(1984) 173



$$R_{\psi(3686)} = (0.66 \pm 0.13 \pm 0.02)\%$$

$$R_{J/\psi} = (21.4 \pm 0.9)\%$$



This can be explained by **PLB 697(2011) 72**, however, it predicts one order lower $\gamma\pi^0$ branching fraction. More studies are needed!

Decay mode	Significance	$\mathcal{B}(\psi(3686)\to\gamma\eta'/\eta/\pi^0)$	Previous results from BESIII
$\psi(3686)\to\gamma\eta'$	$>10\sigma$	$(125.1\pm2.2\pm6.2)\times10^{-6}$	$(126\pm3\pm8)\times10^{-6}$
$\psi(3686)\to\gamma\eta$	7.3σ	$(0.85\pm0.18\pm0.04)\times10^{-6}$	$(1.38\pm0.48\pm0.09)\times10^{-6}$
$\psi(3686)\to\gamma\pi^0$	6.7σ	$(0.95\pm0.16\pm0.05)\times10^{-6}$	$(1.58\pm0.40\pm0.13)\times10^{-6}$

Measurement of higher-order multipole amplitudes in $\psi(3686) \to \gamma\chi_{c1,2}$

PRD 95, 072004 (2017)

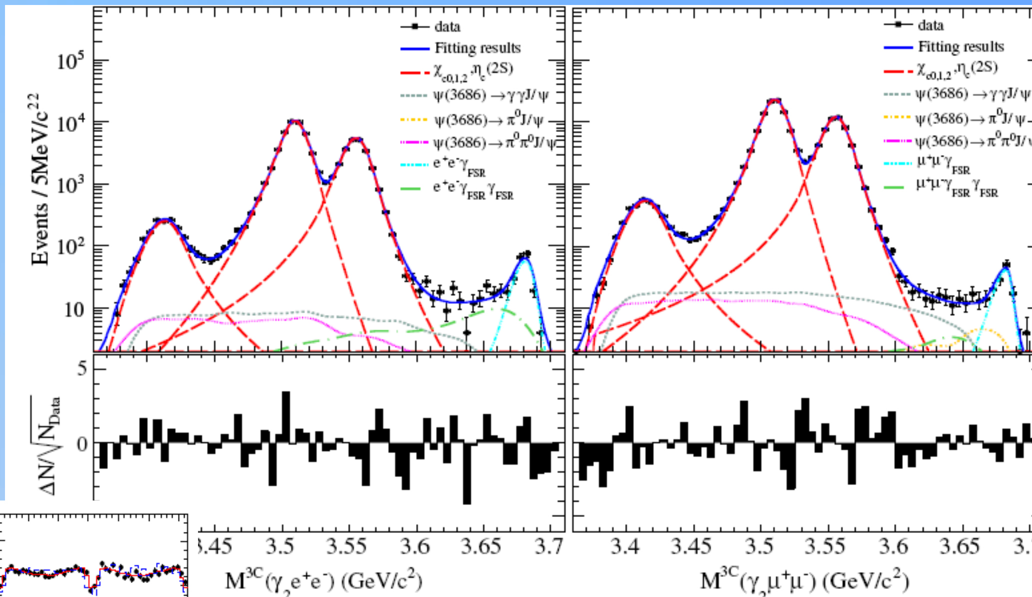
From PRL 45 (1980) 215:

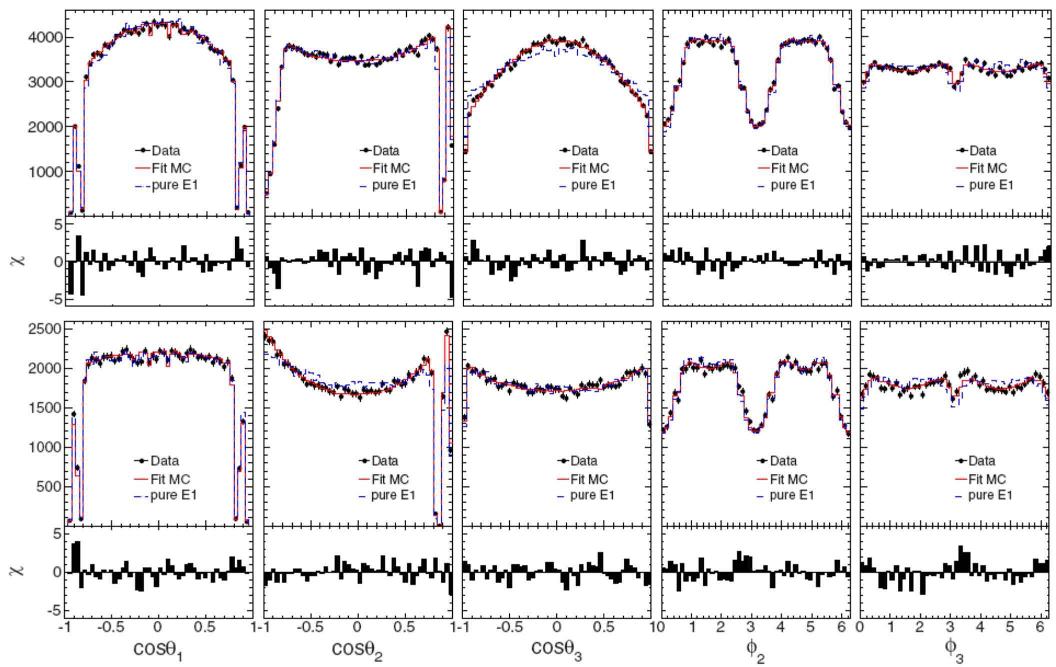
$$b_2^1 = \frac{E_{\gamma_1}[\psi(3686) \to \gamma_1 \chi_{c1}]}{4m_c}(1+\kappa) = 0.029(1+\kappa)$$

$$a_2^1 = -\frac{E_{\gamma_2}[\chi_{c1} \to \gamma_2 J/\psi]}{4m_c}(1+\kappa) = -0.065(1+\kappa)$$

$$b_2^2 = \frac{3}{\sqrt{5}}\frac{E_{\gamma_1}[\psi(3686) \to \gamma_1 \chi_{c2}]}{4m_c}(1+\kappa) = 0.029(1+\kappa)$$

$$a_2^2 = -\frac{3}{\sqrt{5}}\frac{E_{\gamma_2}[\chi_{c2} \to \gamma_2 J/\psi]}{4m_c}(1+\kappa) = -0.096(1+\kappa)$$





