# THREE-BODY SCATTERING IN ISOBAR ANSATZ

*Maxim Mai*
The George Washington University

**Exploration of low and intermediate energy regime → rich spectrum of resonances**

- Many are missing compared to QM prediction
- Properties of established ones are sometimes complex

→ see e.g. review on σ-resonance **Pelaez(2016)**

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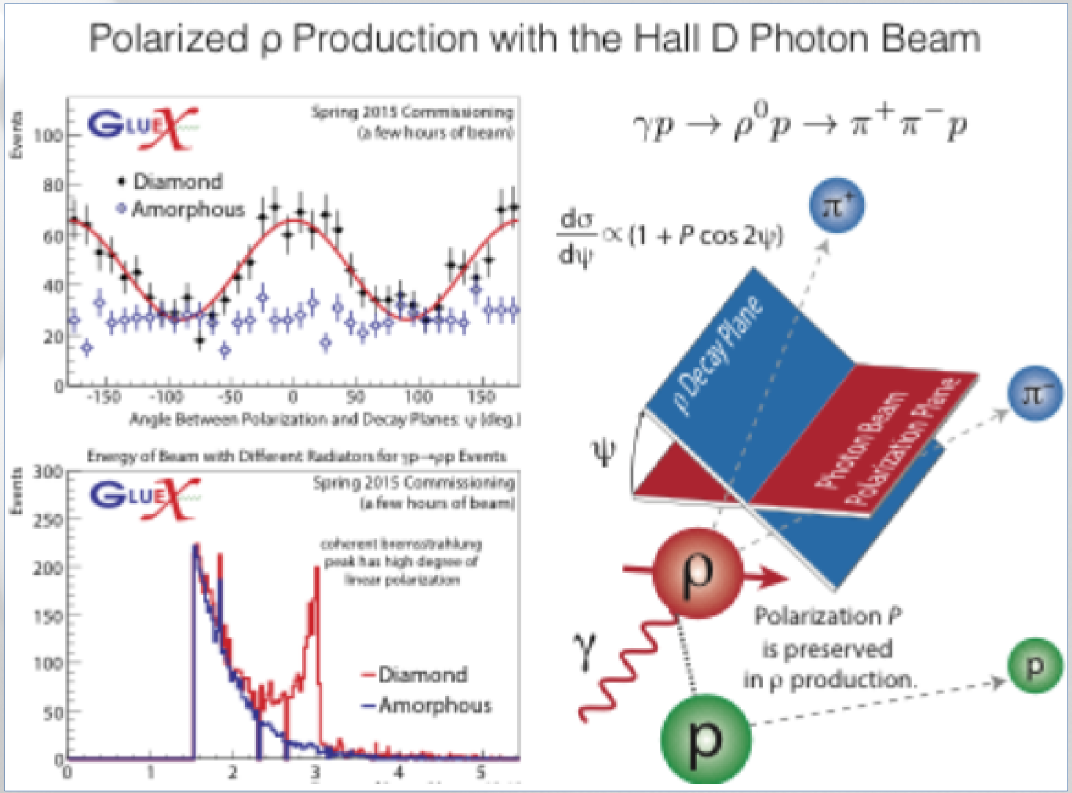

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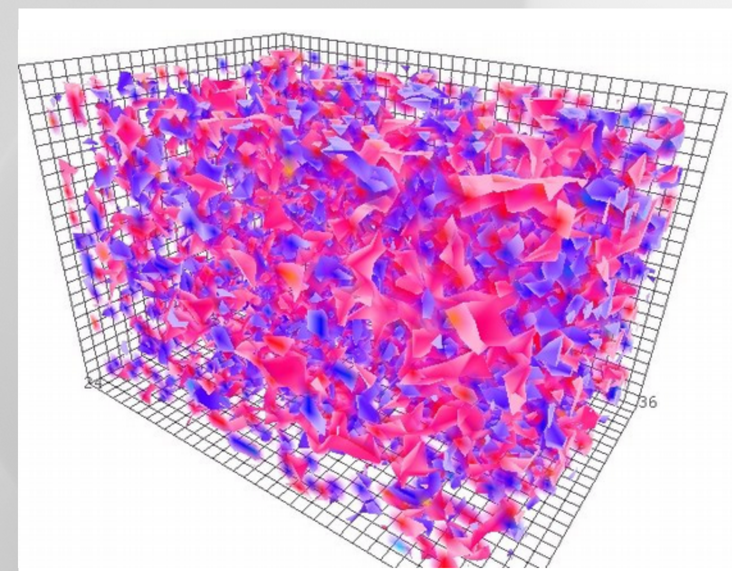
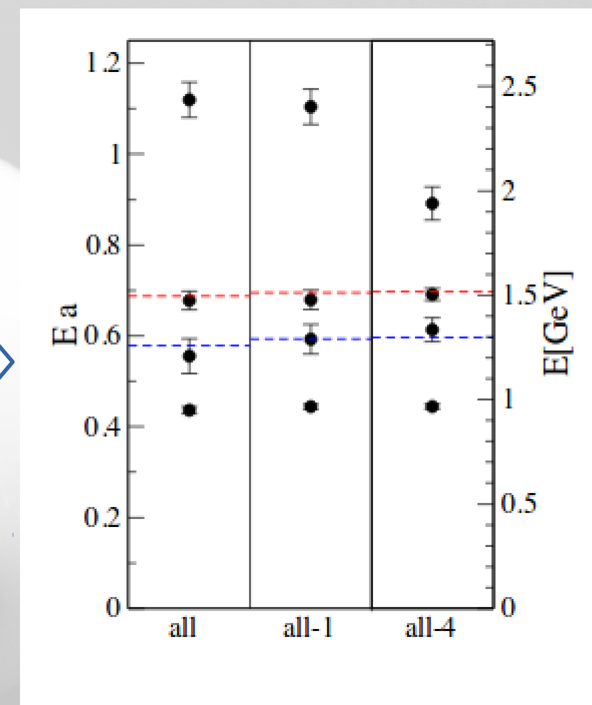
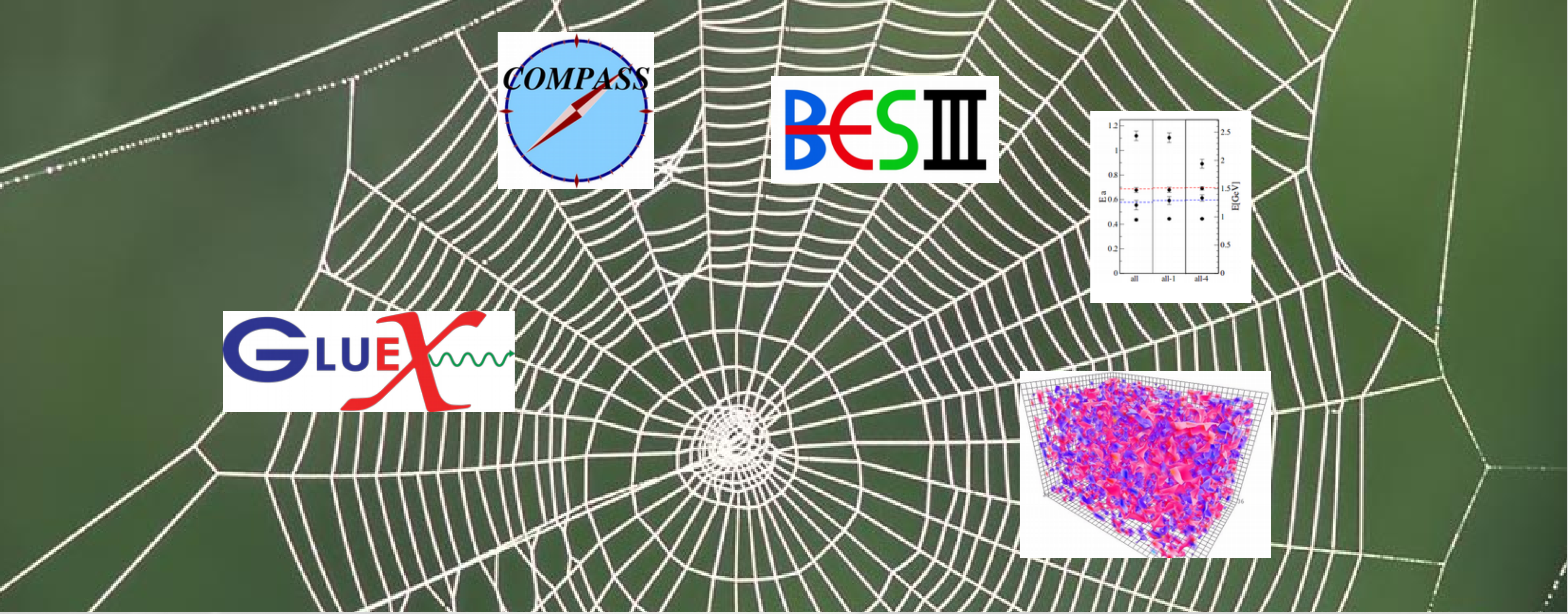


