# Covariant and helicity formalisms

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# Recipes to build an amplitude

The literature abounds with discussions on the optimal approach to construct the amplitudes for the hadronic reactions

- Helicity formalism

  **Jacob, Wick, Annals Phys. 7, 404 (1959)**

- Covariant tensor formalisms

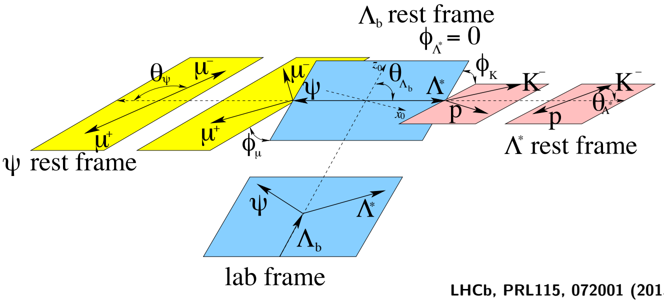
  **Chung, PRD48, 1225 (1993)**
  **Chung, Friedrich, PRD78, 074027 (2008)**
  **Filippini, Fontana, Rotondi, PRD51, 2247 (1995)**
  **Anisovich, Sarantsev, EPJA30, 427 (2006)**

The common lore is that the former one is nonrelativistic, especially when expressed in terms of LS couplings
and the latter takes into account the proper relativistic corrections

# How helicity formalism works

- Helicity formalism enforces the constraints about rotational invariance
- It allows us to fix the angular dependence of the amplitude
- What about energy dependence?

Example: $\Lambda_b \to \psi \Lambda^* \to pK$




**LHCb, PRL115, 072001 (2015)**

Each set of angles is defined in a different reference frame

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Example: $\Lambda_b \to \psi\Lambda^* \to pK$

$$\mathcal{M}^{\Lambda^*}_{\lambda_{\Lambda_b^0},\,\lambda_p,\,\Delta\lambda_\mu} \equiv \sum_n \sum_{\lambda_{\Lambda^*}} \sum_{\lambda_\psi} \mathcal{H}^{\Lambda_b^0\to\Lambda_n^*\psi}_{\lambda_{\Lambda^*},\,\lambda_\psi} D^{\frac{1}{2}}_{\lambda_{\Lambda_b^0},\,\lambda_{\Lambda^*}-\lambda_\psi}(0,\theta_{\Lambda_b^0},0)^* \\ \mathcal{H}^{\Lambda_n^*\to Kp}_{\lambda_p,\,0} D^{J_{\Lambda_n^*}}_{\lambda_{\Lambda^*},\,\lambda_p}(\phi_K,\theta_{\Lambda^*},0)^* \\ R_{\Lambda_n^*}(m_{Kp}) D^{1}_{\lambda_\psi,\,\Delta\lambda_\mu}(\phi_\mu,\theta_\psi,0)^*,$$

**LHCb, PRL115, 072001 (2015)**

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# How tensor formalisms work

The method is based on the construction of explicitly covariant expressions.

- To describe the decay $a \to bc$, we first consider the polarization tensor of each particle, $\varepsilon^{i}_{\mu_1 \dots \mu_{j_i}}(p_i)$
- We combine the polarizations of $b$ and $c$ into a "total spin" tensor $S_{\mu_1 \dots \mu_S}(\varepsilon_b, \varepsilon_c)$
- Using the decay momentum, we build a tensor $L_{\mu_1 \dots \mu_L}(p_{bc})$ to represent the orbital angular momentum of the $bc$ system, orthogonal to the total momentum of $p_a$
- We contract $S$ and $L$ with the polarization of $a$

Tensor $\times R_X(m)$ which contain resonances and form factors

# What do we know?

- Energy dependence is not constrained by symmetry
- Still, there are some known properties one can enforce

$$R_X(m) = B'_{L^X_{\Lambda_b^0}}(p, p_0, d) \left(\frac{p}{M_{\Lambda_b^0}}\right)^{L^X_{\Lambda_b^0}}$$
$$\mathrm{BW}(m|M_{0X}, \Gamma_{0X})\, B'_{L_X}(q, q_0, d) \left(\frac{q}{M_{0X}}\right)^{L_X}$$

- Kinematical singularities: e.g. barrier factors (known)
- Left hand singularities (need model, e.g. Blatt-Weisskopf)
- Right hand singularities = resonant content (Breit Wigner, K-matrix...)