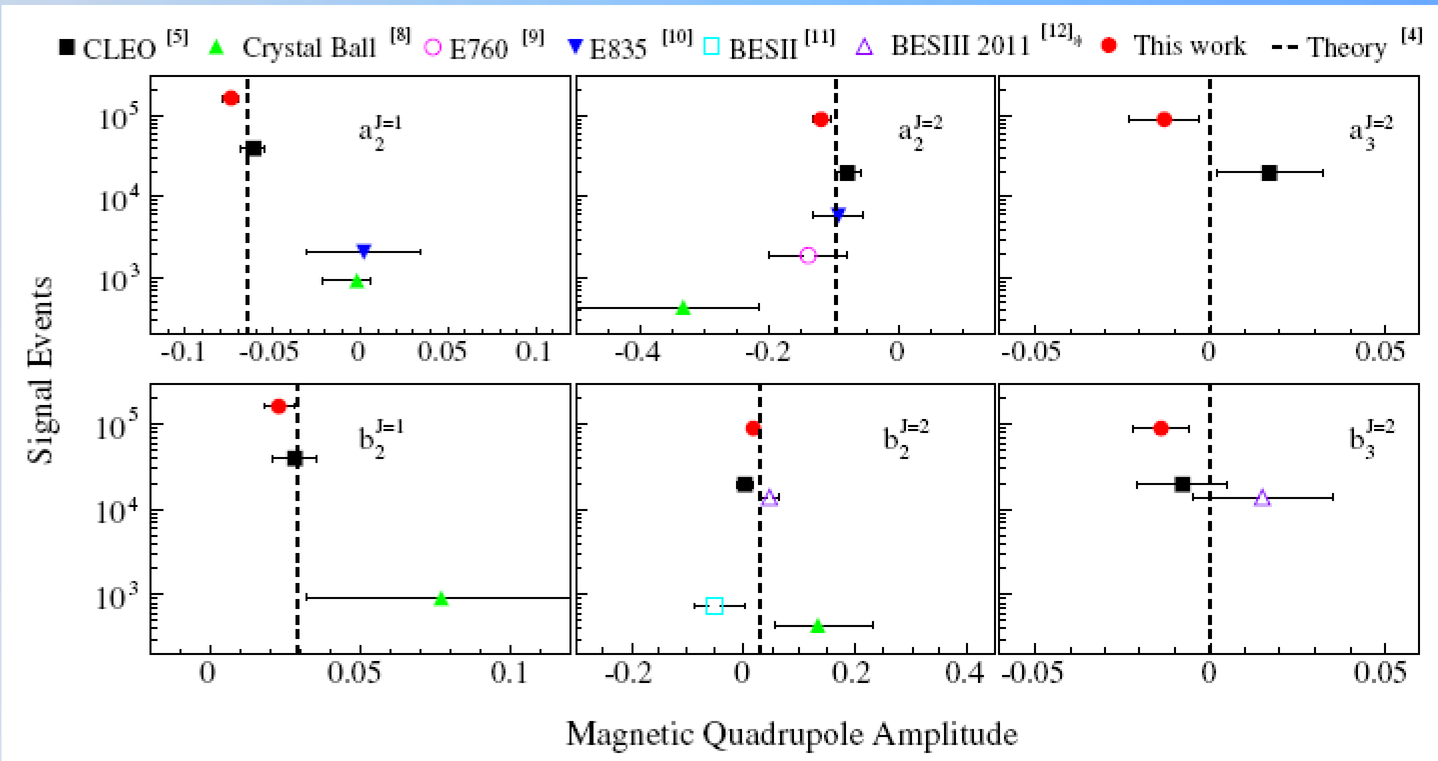
$$W_{\chi_{cJ}}(\theta_1, \theta_2, \phi_2, \theta_3, \phi_3, a_{2,3}^J, b_{2,3}^J) = \sum_n a_n A_{|\nu|}^J A_{|\tilde{\nu}|}^J B_{|\nu'|}^J B_{|\tilde{\nu}'|}^J$$

$$\begin{pmatrix} A_0^1 \\ A_1^1 \end{pmatrix} = \begin{pmatrix} \sqrt{0.5} & \sqrt{0.5} \\ \sqrt{0.5} & -\sqrt{0.5} \end{pmatrix} \begin{pmatrix} a_1^1 \\ a_2^1 \end{pmatrix}$$

$$\begin{pmatrix} B_0^1 \\ B_1^1 \end{pmatrix} = \begin{pmatrix} \sqrt{0.5} & \sqrt{0.5} \\ \sqrt{0.5} & -\sqrt{0.5} \end{pmatrix} \begin{pmatrix} b_1^1 \\ b_2^1 \end{pmatrix}$$

$$\begin{pmatrix} A_0^2 \\ A_1^2 \\ A_2^2 \end{pmatrix} = \frac{1}{\sqrt{30}} \begin{pmatrix} \sqrt{3} & \sqrt{15} & 2\sqrt{3} \\ 3 & \sqrt{5} & -4 \\ 3\sqrt{2} & -\sqrt{10} & \sqrt{2} \end{pmatrix} \begin{pmatrix} a_1^2 \\ a_2^2 \\ a_3^2 \end{pmatrix}$$