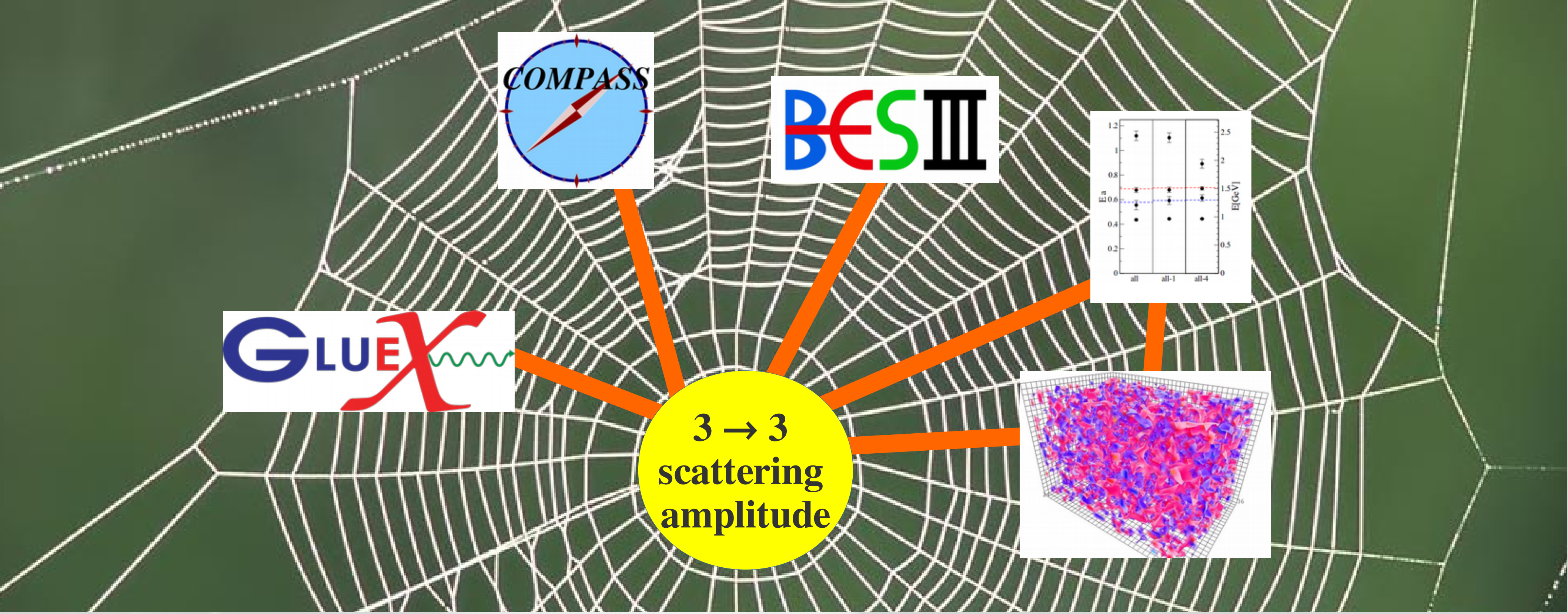
*Monte-Carlo sampling of QCD*

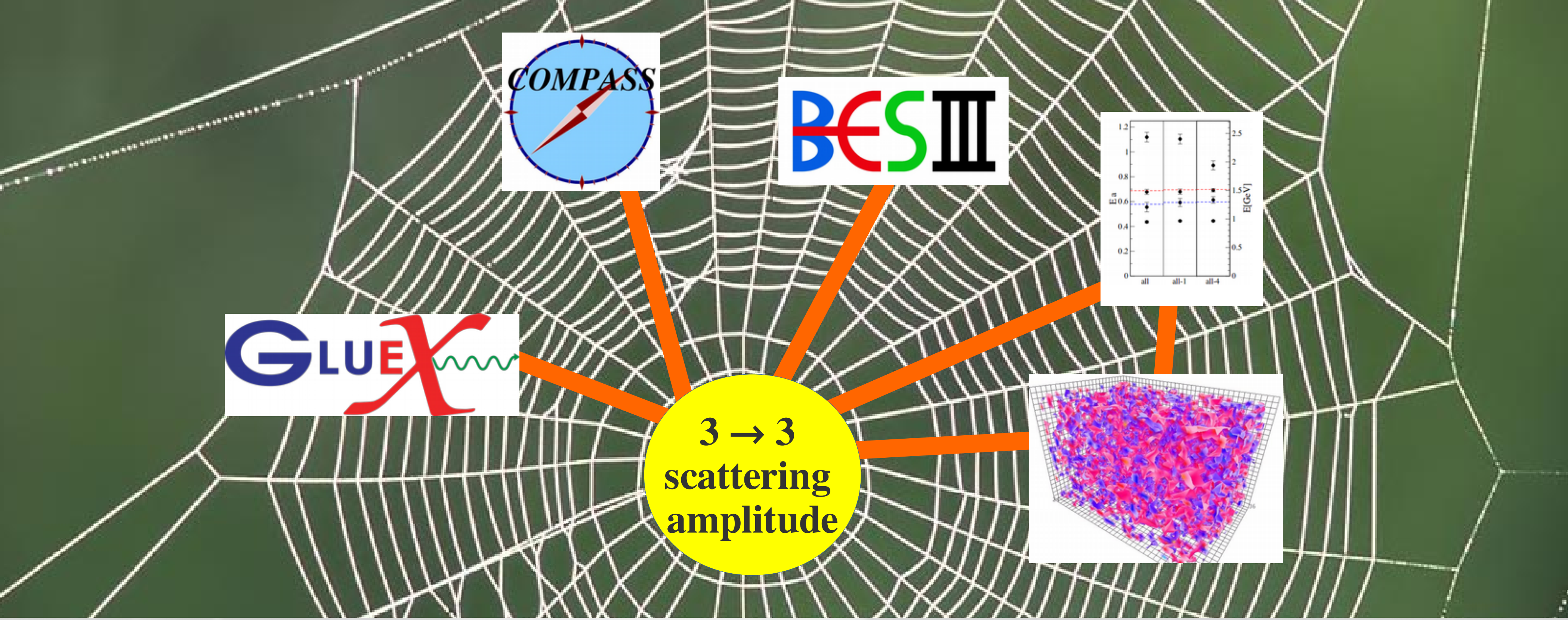
$O_1^{N\pi}$, $O_3^{N_n}$, $O_{6,8}^{N_w}$, $O_9^{N\sigma}$

**Lang et al. (2017)**




COMPASS
BESIII
GLUEX
Ea
E[GeV]
all
all-1
all-4


COMPASS
BESIII
GlueX
3 → 3
scattering
amplitude




**Theoretical tools**

- *Faddeev equations (FE)* — **Faddeev(1959)**
- *FE* in fixed-center approximation — **Chand/Daltz(1962)**

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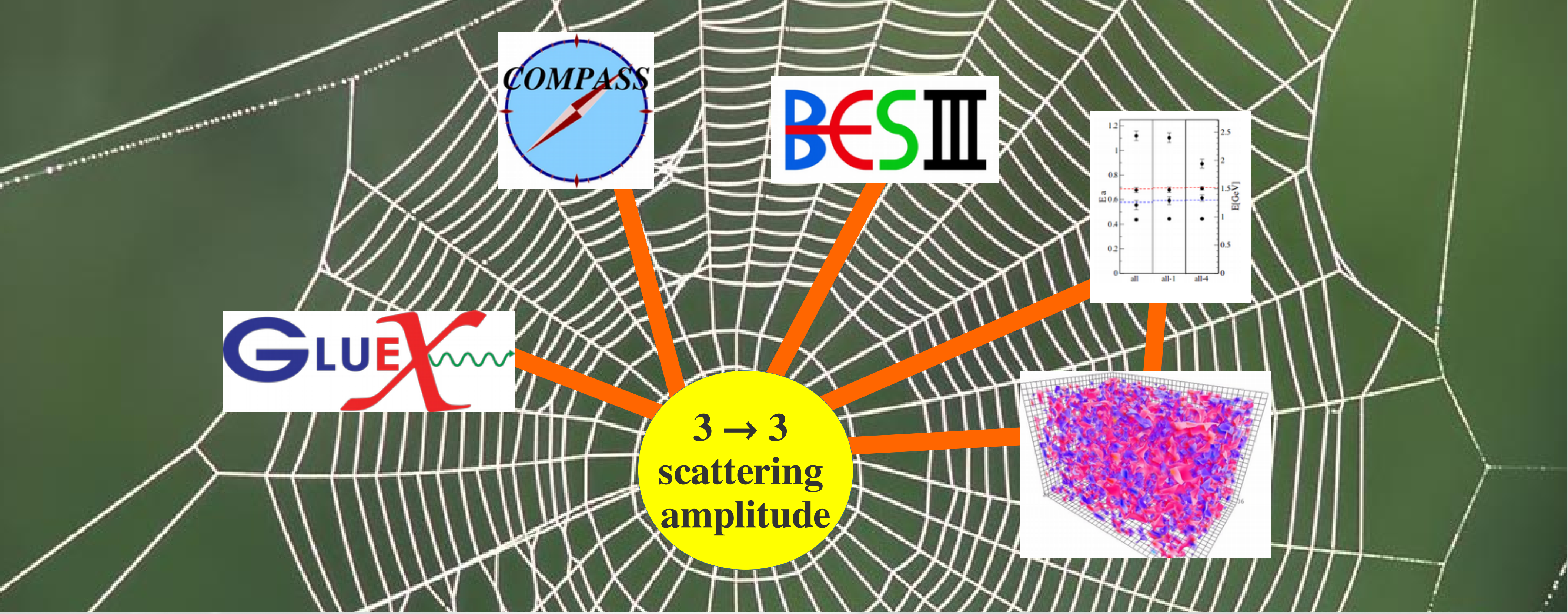


## Theoretical tools

- *Faddeev equations (FE)* **Faddeev(1959)**
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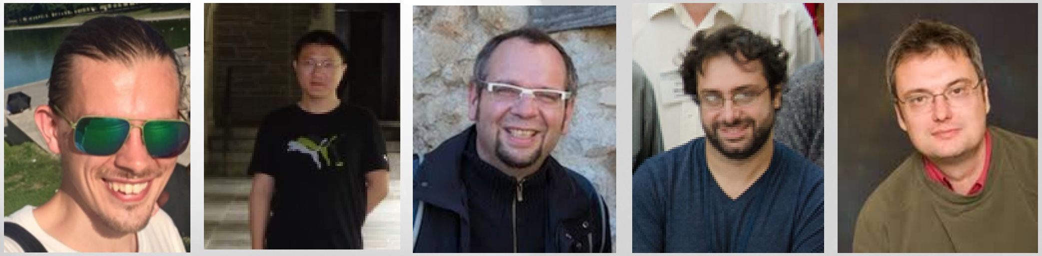
- ***FE in isobar parametrization*** **Omnes(1964) Aaron(1967)**
  → re-parametrization of two-body amplitude **Bedaque(1999)**

# FADDEEV EQUATIONS IN ISOBAR PARAMETRIZATION

**MM, Hu, Döring, Pilloni, Szczepaniak**

**Eur.Phys.J. A53 (2017) no.9, 177**



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**Original study by Amado/Aaron/Young** **AAY(1968)**