# Kinematical singularities

- Kinematical singularities appear because of the spin of the external particle involved
- We can write the most general covariant parametrization of the amplitude as
  tensor of external polarizations $\otimes$ scalar amplitudes
- Scalar amplitudes must be kinematical singularities free
- They can be matched to the helicity amplitudes
- We can get the minimal energy dependent factor
- Any other additional energy factor would be model-dependent

# Crossing symmetry in tensor formalisms

- The process $B \to \bar{D}\pi\pi$ is composed of scalar particles only

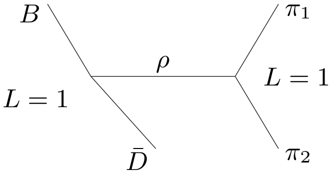
  **LHCb, PRD92, 032002 (2015)**

- One defines the helicity angle in the isobar rest frame, then the amplitude is Lorentz Invariant
- Let's consider the $\rho$ intermediate state, $B \to \bar{D}\rho(\to \pi\pi)$

$$A = \frac{m_B^2 + s - m_D^2}{2m_B^2} \cos\theta \times qp = \frac{E_\rho^{(B)}}{m_B} \cos\theta \times qp$$

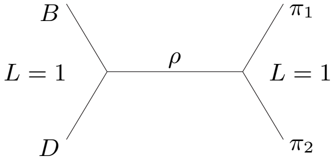
- The factors $p$ and $q$ are the $L = 1$ expected barrier factors. The additional factor is analytical in $s$, not a kinematical singularity. Why is it there?

# Crossing symmetry in tensor formalisms




The tensor amplitude is given by $p_D^{(B)} \cdot p_\pi^{(\rho)}$, where $p_D^{(B)}$ is the breakup momentum in the B frame, and $p_\pi^{(\rho)}$ the decay momentum in the isobar frame

$$A = \frac{m_{B^0}^2 + s - m_{h_3}^2}{2m_{B^0}^2} pq \cos\theta$$




However, one can consider the scattering process just in the isobar rest frame.

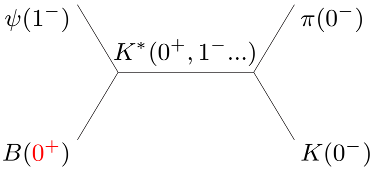
$$A = pq \cos\theta$$

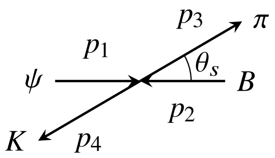
By crossing symmetry the amplitudes must be the same.

The usual implementation fails crossing symmetry

# Example: $B \to \psi\pi K$

To consider the effect of spin, let's consider $B \to \psi\pi K$
We focus on the parity violating amplitude for the $K^*$ isobars, scattering kinematics




$$p = \frac{\lambda_{12}^{1/2}}{2\sqrt{s}}, \quad q = \frac{\lambda_{34}^{1/2}}{2\sqrt{s}}, \quad z_s = \frac{s(t-u) + (m_1^2 - m_2^2)(m_3^2 - m_4^2)}{\lambda_{12}^{1/2}\lambda_{34}^{1/2}}$$

## Helicity amplitude

$$A_\lambda = \frac{1}{4\pi} \sum_{j=|\lambda|}^{\infty} (2j+1) A^j_\lambda(s)\, d^j_{\lambda 0}(z_s)$$

$$d^j_{\lambda 0}(z_s) = \hat{d}^j_{\lambda 0}(z_s) \xi_{\lambda 0}(z_s), \qquad \xi_{\lambda 0}(z_s) = \left(\sqrt{1 - z_s^2}\right)^\lambda$$

$\hat{d}^j_{\lambda 0}(z_s)$ is a polynomial of order $j - |\lambda|$

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$\hat{d}^j_{\lambda 0}(z_s)$ is a polynomial of order $j-|\lambda|$
The kinematical singularities of $A^j_\lambda(s)$ can be isolated by writing

$$A^j_0 = \frac{m_1}{p\sqrt{s}}\,(pq)^j\,\hat{A}^j_0 \qquad \text{for } j \geq 1,$$
$$A^j_\pm = q\,(pq)^{j-1}\,\hat{A}^j_\pm \qquad \text{for } j \geq 1,$$
$$A^0_0 = \frac{p\sqrt{s}}{m_1}\hat{A}^0_0 \qquad \text{for } j = 0,$$

# Identify covariants

Two helicity couplings → two independent covariant structures
Important: we are not imposing any intermediate isobar

$$A_\lambda(s,t) = \varepsilon_\mu(\lambda, p_1)\left[(p_3 - p_4)^\mu - \frac{m_3^2 - m_4^2}{s}(p_3+p_4)^\mu\right] C(s,t) + \varepsilon_\mu(\lambda, p_1)(p_3+p_4)^\mu B(s,t)$$