$$\begin{pmatrix} B_0^2 \\ B_1^2 \\ B_2^2 \end{pmatrix} = \frac{1}{\sqrt{30}} \begin{pmatrix} \sqrt{3} & \sqrt{15} & 2\sqrt{3} \\ 3 & \sqrt{5} & -4 \\ 3\sqrt{2} & -\sqrt{10} & \sqrt{2} \end{pmatrix} \begin{pmatrix} b_1^2 \\ b_2^2 \\ b_3^2 \end{pmatrix}$$



$$(b_2^1/b_2^2)_{\rm th} = 1.000 \pm 0.015$$

$$(a_2^1/a_2^2)_{\rm th} = 0.676 \pm 0.071$$

$$b_2^1/b_2^2 = 1.35 \pm 0.72$$

$$a_2^1/a_2^2 = 0.617 \pm 0.083$$

Nonzero magnetic quadrupole amplitudes for the transitions $\psi(3686) \rightarrow \gamma\chi_{c1,2}$

Observation of $\psi(3686)\to e^+e^-\chi_{cJ}$ and $\chi_{cJ}\to e^+e^-J/\psi$

PRL 118, 221802 (2017)

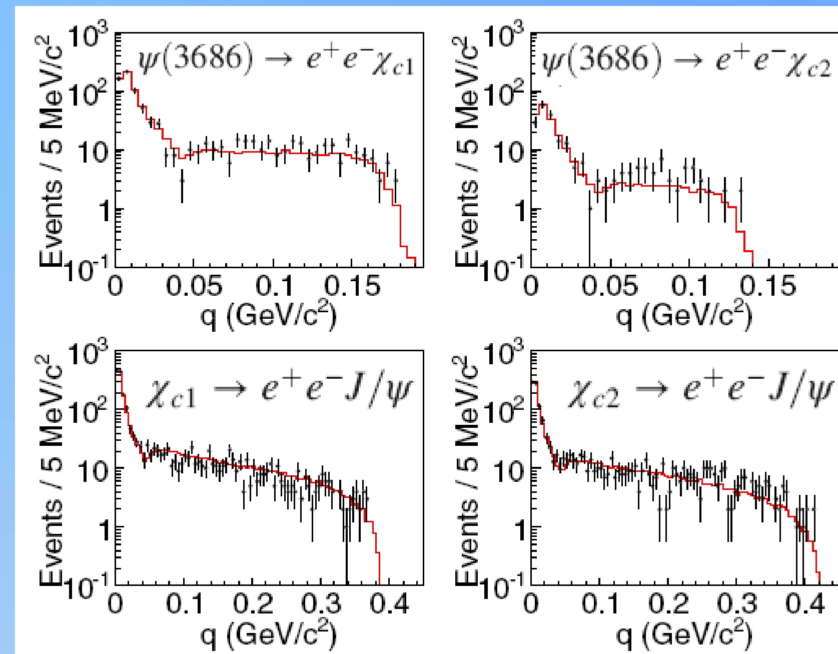
We just mentioned the higher-order multipole amplitudes in $\psi(3686)\to\gamma\chi_{c1,2}$ are small, the E1 contribution is dominant.

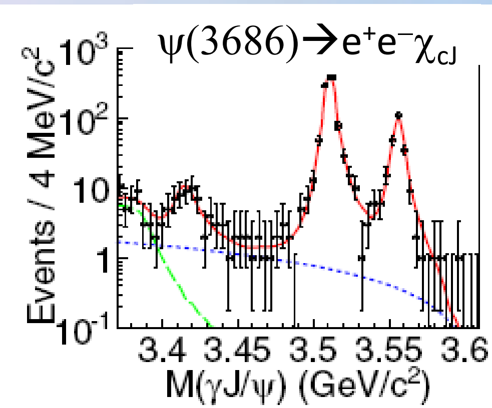
the q^2-dependent Transition Form Factor can:

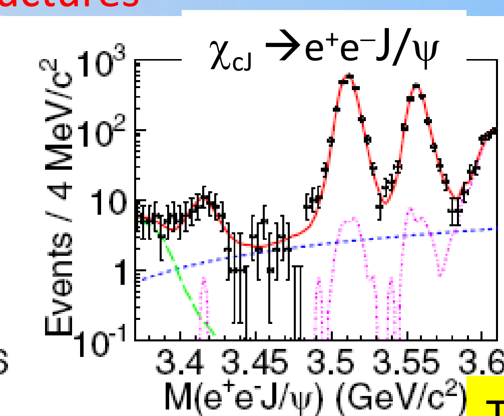
Probe χ_{cJ} internal structures

Possibly distinguish the transition mechanisms based on the $c\bar{c}$ scenario and other solutions

Probe exotic hadron structures







The measured q distributions are consistent with assumption of a pointlike meson

Mode	Branching fraction	$\mathcal{B}(\psi(36$ … / $\mathcal{B}(\psi(3686)\to\gamma\chi_{cJ})$	… / $\mathcal{B}(\chi_{cJ}\to\gamma J/\psi)$
$\psi(3686)\to e^+e^-\chi_{c0}$	$(11.7\pm2.5\pm1.0)\times10^{-4}$	$(9.4\pm1.9\pm0.6)\times10^{-3}$	⋯
$\psi(3686)\to e^+e^-\chi_{c1}$	$(8.6\pm0.3\pm0.6)\times10^{-4}$	$(8.3\pm0.3\pm0.4)\times10^{-3}$	⋯
$\psi(3686)\to e^+e^-\chi_{c2}$	$(6.9\pm0.5\pm0.6)\times10^{-4}$	$(6.6\pm0.5\pm0.4)\times10^{-3}$	⋯
$\chi_{c0}\to e^+e^-J/\psi$	$(1.51\pm0.30\pm0.13)\times10^{-4}$	⋯	$(9.5\pm1.9\pm0.7)\times10^{-3}$
$\chi_{c1}\to e^+e^-J/\psi$	$(3.73\pm0.09\pm0.25)\times10^{-3}$	⋯	$(10.1\pm0.3\pm0.5)\times10^{-3}$
$\chi_{c2}\to e^+e^-J/\psi$	$(2.48\pm0.08\pm0.16)\times10^{-3}$	⋯	$(11.3\pm0.4\pm0.5)\times10^{-3}$