- 3-dimensional integral equation from unitarity constraint & BSE ansatz
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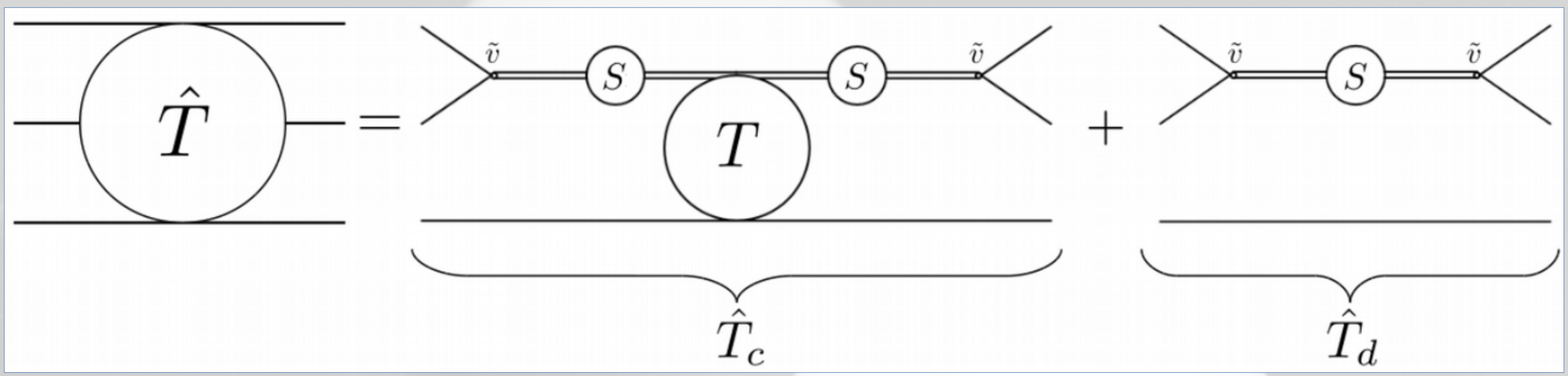


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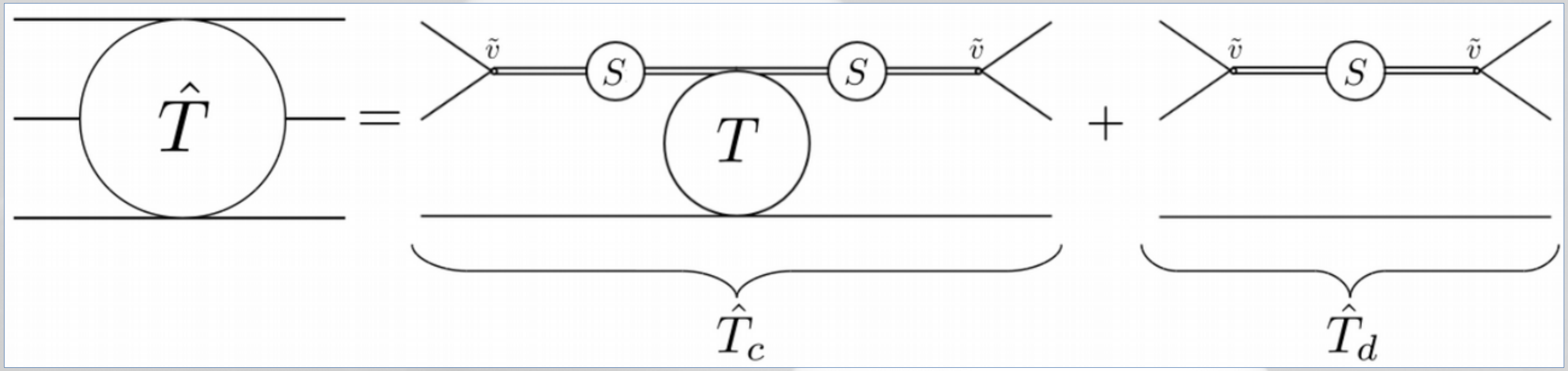


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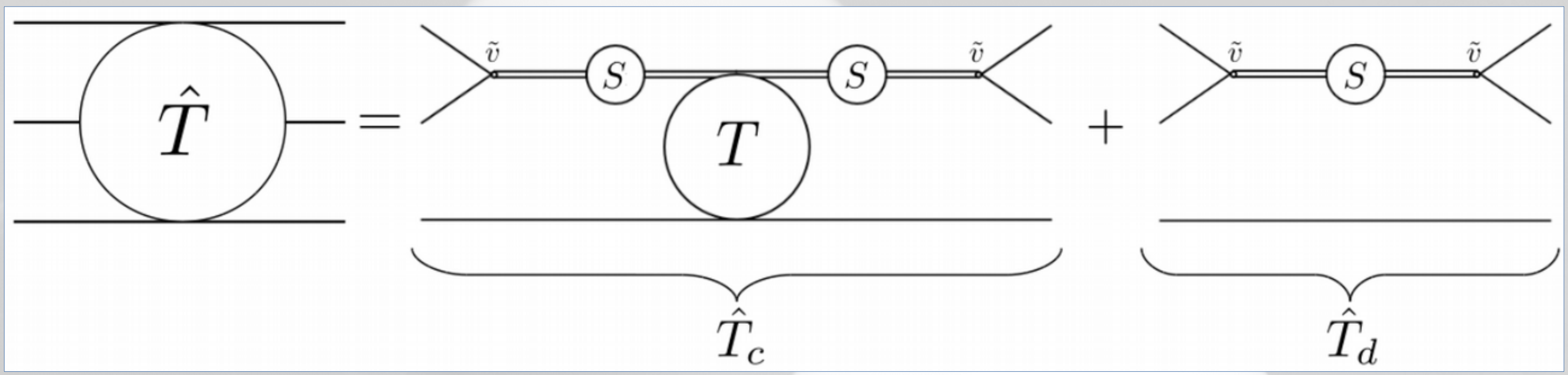


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- ***S*** **and** ***T*** **are yet unknown functions**

# *Unitarity & Matching*

**3-body Unitarity (normalization condition ↔ phase space integral)**

$$\langle q_1,q_2,q_3|(\hat{T}-\hat{T}^\dagger)|p_1,p_2,p_3\rangle \quad = \quad i\int_P \langle q_1,q_2,q_3|\hat{T}^\dagger|k_1,k_2,k_3\rangle\langle k_1,k_2,k_3|\hat{T}|p_1,p_2,p_3\rangle$$

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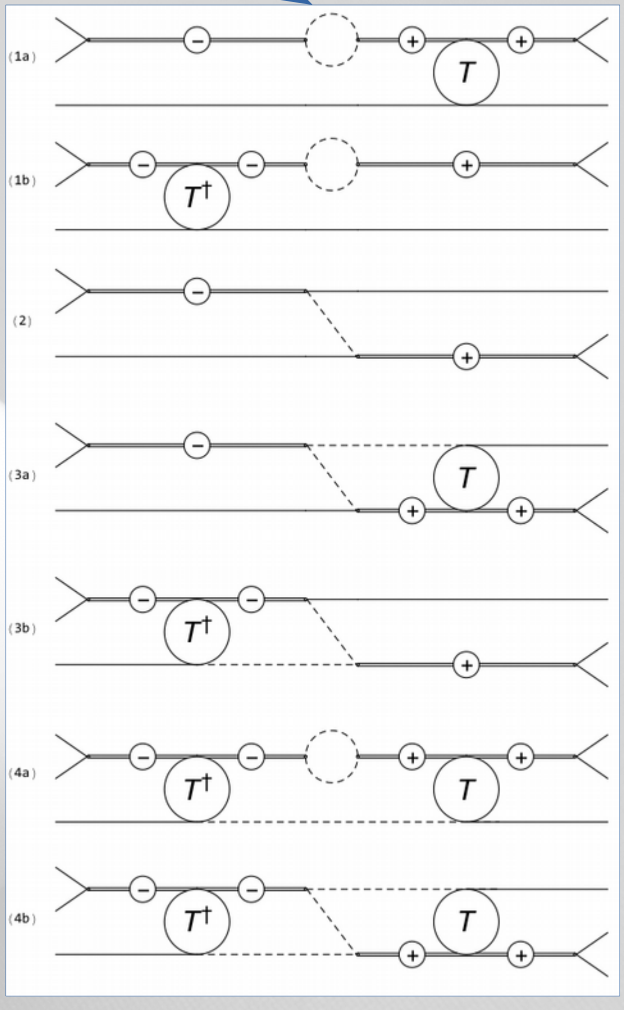

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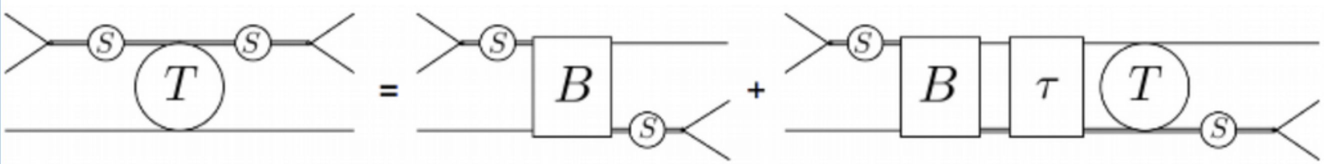
**A general ansatz for the Isobar-spectator interaction**

**→ B & τ are unknown!!!**



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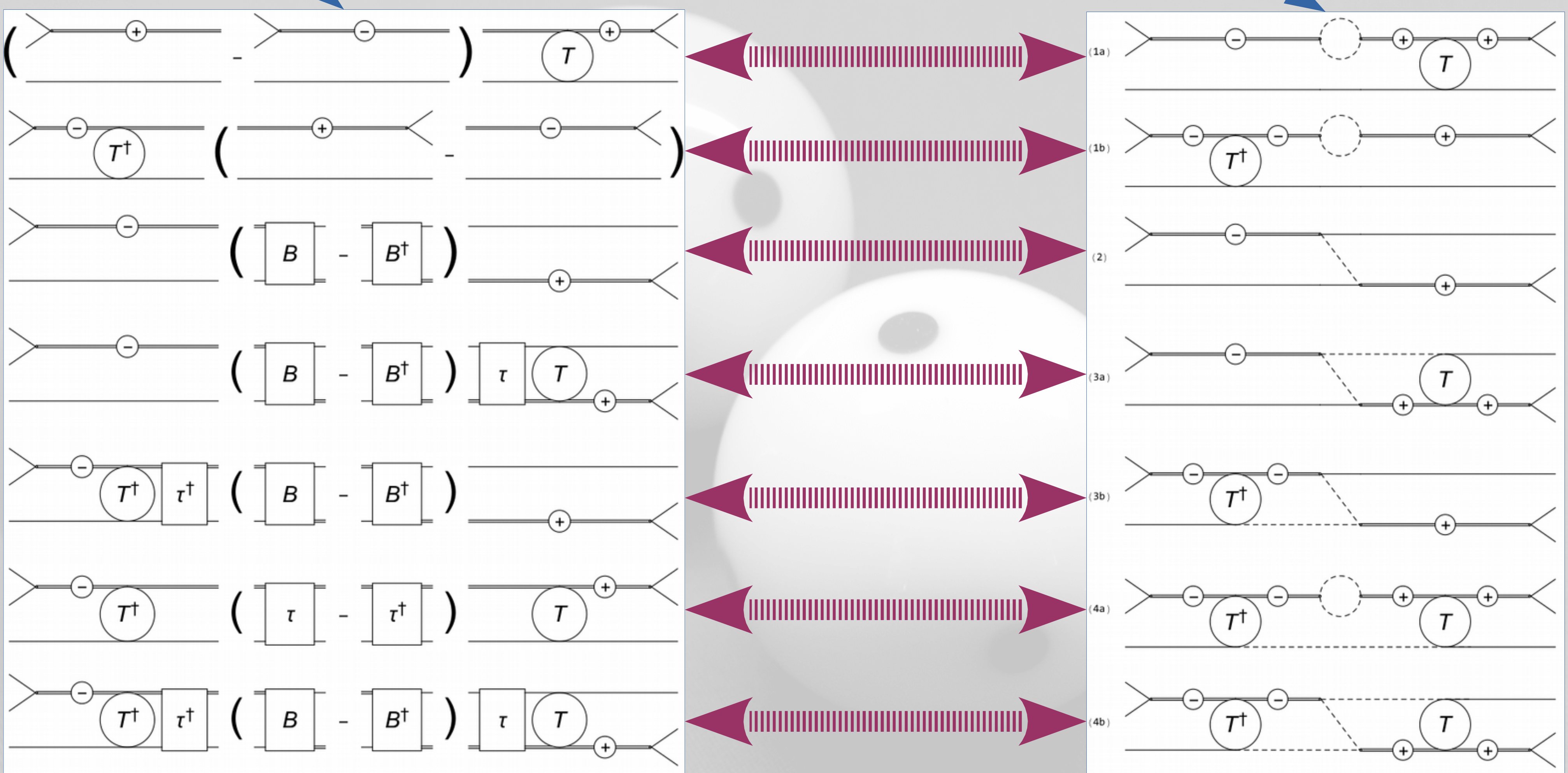