# Identify covariants

Two helicity couplings $\to$ two independent covariant structures
Important: we are not imposing any intermediate isobar

$$C(s,t) = \frac{1}{4\pi\sqrt{2}} \sum_{j>0} (2j+1)(pq)^{j-1} \hat{A}^j_+(s)\, \hat{d}^j_{10}(z_s)$$

$$B(s,t) = \frac{1}{4\pi} \hat{A}^0_0 + \frac{1}{4\pi} \frac{4m_1^2}{\lambda_{12}} \sum_{j>0} (2j+1)(pq)^j \times \left[ \hat{A}^j_0(s) \hat{d}^j_{00}(z_s) + \frac{s+m_1^2-m_2^2}{\sqrt{2} m_1^2} \hat{A}^j_+(s)\, z_s \hat{d}^j_{10}(z_s) \right].$$

Everything looks fine but the $\lambda_{12}$ in the denominator
The brackets must vanish at $\lambda_{12} = 0 \Rightarrow s = s_\pm = (m_1 \pm m_2)^2$,
$\hat{A}^j_+$ and $\hat{A}^j_0$ cannot be independent

# General form and comparisons

$$\hat{A}^j_+ = \langle j-1, 0; 1, 1 | j, 1 \rangle g_j(s) + f_j(s)$$

$$\hat{A}^j_0 = \langle j-1, 0; 1, 0 | j, 0 \rangle \frac{s + m_1^2 - m_2^2}{2m_1^2} g'_j(s) + f'_j(s)$$

$g_j(s_\pm) = g'_j(s_\pm)$, and $f_j(s), f'_j(s) \sim O(s - s_\pm)$
All these four functions are free of kinematic singularity.

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Comparison with ordinary LS

$$g_j(s) = g'_j(s)\frac{s+m_1^2-m_2^2}{2m_1\sqrt{s}} = \left(\frac{2j-1}{2j+1}\right)^{1/2} \hat{G}^j_{j-1}(s),$$

$$f_j(s) = \frac{m_1}{\sqrt{s}} f'_j(s) = p^2 \left(\frac{2j+3}{2j+1}\right)^{1/2} \hat{G}^j_{j+1}(s).$$

If the $\hat{G}^j_L$ are the usual Breit-Wigner, the $g', f'$ have singularities

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Comparison with tensor formalisms ($j = 1$)

$$A_\lambda = \varepsilon_\mu(\lambda, p_1) \left( -g^{\mu\nu} + \frac{P^\mu P^\nu}{s} \right) X_\nu(q) g_S(s) + \varepsilon^\rho(\lambda, p_1) X_{\rho\mu}(p) \left( -g^{\mu\nu} + \frac{P^\mu P^\nu}{s} \right) X_\nu(q) g_D(s)$$

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$g_j(s_\pm) = g'_j(s_\pm)$, and $f_j(s), f'_j(s) \sim O(s-s_\pm)$
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Comparison with tensor formalisms ($j=1$)

$$g_1 = g'_1 = \frac{4\pi}{3} g_S, \quad f_1 = \frac{2\pi\lambda_{12}}{3s} g_D, \quad f'_1 = -\frac{4\pi\lambda_{12}}{3s}\frac{s+m_1^2-m_2^2}{m_1^2} g_D.$$

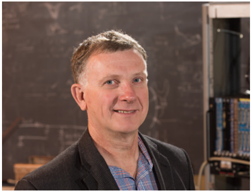
If the $g_S, g_D$ are the usual Breit-Wigner, the $g', f'$ are fine
but for singularities at $s=0$

# Few words on form factors and Blatt-Weisskopf

- Left hand singularities are due to dynamics $\rightarrow$ model-dependent!
- They originate from the cross-channel exchanges in scattering processes
- These singularities are far from the physical region, but modify the high-energy behaviour (e.g. suppress the barrier factors)
- In nonrelativistic potential theory, they can be univocally calculated depending on the potential shape, e.g. Blatt-Weisskopf for square well+centrifugal potential
- Best practice: explore different form factor models, check the stability of the resonant parameters

# Conclusions

- There is no God-given recipe to build the right amplitude, but one can ensure the right singularities to be respected
- The LS couplings are perfectly relativistic, although the naive implementation fails to respect the analyticity requirements
- The tensor formalisms have some of the right features, but explicitly violates crossing symmetry


Misha Mikhasenko, Tomasz Skwarnicki, Adam Szczepaniak