Observation of $\chi_{c0,2} \rightarrow \eta\eta', \eta'\eta'$

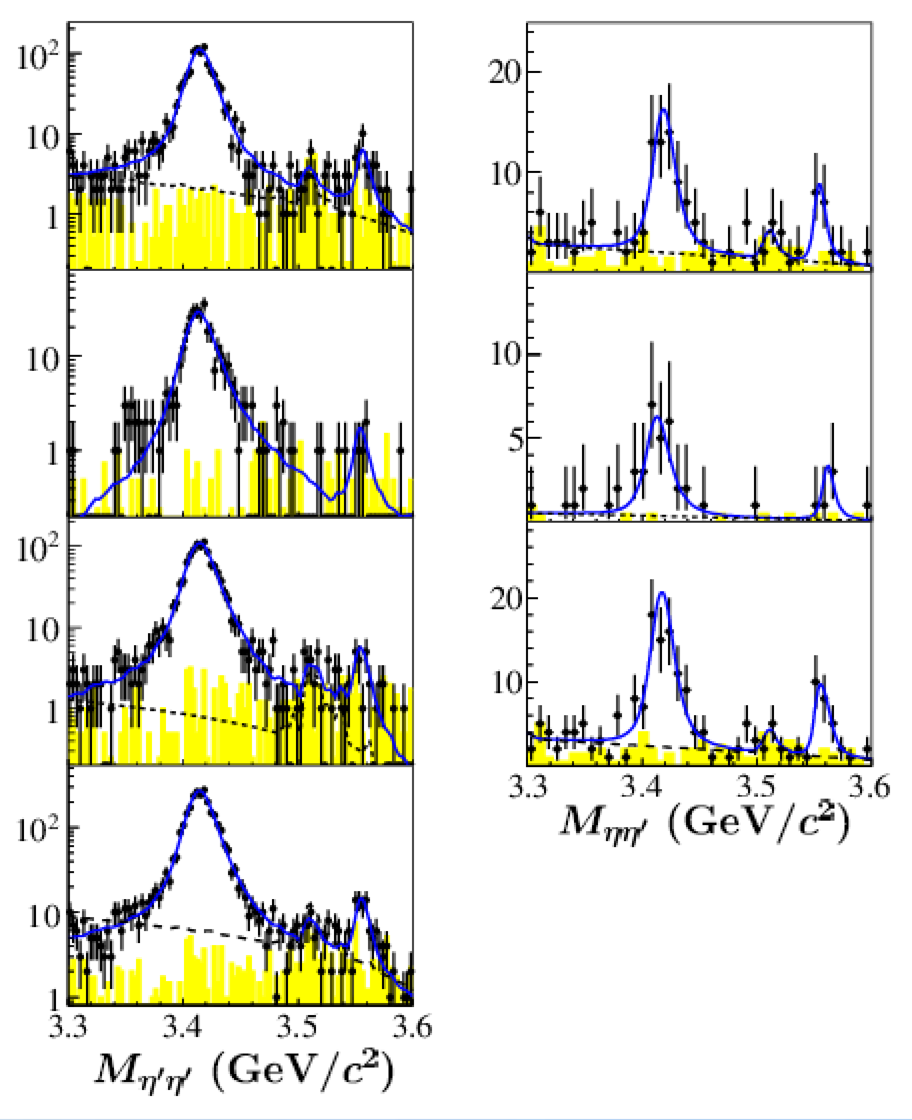
hep-ex:1707.07042

Doubly OZI violating amplitude play a crucial role in isospin-0 light meson pairs? Such as $\chi_{cJ} \rightarrow$SS, PP and VV should be studied for insights into the mechanisms for the pair production.

η-η' mixing mechanism in χ_{cJ} decays

Channel	This work (10^{-5})	PDG (10^{-5})
$\chi_{c0} \rightarrow \eta'\eta'$	$219 \pm 3 \pm 14$	196 ± 21
$\chi_{c2} \rightarrow \eta'\eta'$	$4.76 \pm 0.56 \pm 0.38$	<10
$\chi_{c0} \rightarrow \eta\eta'$	$8.92 \pm 0.84 \pm 0.65$	<23
$\chi_{c2} \rightarrow \eta\eta'$	$2.27 \pm 0.43 \pm 0.25$	<6.0

The results give relative small DOZI contribution in $\chi_{c0,2} \rightarrow$PP



Observation of HSR suppressed processes $\chi_{c2} \to K(892)^{*}K$ and study of $\chi_{c2} \to \rho^{\pm}\pi$