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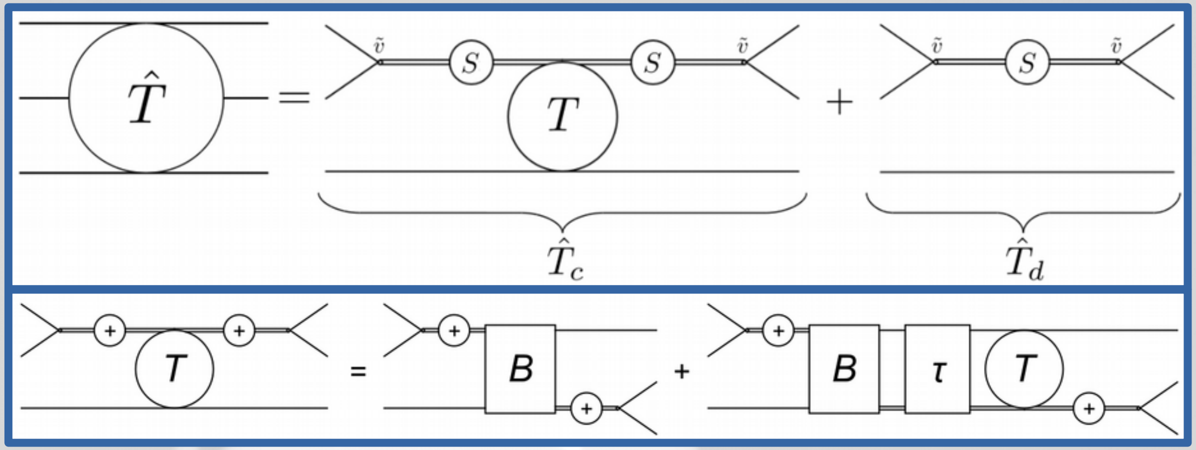

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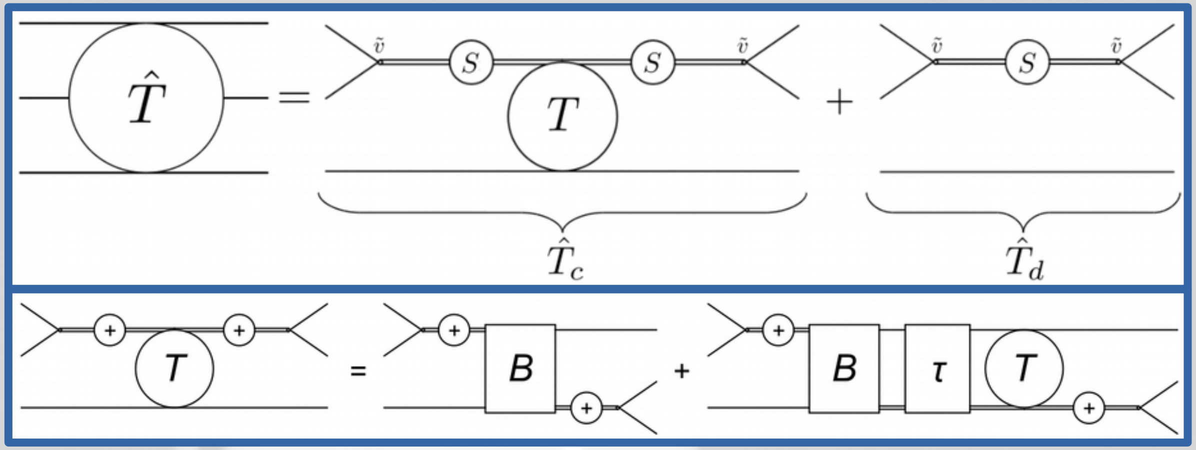


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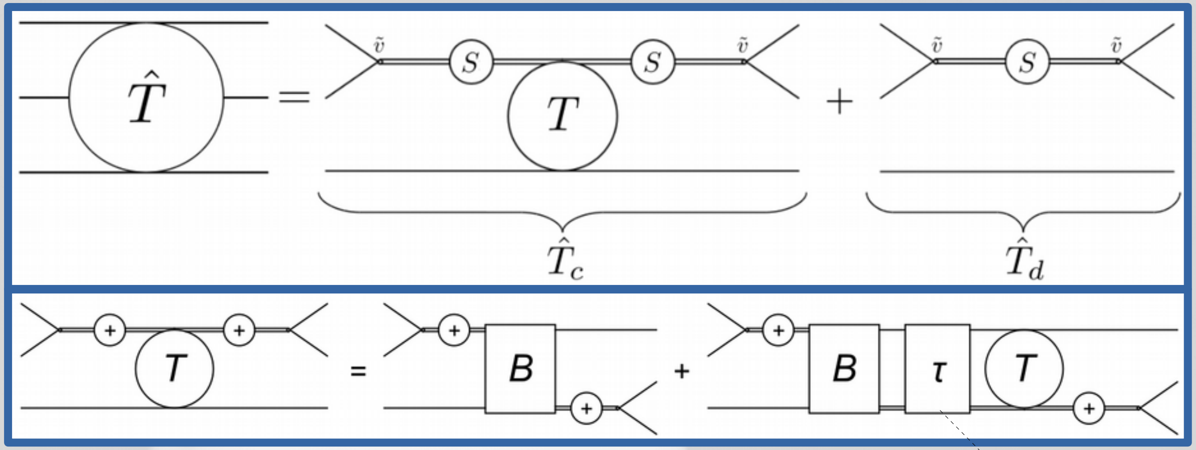


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$$\tau(\sigma(k)) = (2\pi)\delta^+(k^2 - m^2)S(\sigma(k))$$

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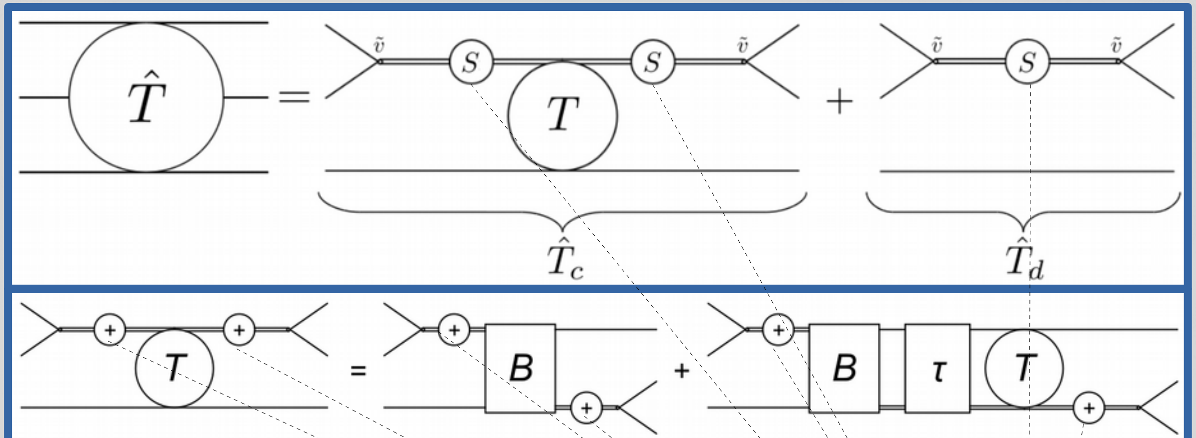


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$$-\frac{1}{S(P^2)} = \sigma(k) - M_0^2 - \frac{1}{(2\pi)^3}\int d^3\ell\,\frac{\lambda^2}{2E_\ell(\sigma(k)-4E_\ell^2+i\epsilon)}$$

- in the rest-frame of isobar (***Lorentz invariance***!)
- twice subtracted dispersion relation in ***σ(k)=(P-k)²***

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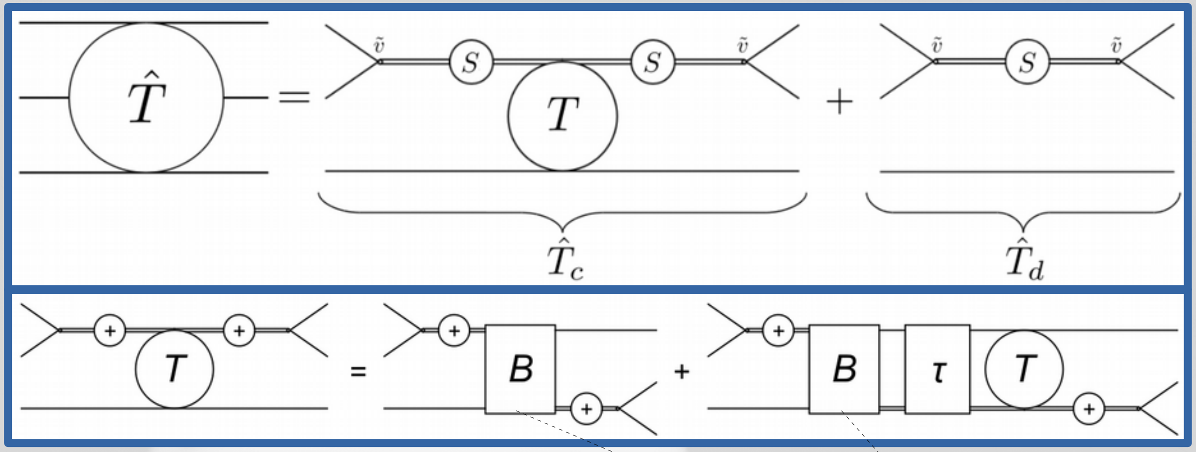