hep-ex: 1612.07398

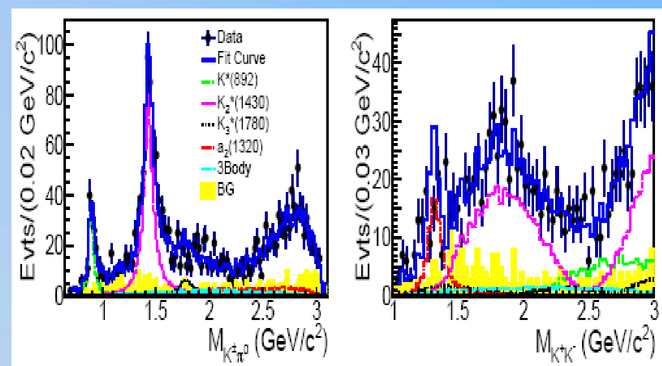
pQCD dominance is accepted

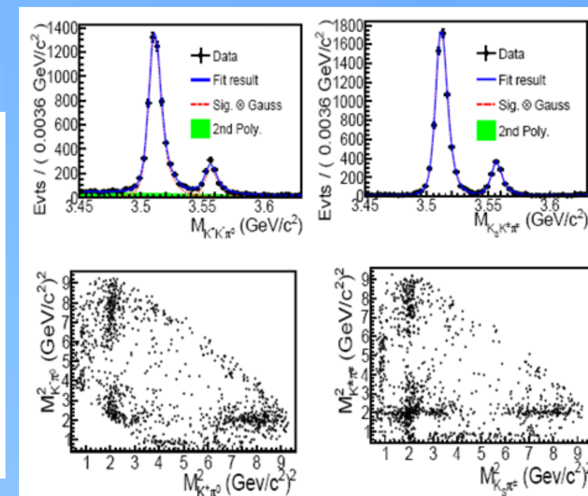
How important of non-perturbative mechanisms in charmonium region?

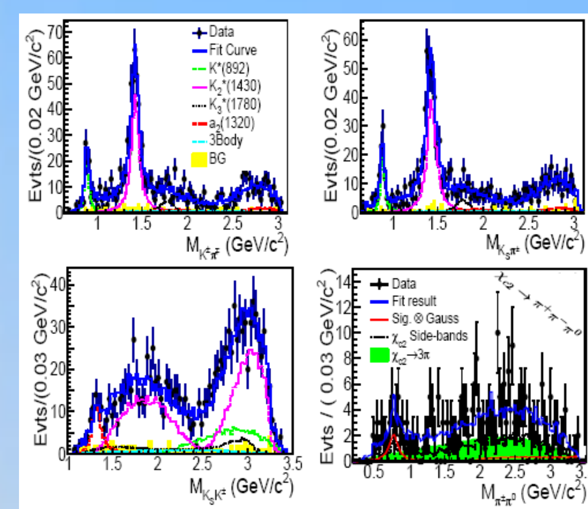
$\chi_{c2} \to$ VP can :

testing the HSR;

pinning down the mechanisms violating the leading pQCD approximation





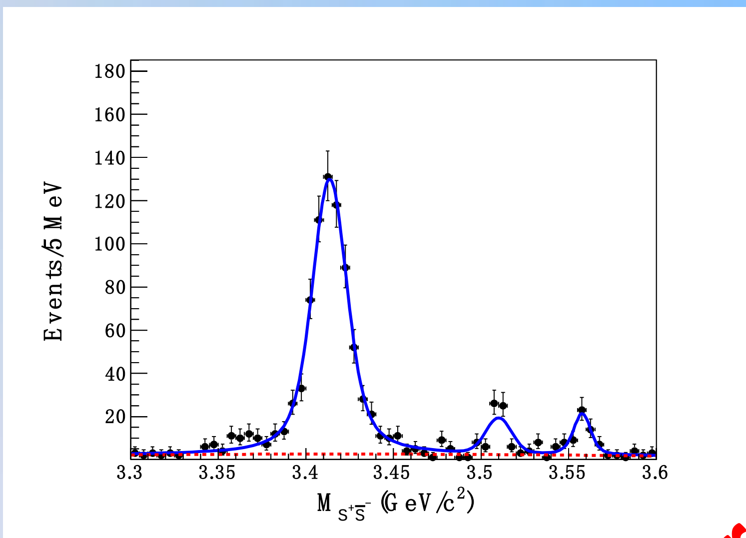


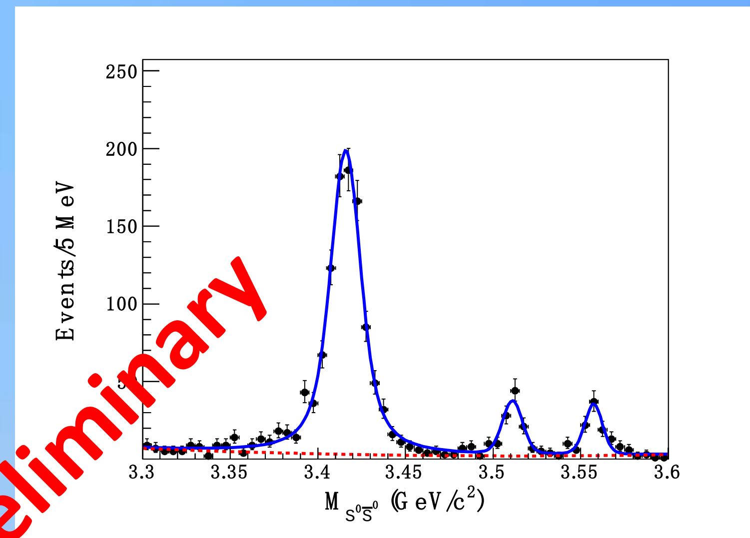
Mode	$K^+K^-\pi^0$	$K_SK^{\pm}\pi^{\mp}$	Combined
$K^{*\pm}K^{\mp}$	$1.8 \pm 0.2 \pm 0.2$	$1.4 \pm 0.2 \pm 0.2$	$1.5 \pm 0.1 \pm 0.2$
$K^{*0}\overline{K}^0$	—	$1.3 \pm 0.2 \pm 0.2$	—
$K_2^{*\pm}K^{\mp}$	$18.2 \pm 0.8 \pm 1.6$	$13.6 \pm 0.8 \pm 1.4$	$15.5 \pm 0.6 \pm 1.2$
$K_2^{*0}\overline{K}^0$	—	$13.0 \pm 1.0 \pm 1.5$	—
$K_3^{*\pm}K^{\mp}$	$5.3 \pm 0.5 \pm 0.9$	$5.9 \pm 1.1 \pm 1.5$	$5.4 \pm 0.5 \pm 0.7$
$K_3^{*0}\overline{K}^0$	—	$5.9 \pm 1.6 \pm 1.5$	—
$a_2^0\pi^0$	$13.5 \pm 1.6 \pm 3.2$	—	—
$a_2^{\pm}\pi^{\mp}$	—	$18.4 \pm 3.3 \pm 5.5$	—

Measurement of $\chi_{cJ} \rightarrow \Sigma^+\bar{\Sigma}^-$, $\Sigma^0\bar{\Sigma}^0$

Test the color octet model; Provide information for helicity selection rule

Test the iso-spin symmetry





BESIII Preliminary

Decay Channel	This work (10^{-5})	PDG (10^{-5})	Ratio(charge/neutral)
$\chi_{c0} \rightarrow \Sigma^+\Sigma^-$	51.8±2.6±3.0	39±7	1.09±0.07±0.09
$\chi_{c0} \rightarrow \Sigma^0\Sigma^0$	47.7±1.9±3.6	44±4	
$\chi_{c1} \rightarrow \Sigma^+\Sigma^-$	3.8±0.6±0.3	<6	0.80±0.18±0.08
$\chi_{c1} \rightarrow \Sigma^0\Sigma^0$	3.7±1.0±0.5	<4	
$\chi_{c2} \rightarrow \Sigma^+\Sigma^-$	3.6±0.7±0.3	<7	0.92±0.21±0.11
$\chi_{c2} \rightarrow \Sigma^0\Sigma^0$	3.8±1.0±0.5	<6	

Improved measurement of $\eta_c \to \phi\phi, \omega\phi$

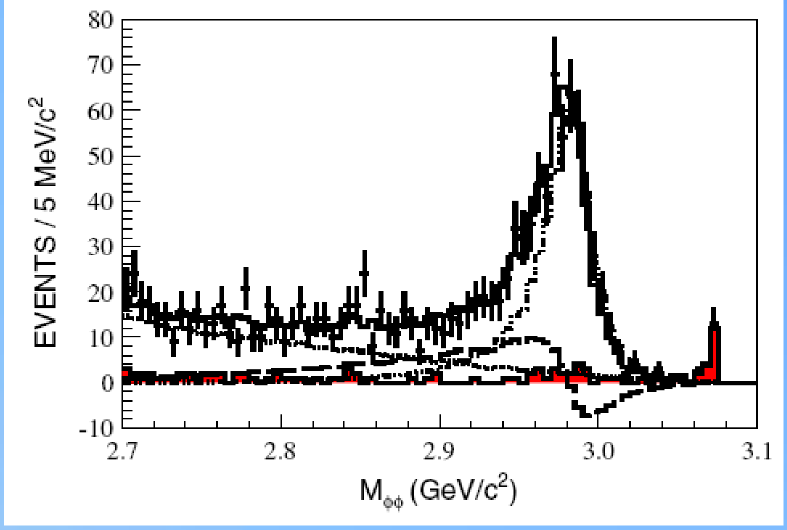
PRD 95, 092004 (2017)

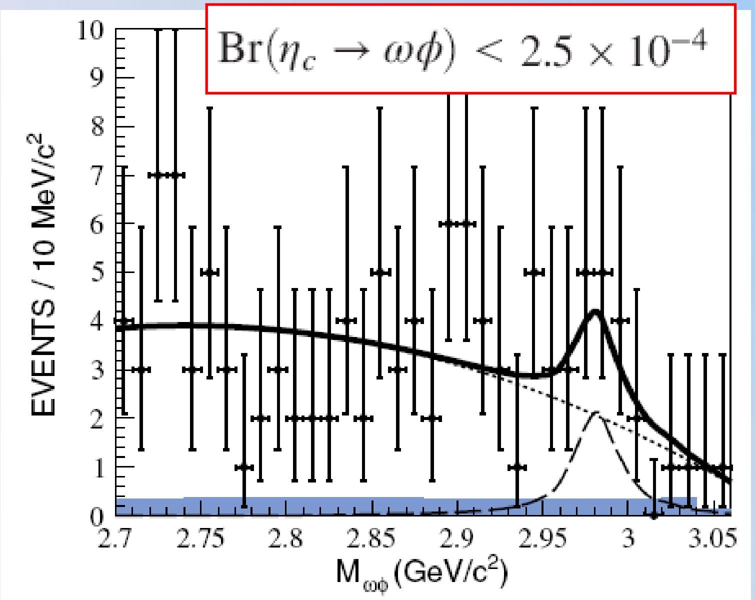
$\eta_c \to VV$ are helicity selection rule suppressed decays

Many calculations:

HSR evasion scenario, next-to-leading order and relativistic corrections in QCD, light quark mass corrections, $^3P_0$ quark pair creation mechanism, long-distance intermediate meson loop effect

Previous measurement is much larger than those of theoretical predictions but with relative large uncertainty.