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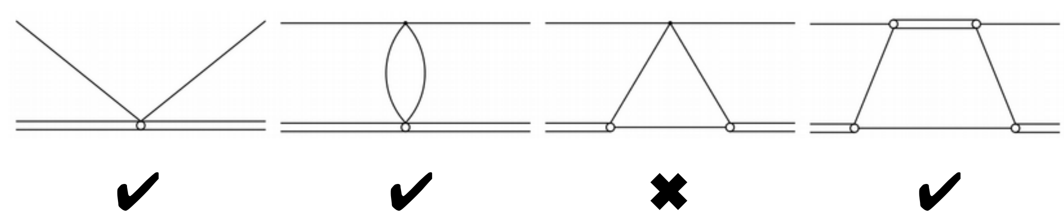
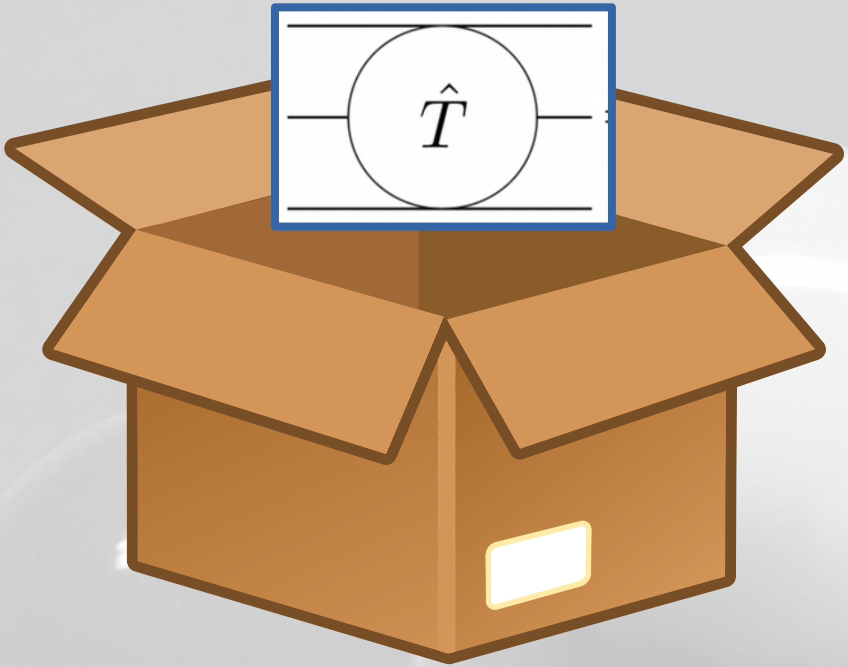
- un-subtracted dispersion relation
- one-π exchange in TOPT
- real contributions can be added to B






# THREE-BODY AMPLITUDE IN A BOX

**MM, Döring**

**Today on arxiv: 1709.08222**

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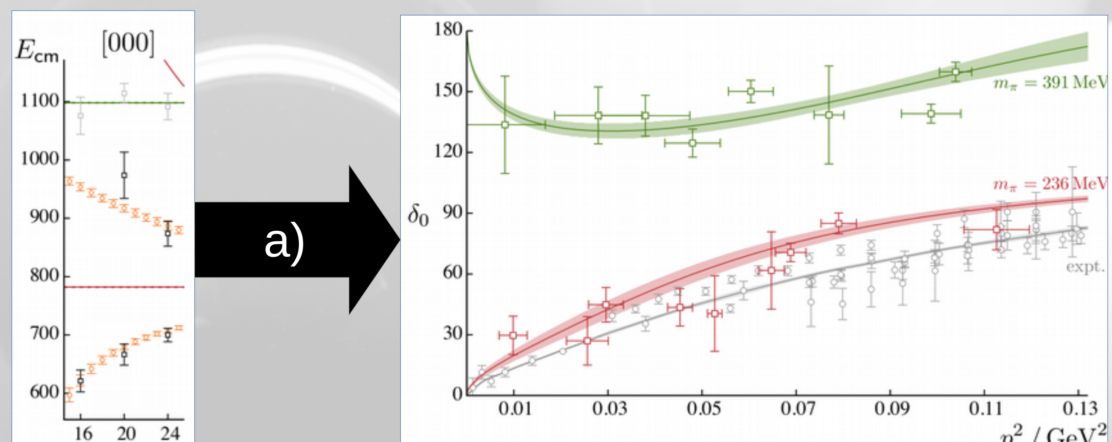


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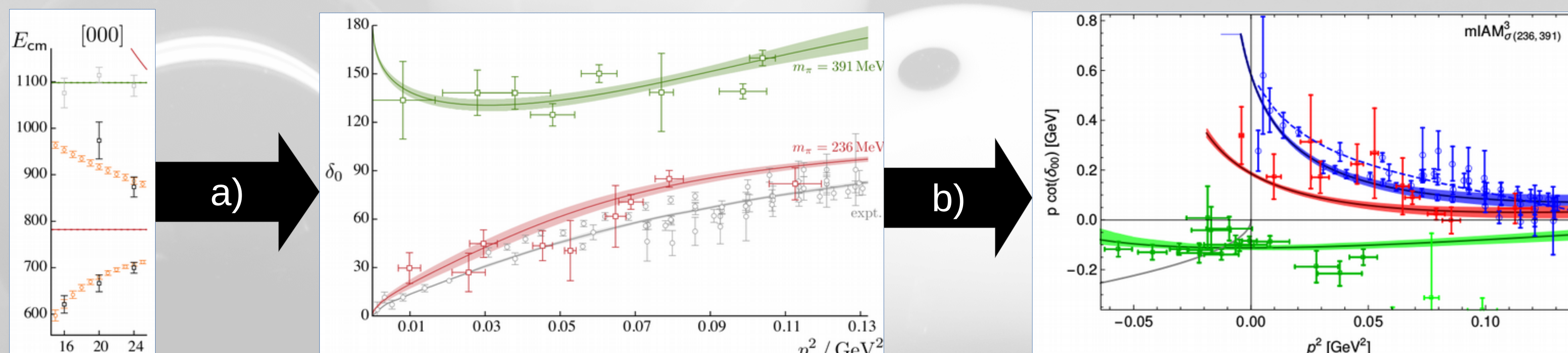


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  **⇒ 3 → 3 scattering amplitude in isobar formulation**

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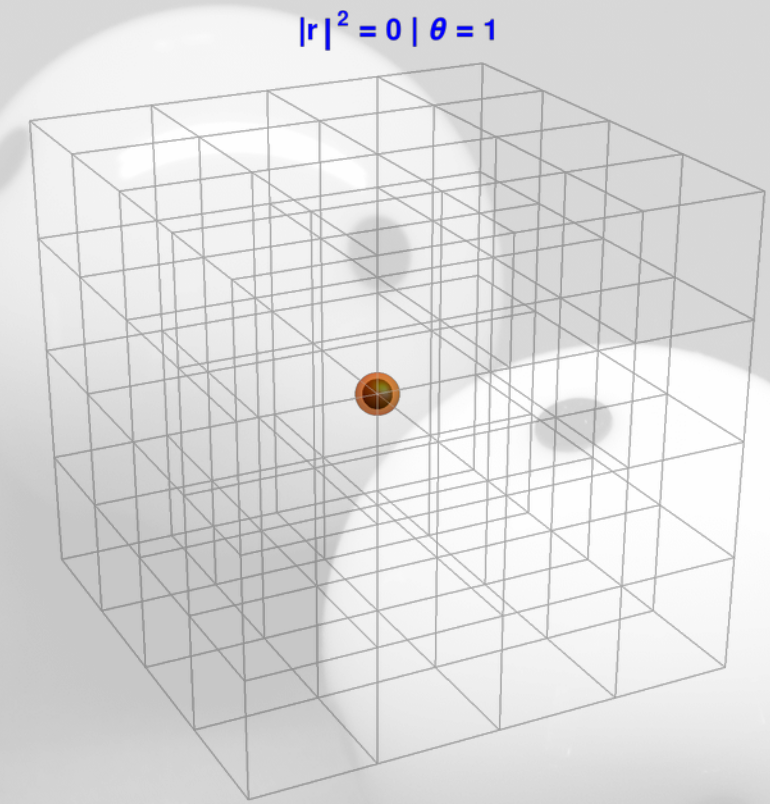

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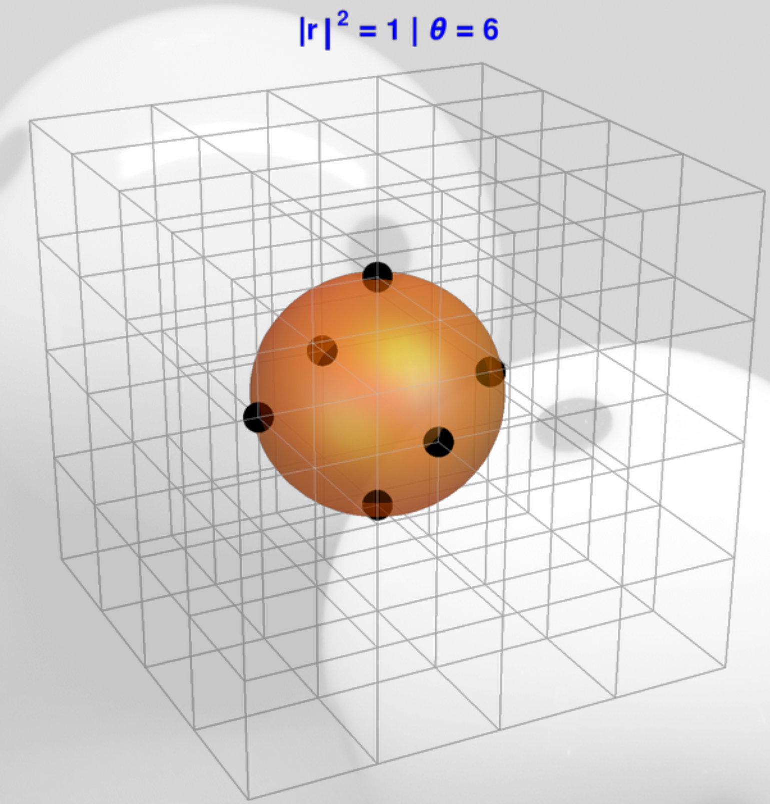