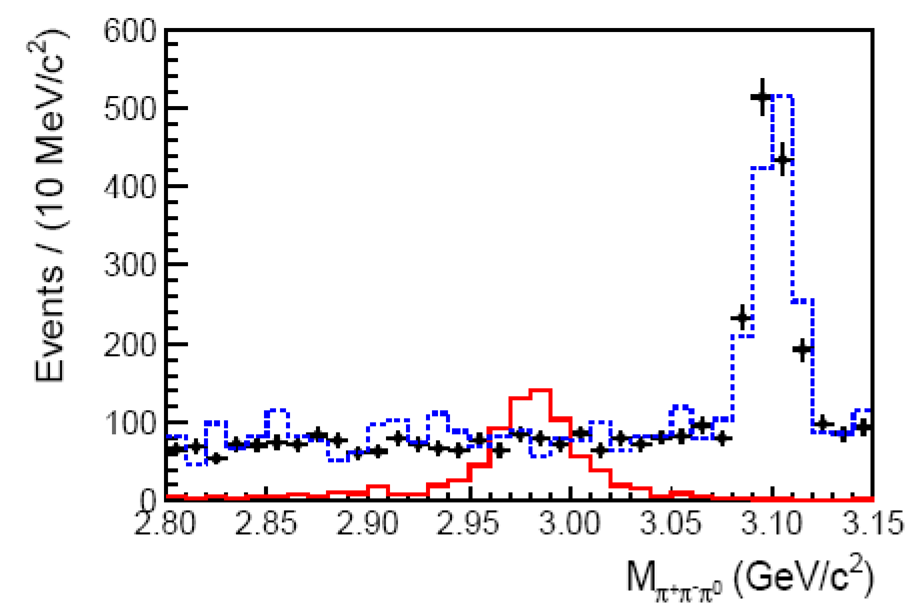
Experiment	$Br(\eta_c \to \phi\phi)(\times 10^{-3})$
BESIII	$2.5 \pm 0.3^{+0.3}_{-0.7} \pm 0.6$
BESII	1.9 ± 0.6
DM2	2.3 ± 0.8
Theoretical	
pQCD	(0.7–0.8)
$^3P_0$ quark model	(1.9–2.0)
Charm meson loop	2.0

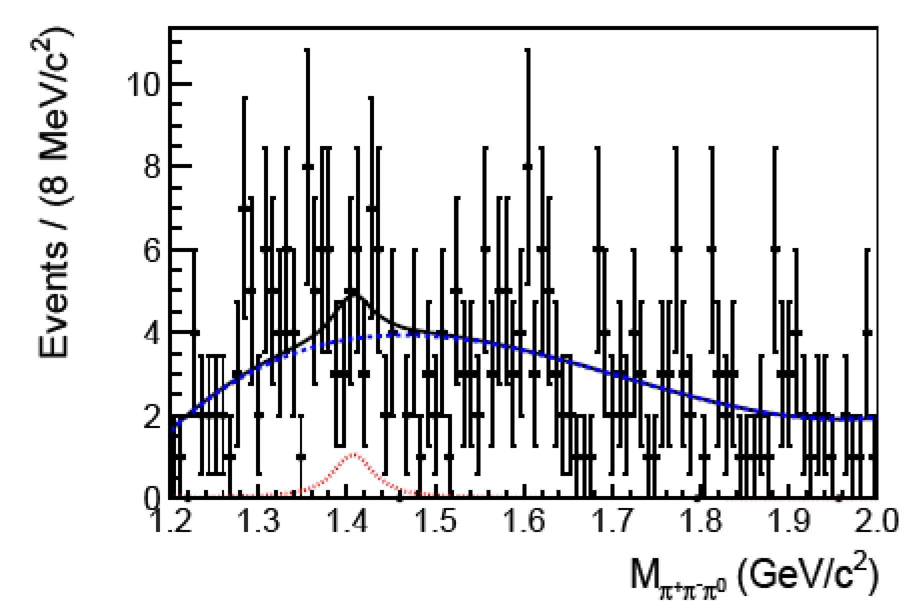
Search for $\eta_c/\eta(1405) \rightarrow \pi^+\pi^-\pi^0$

The lowest charmonium, annihilation into two gluons, much information on gluon dynamics can be obtained by studying η_c decays

hep-ex: 1707.0517

Search for the isospin violating decay





$Br(\eta_c \rightarrow \pi^+\pi^-\pi^0) < 1.6 \times 10^{-6}$

$Br(\eta(1405) \rightarrow \pi^+\pi^-\pi^0) < 5.2 \times 10^{-7}$

Summary

A lot of charmonium decays, J/ψ, $\psi(3686)$, η_c, $\eta_c(2S)$ are reported by BESIII, only some of them are listed in this talk, and still many of them:

Measurement of BF($\psi(3686)\to\gamma\chi_{cJ}$) **PRD96 (2017) 032001**

Observation of $h_c\to\gamma\eta'$ and $\gamma\eta$ **PRL116 (2016) 251802**

Study of $\psi\to\Xi^-\bar{\Xi}^+,\Sigma^{+/-}\bar{\Sigma}^{-/+}$ **PRD93 (2016) 072003**

……

World largest e+e- collision charmonium data sets at BESIII provide ideal laboratory to study charmonium decays: **high statistics, low background**

Many results have been published, many analysis are still ongoing, more results are promising. Expecting more theoretical attentions on these topics and more communications

Thanks for your attention!

Amplitude analysis of J/ψ→γφφ, γωφ

$$A_{\eta_c}(\lambda_0,\lambda_\gamma,\lambda_1,\lambda_2) = F^{\psi}_{\lambda_\gamma}(r_1) D^{1*}_{\lambda_0,-\lambda_\gamma}(\theta_0,\phi_0) BW_j(m_{\phi\phi}) \times F^{\eta_c}_{\lambda_1,\lambda_2}(r_2) D^{0*}_{0,\lambda_1-\lambda_2}(\theta_1,\phi_1) \frac{\mathcal{F}(E_\gamma)}{\mathcal{F}(E^0_\gamma)}$$

$$F^{\psi}_{1} = -F^{\psi}_{-1} = \frac{g_{11}}{\sqrt{2}} r_1 \frac{B_1(r_1)}{B_1(r^0_1)},$$

$$F^{\eta_c}_{1,1} = -F^{\eta_c}_{-1,-1} = \frac{g'_{11}}{\sqrt{2}} r_2 \frac{B_1(r_2)}{B_1(r^0_2)},$$