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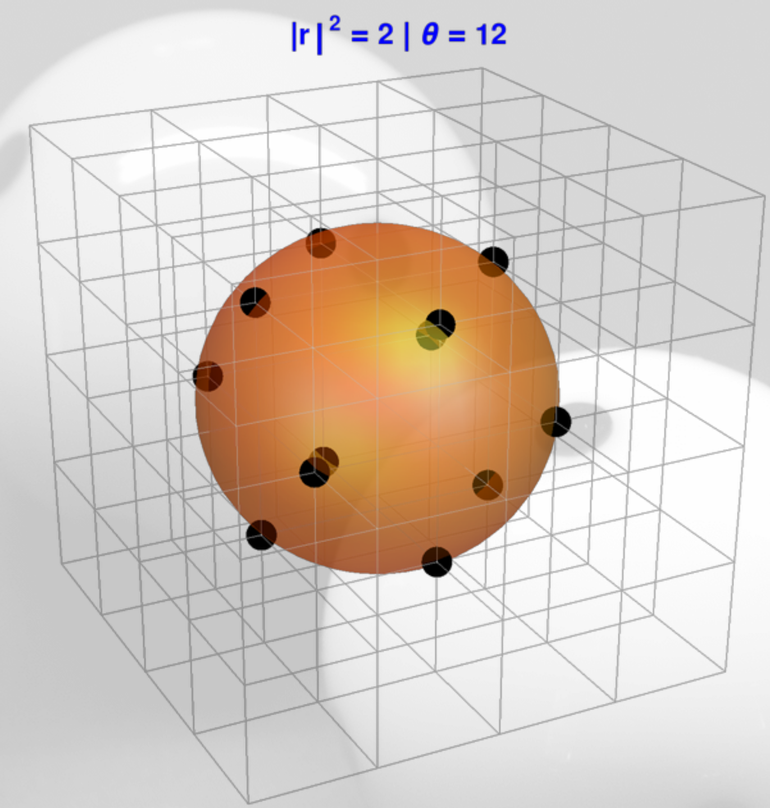

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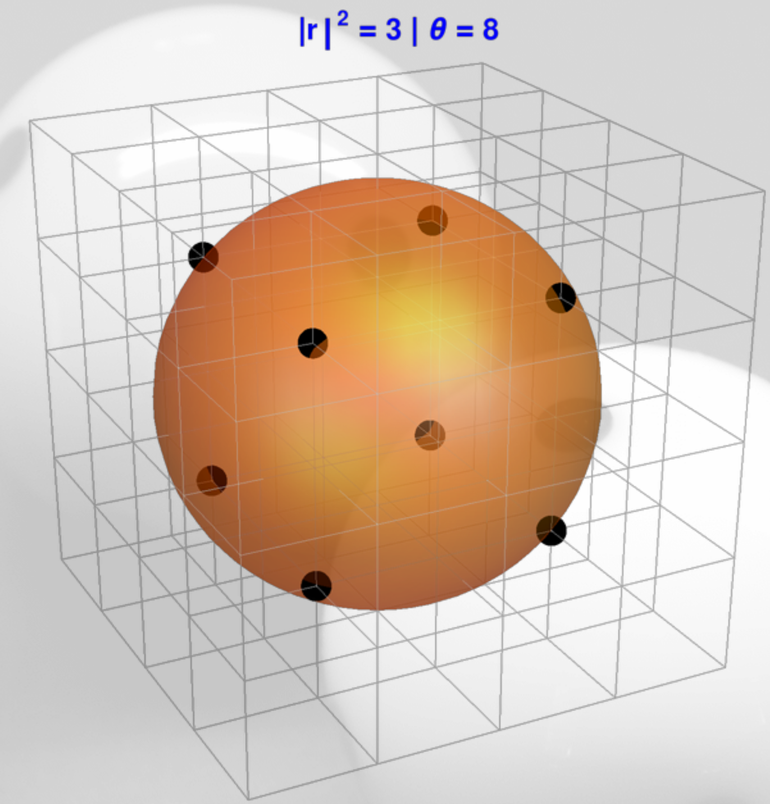

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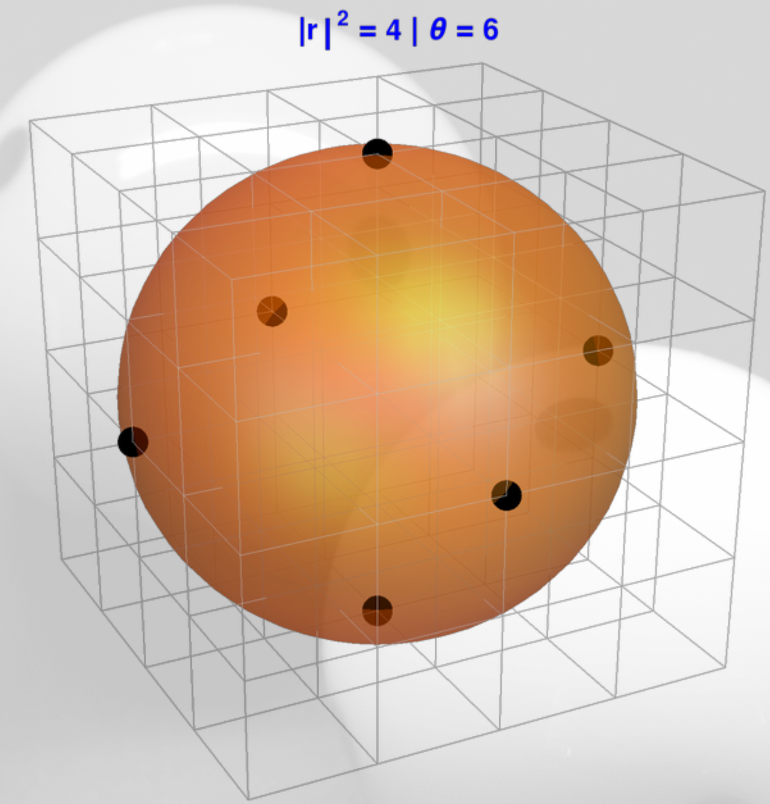

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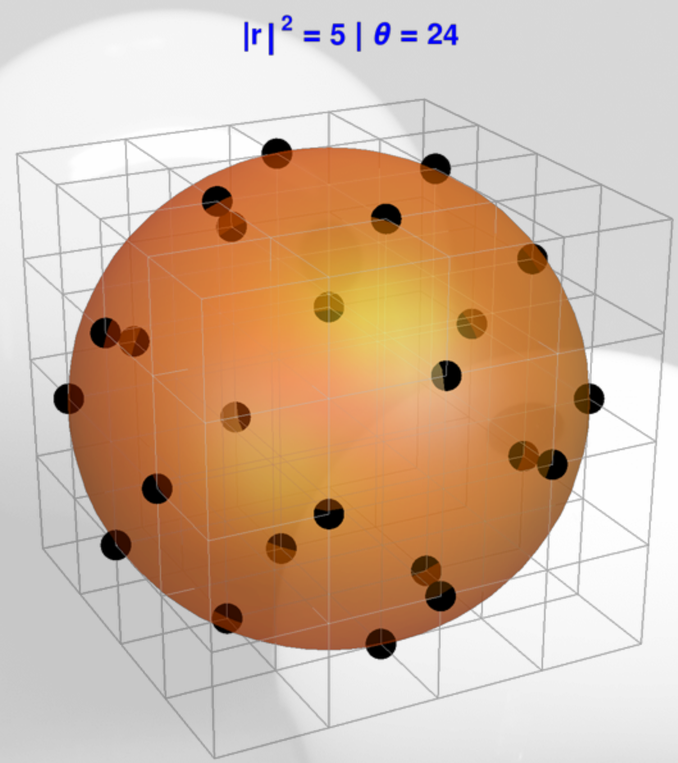

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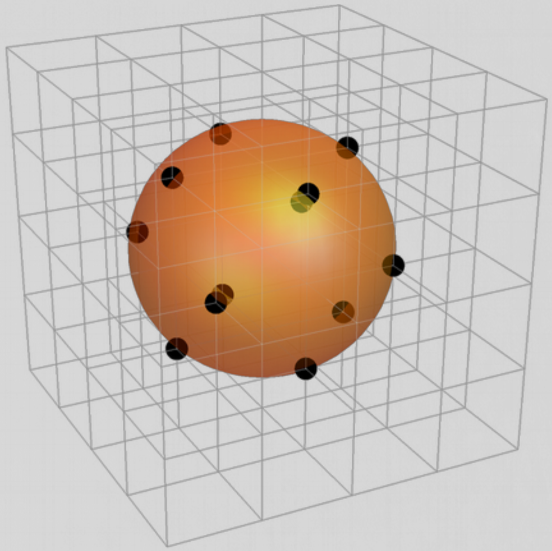

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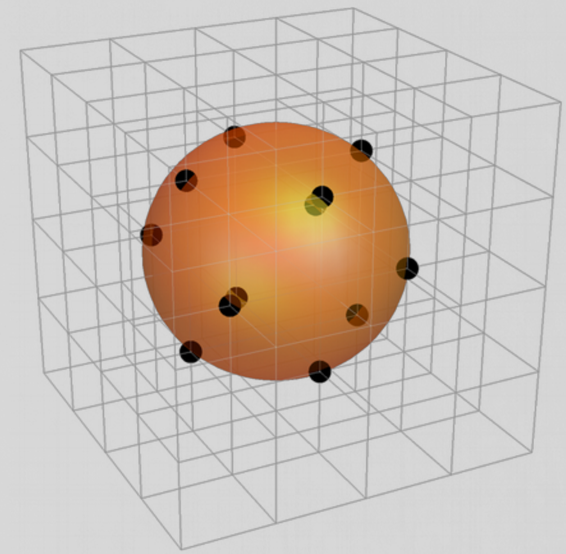

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**3) Spherical & Lorentz symmetry is broken on the lattice**

**→ self-energy part of isobar-propagator has to be boosted to 3-body cms!**

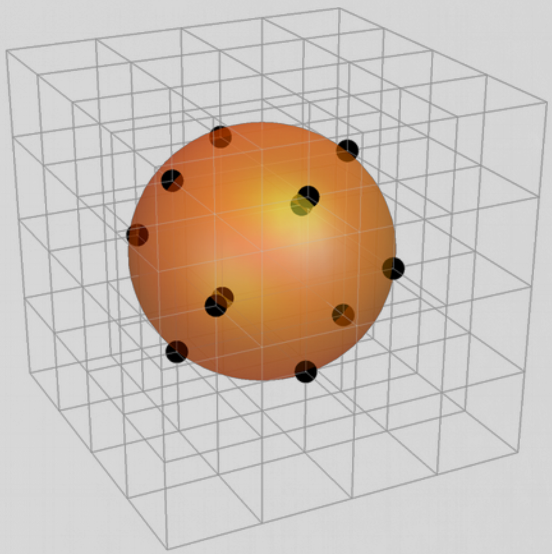
**→ ensures correct cancellations of *non-3-body* singularities**

$$-\frac{1}{S(P^2)} = \sigma(k) - M_0^2 - \frac{1}{(2\pi)^3}\int d^3\boldsymbol{\ell}\,\frac{\lambda^2}{2E_\ell(\sigma(k)-4E_\ell^2+i\epsilon)}$$

# DISCRETIZATION

3→3 scattering amplitude in isobar formulation gives access to fin-vol. spectrum

1) Discretized momenta ordered in shells



$$\boldsymbol{q}_{ni} = \frac{2\pi}{L}\,\boldsymbol{r}_i \text{ for } \{\boldsymbol{r}_i \in Z^3 | \boldsymbol{r}_i^2 = n, i = 1, \ldots, \vartheta(n)\}$$

2) "PW-expansion" in finite volume

→ $A_1^+$ representation (S-/G-/…-waves)

→ one basis vector $\chi^{A_1^+} = Y_{00}$

3) Spherical & Lorentz symmetry is broken on the lattice

→ self-energy part of isobar-propagator has to be boosted to 3-body cms!

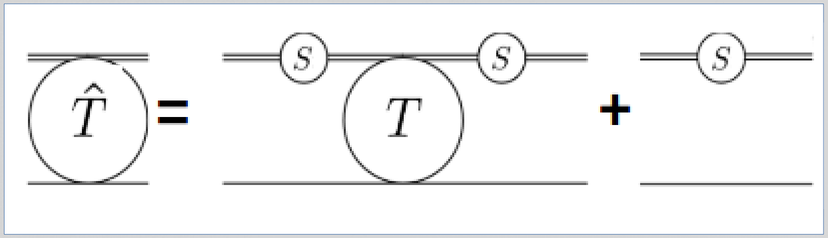
→ ensures correct cancellations of *non-3-body* singularities

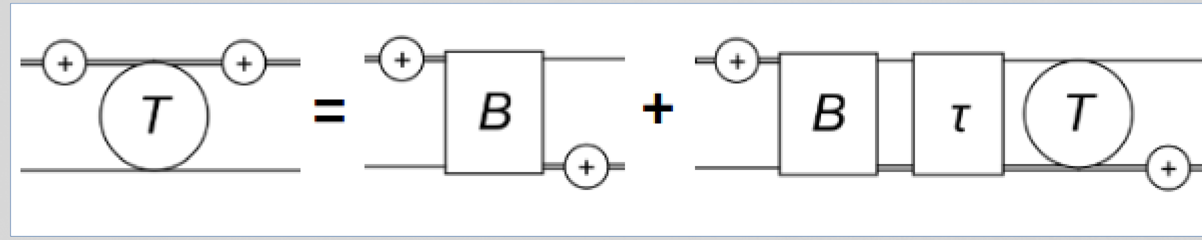
**4) Replace integrals by sums**

$$\int \frac{d^3\mathbf{q}}{(2\pi)^3} \to \frac{1}{L^3} \sum_{n \in set_8} \sum_{i=1}^{\vartheta(n)}$$

# RESULTS

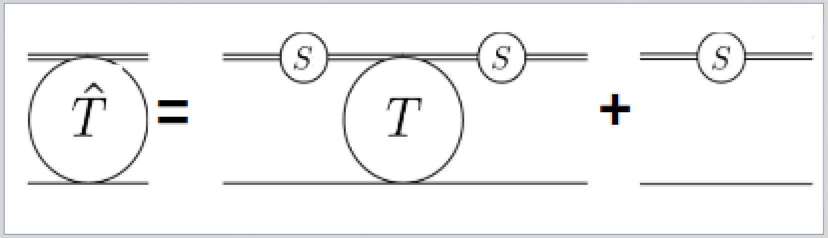
- *$v$* **is cut-free → irrelevant for finite volume spectrum**

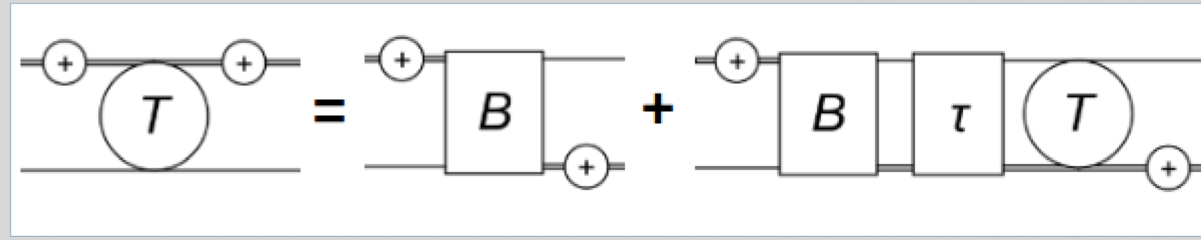



# RESULTS

- *$\upsilon(.,.)$* is cut-free → irrelevant for finite volume spectrum



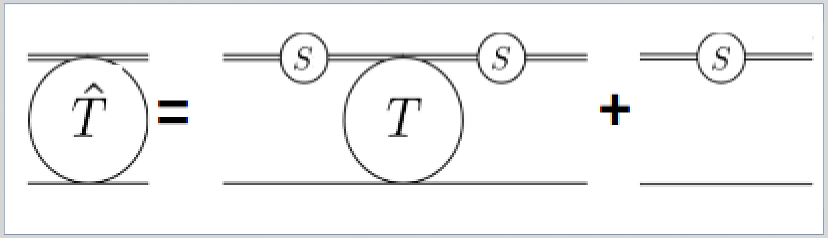

- **Projection to $A_1^+$ representation (S-/G-/…-waves)**
  **→ Scattering amplitude is a matrix in {shell} x {shell} space**

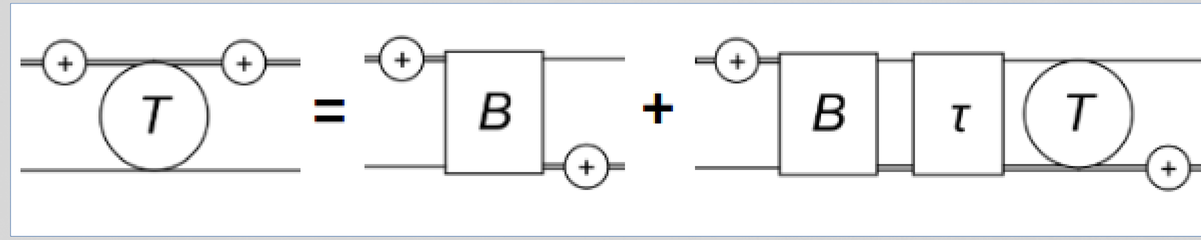
$$\hat{T}^{A_1^+}(s) = \left[X(s)B^{A_1^+}(s)X(s) + X(s)\tau(s)^{-1}\right]^{-1}$$

$$\text{for} \quad X(s) := \mathrm{Diag}_{n\in set_8}\left(\frac{\vartheta(n)}{2E_n(s)L^3}\right)$$

# RESULTS

- *$v(.,.)$* is cut-free → irrelevant for finite volume spectrum




- Projection to $A_1^+$ representation (S-/G-/…-waves)
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$$\text{for} \quad X(s) := \mathrm{Diag}_{n\in sets}\left(\frac{\vartheta(n)}{2E_n(s)L^3}\right)$$

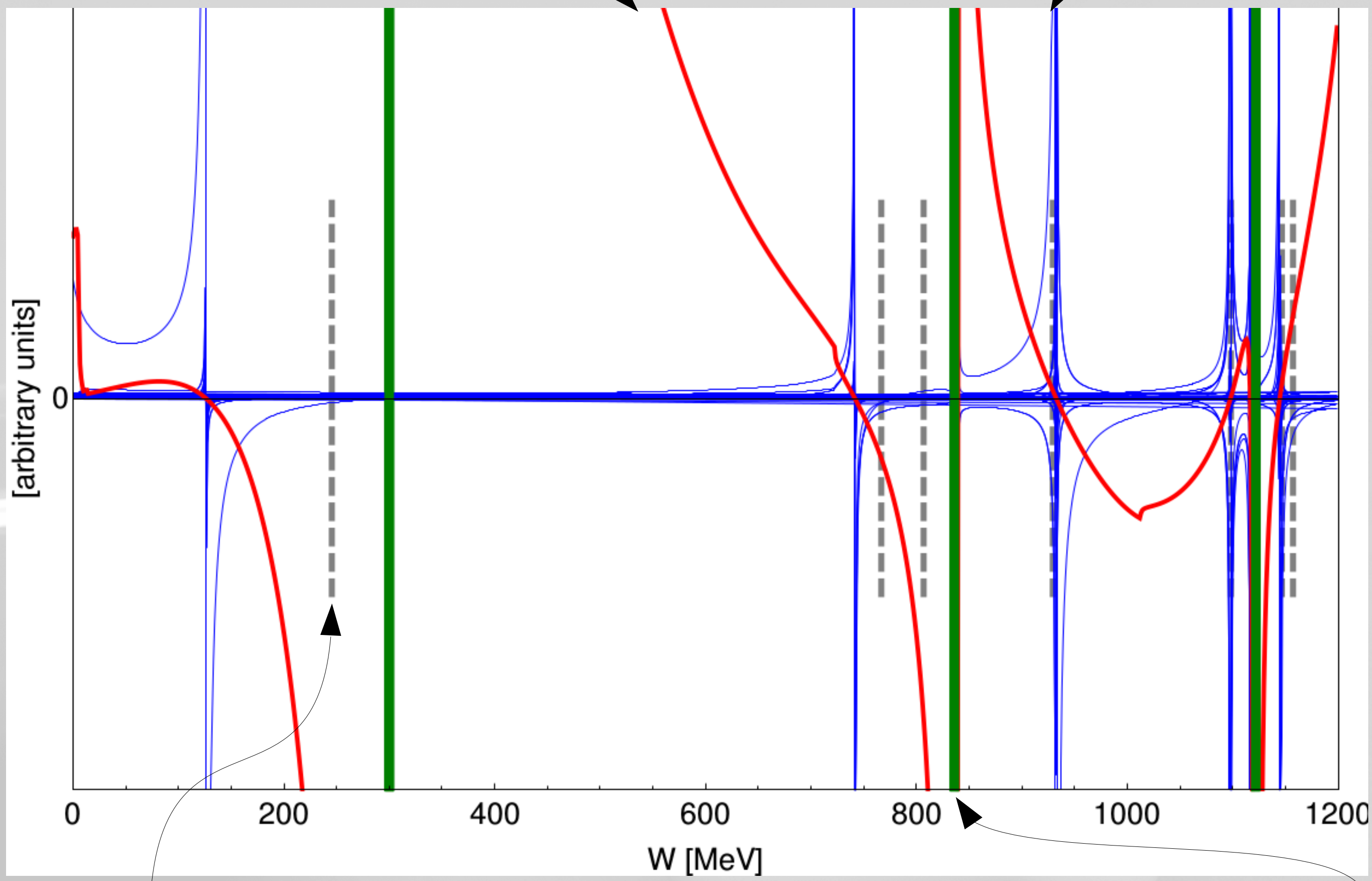
- **Poles = positions of energy-eigenvalues in finite volume. IFF**