$$F^{\eta_c}_{0,0} = 0,$$

$$A^{0^-}_{\rm NR}(\lambda_0,\lambda_\gamma,\lambda_1,\lambda_2) = F^{\psi}_{\lambda_\gamma,0} D^{1*}_{\lambda_0,-\lambda_\gamma}(\theta_0,\phi_0) F^{0^-}_{\lambda_1,\lambda_2} \times D^{0*}_{0,\lambda_1-\lambda_2}(\theta_1,\phi_1) \quad \text{for } 0^-,$$

$$A^{0^+}_{\rm NR}(\lambda_0,\lambda_\gamma,\lambda_1,\lambda_2) = F^{\psi}_{\lambda_\gamma,0} D^{1*}_{\lambda_0,-\lambda_\gamma}(\theta_0,\phi_0) F^{0^+}_{\lambda_1,\lambda_2} \times D^{0*}_{0,\lambda_1-\lambda_2}(\theta_1,\phi_1) \quad \text{for } 0^+,$$

$$A^{2^+}_{\rm NR}(\lambda_0,-\lambda_\gamma,\lambda_1,\lambda_2) = \sum_{\lambda_J} F^{\psi}_{\lambda_\gamma,\lambda_J} D^{1*}_{\lambda_0,\lambda_J-\lambda_\gamma}(\theta_0,\phi_0) \times F^{2^+}_{\lambda_1,\lambda_2} D^{2*}_{\lambda_J,\lambda_1-\lambda_2}(\theta_1,\phi_1) \quad \text{for } 2^+.$$

The total amplitude is expressed by

$$A(\lambda_0,\lambda_\gamma,\lambda_1,\lambda_2) = A_{\eta_c}(\lambda_0,\lambda_\gamma,\lambda_1,\lambda_2) + \sum_{J^P} A^{J^P}_{\rm NR}(\lambda_0,\lambda_\gamma,\lambda_1,\lambda_2)$$

For the 0^+ case, helicity amplitudes are taken as

$$F^{\psi}_{1} = F^{\psi}_{-1} = \frac{g_{21} r^2_1}{\sqrt{6}} \frac{B_2(r_1)}{B_2(r^0_1)} + \frac{g_{01}}{\sqrt{3}},$$

$$F^{0^+}_{11} = F^{0^+}_{11} = \frac{g'_{22} r^2_2}{\sqrt{6}} \frac{B_2(r_2)}{B_2(r^0_2)} + \frac{g'_{00}}{\sqrt{3}},$$

$$F^{0^+}_{00} = \sqrt{\frac{2}{3}} r^2_2 g'_{22} \frac{B_2(r_2)}{B_2(r^0_2)} - \frac{g'_{00}}{\sqrt{3}}.$$

For the 2^+ case, helicity amplitudes are taken as

$$F^{\psi}_{12} = F^{\psi}_{-1-2} = \frac{g_{43} r^4_1}{\sqrt{70}} \frac{B_4(r_1)}{B_4(r^0_1)} + \frac{g_{21} r^2_1}{\sqrt{10}} \frac{B_2(r_1)}{B_2(r^0_1)} - \frac{g_{22} r^2_1}{\sqrt{6}} \frac{B_2(r_1)}{B_2(r^0_1)} + \sqrt{\frac{2}{105}} g_{23} r^2_1 \frac{B_2(r_1)}{B_2(r^0_1)} + \frac{g_{01}}{\sqrt{5}},$$

$$F^{\psi}_{11} = F^{\psi}_{-1-1} = \frac{-2 g_{43} r^4_1}{\sqrt{35}} \frac{B_4(r_1)}{B_4(r^0_1)} - \frac{g_{21} r^2_1}{\sqrt{5}} \frac{B_2(r_1)}{B_2(r^0_1)} + \sqrt{\frac{3}{35}} g_{23} r^2_1 \frac{B_2(r_1)}{B_2(r^0_1)} + \frac{g_{01}}{\sqrt{10}},$$

$$F^{\psi}_{10} = F^{\psi}_{-10} = \sqrt{\frac{3}{35}} g_{43} r^4_1 \frac{B_4(r_1)}{B_4(r^0_1)} + \frac{g_{21} r^2_1}{2\sqrt{15}} \frac{B_2(r_1)}{B_2(r^0_1)} + \frac{1}{2} g_{22} r^2_1 \frac{B_2(r_1)}{B_2(r^0_1)} + \frac{2 g_{23} r^2_1}{\sqrt{35}} \frac{B_2(r_1)}{B_2(r^0_1)} + \frac{g_{01}}{\sqrt{30}},$$

$$F^{2^+}_{11} = F^{2^+}_{-1-1} = \sqrt{\frac{3}{35}} g'_{42} \frac{B_4(r)}{B_4(r')} r^4 + \frac{g'_{20} r^2_2}{\sqrt{3}} \frac{B_2(r_2)}{B_2(r^0_2)} - \frac{g'_{22} r^2_2}{\sqrt{21}} \frac{B_2(r_2)}{B_2(r^0_2)} + \frac{g'_{02}}{\sqrt{30}},$$

$$F^{2^+}_{10} = F^{2^+}_{-10} = -\frac{2}{\sqrt{35}} g'_{42} r^4 \frac{B_4(r)}{B_4(r')} - \frac{1}{2} g'_{21} r^2_2 \frac{B_2(r_2)}{B_2(r^0_2)} - \frac{g'_{22} r^2_2}{2\sqrt{7}} \frac{B_2(r_2)}{B_2(r^0_2)} + \frac{g'_{02}}{\sqrt{10}},$$

$$F^{2^+}_{1-1} = F^{2^+}_{-11} = \frac{g_{42} r^4}{\sqrt{70}} \frac{B_4(r)}{B_4(r')} + \sqrt{\frac{2}{7}} g'_{22} r^2_2 \frac{B_2(r_2)}{B_2(r^0_2)} + \frac{g'_{02}}{\sqrt{5}}.$$