$$\mathrm{Det}[B^{A_1^+}(s)X(s) + \tau(s)^{-1}] = 0\,.$$

# RESULTS *(L=3 fm, M=100 MeV)*

$\mathrm{Det}[B^{A_1^+}(s)X(s) + \tau(s)^{-1}]$

$\hat{T}^{A_1^+}(s)$




Isobar propagator poles

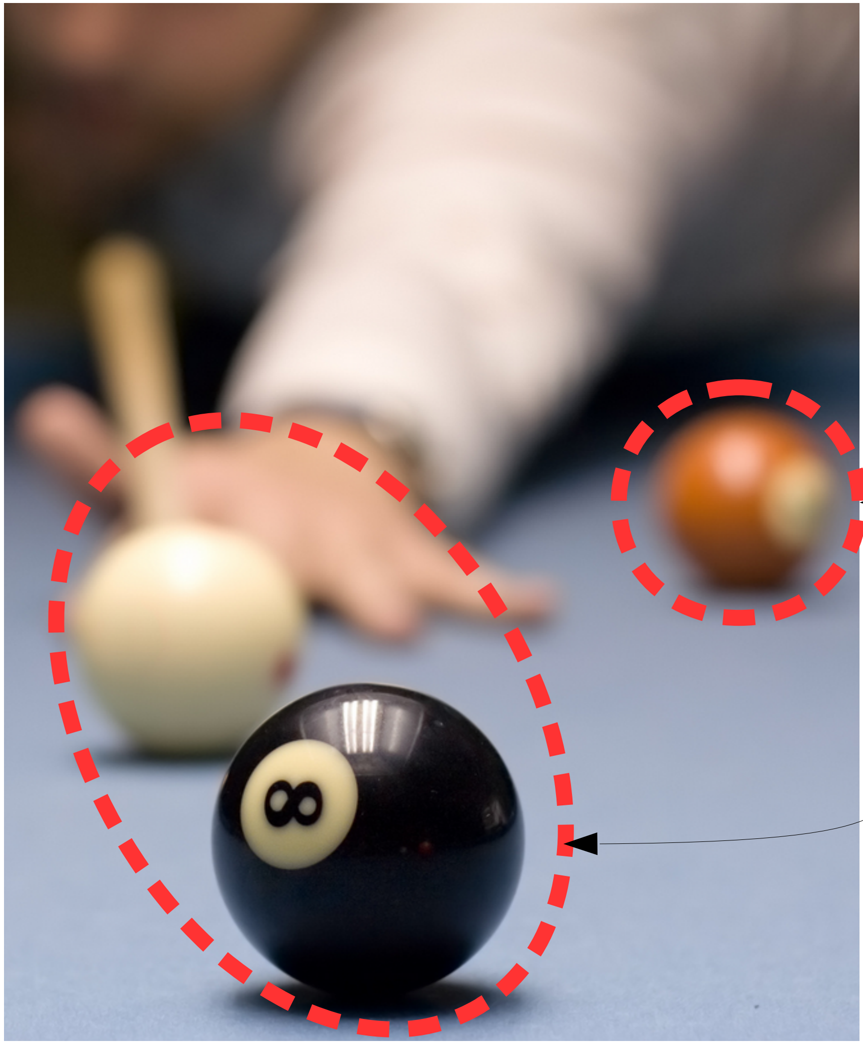
Free energy eigenvalues


# SUMMARY

- $3 \to 3$ scattering amplitude formulated in the isobar Ansatz
- 3-body Unitarity dictates im-parts of the driving term & isobar propagator
- Result: 3-dimensional relativistic integral equation.

- Finite volume investigation reveals
  - Multiple cancellation mechanisms
  - Quantization condition

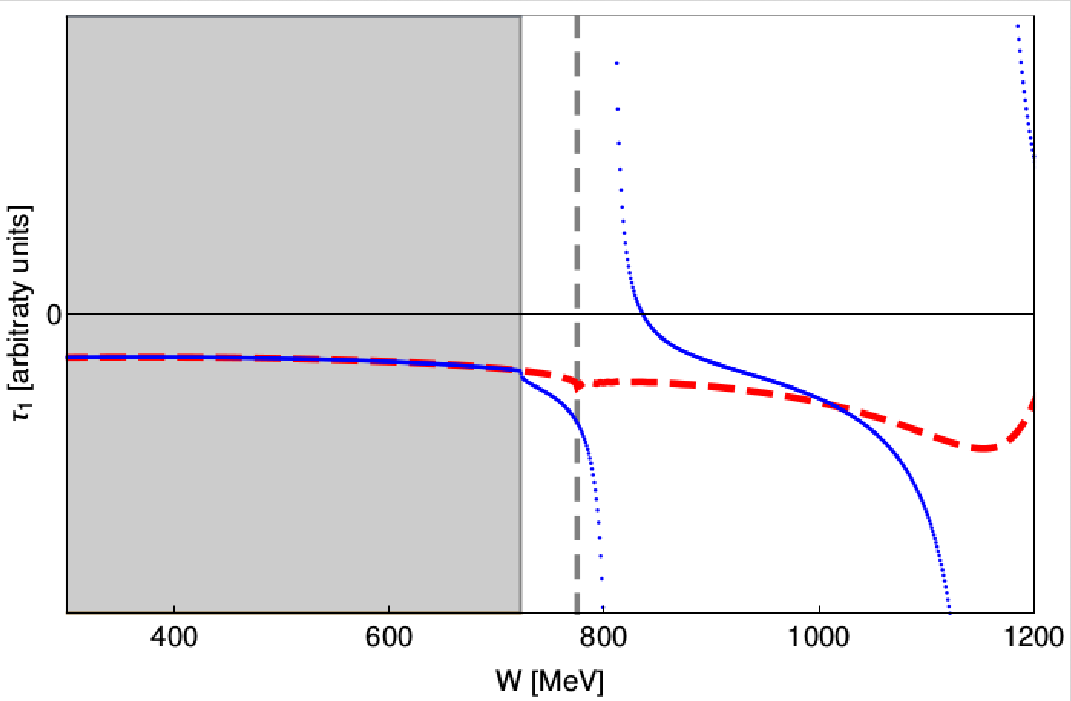
# Thank you

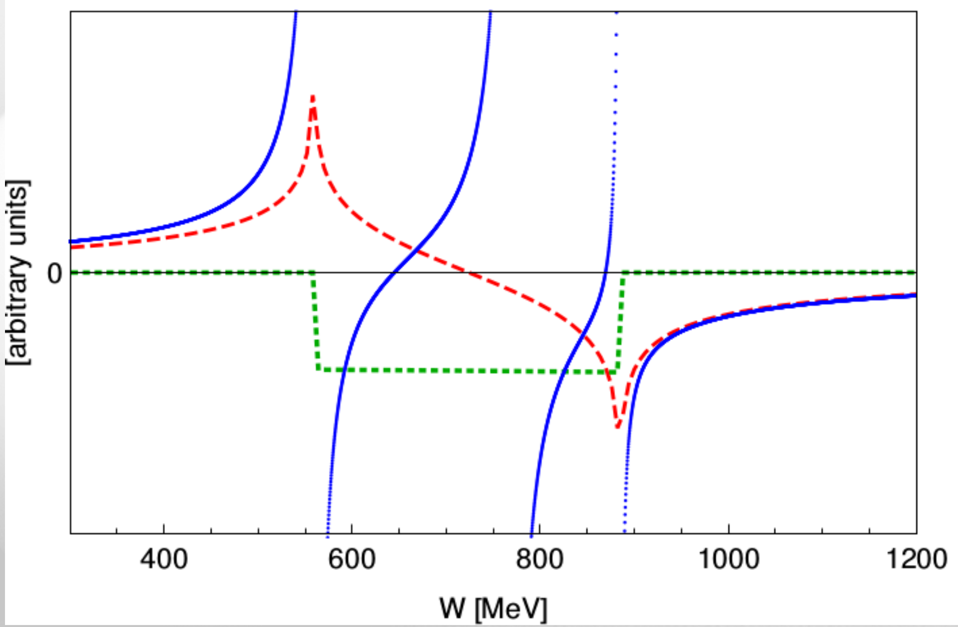


Spectator

Isobar

# SPARES


$\tau_1$ [arbitraty units]
0
400
600
800
1000
1200
W [MeV]


[arbitrary units]
0
400
600
800
1000
1200
W [MeV]

# *Unitarity & Matching*

- 3-body Unitarity (normalization condition ↔ phase space integral)

$$\langle q_1, q_2, q_3|(\hat{T}-\hat{T}^\dagger)|p_1, p_2, p_3\rangle \quad = \quad i\int_P \langle q_1, q_2, q_3|\hat{T}^\dagger|k_1, k_2, k_3\rangle\langle k_1, k_2, k_3|\hat{T}|p_1, p_2, p_3\rangle$$