Covariant and helicity formalisms

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Salamanca – September 25, 2017
Recipes to build an amplitude

The literature abounds with discussions on the optimal approach to construct the amplitudes for the hadronic reactions

- **Helicity formalism**
  
  Jacob, Wick, Annals Phys. 7, 404 (1959)

- **Covariant tensor formalisms**
  
  Chung, PRD48, 1225 (1993)
  Chung, Friedrich, PRD78, 074027 (2008)
  Anisovich, Sarantsev, EPJA30, 427 (2006)

The common lore is that the former one is nonrelativistic, especially when expressed in terms of LS couplings and the latter takes into account the proper relativistic corrections.
How helicity formalism works

- Helicity formalism enforces the constraints about rotational invariance
- It allows us to fix the **angular dependence** of the amplitude
- What about **energy dependence**?

Example: $\Lambda_b \rightarrow \psi \Lambda^* \rightarrow pK$

Each set of angles is defined in a different reference frame
How helicity formalism works

- Helicity formalism enforces the constraints about rotational invariance
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- What about energy dependence?

Example: \( \Lambda_b \rightarrow \psi \Lambda^* \rightarrow pK \)

\[
M^{\Lambda^*_b \rightarrow \Lambda^*_n \rightarrow pK}_{\lambda_{\Lambda^*_b}}, \lambda_p, \Delta \lambda \mu = \sum_n \sum_{\lambda_{\Lambda^*}} \sum_{\lambda_{\psi}} H^{\Lambda^*_b \rightarrow \Lambda^*_n \psi}_{\lambda_{\Lambda^*_b}, \lambda_{\psi}} D_{\lambda_{\Lambda^*_b}, \lambda_{\Lambda^*}, \lambda_{\psi}}^{\frac{1}{2}} (0, \theta_{\Lambda^*_b}, 0)^* \\
H^{\Lambda^*_n \rightarrow \Lambda^*_p}_{\lambda_p, 0} D_{\lambda_{\Lambda^*_n}, \lambda_p} (\phi_K, \theta_{\Lambda^*}, 0)^* \\
R^{\Lambda^*_n} (m_{\Lambda^*_p}) D_{\lambda_{\psi}, \Delta \lambda \mu}^{\frac{1}{2}} (\phi_\mu, \theta_\psi, 0)^* ,
\]

Each set of angles is defined in a different reference frame

LHCb, PRL115, 072001 (2015)
How tensor formalisms work

The method is based on the construction of explicitly covariant expressions.

- To describe the decay $a \to bc$, we first consider the polarization tensor of each particle, $\epsilon^{i \mu_1 ... \mu_j}_{p_i}$

- We combine the polarizations of $b$ and $c$ into a “total spin” tensor $S_{\mu_1 ... \mu_S}(\epsilon_b, \epsilon_c)$

- Using the decay momentum, we build a tensor $L_{\mu_1 ... \mu_L}(p_{bc})$ to represent the orbital angular momentum of the $bc$ system, orthogonal to the total momentum of $p_a$

- We contract $S$ and $L$ with the polarization of $a$

Tensor $\times R_X(m)$ which contain resonances and form factors
What do we know?

- Energy dependence is not constrained by symmetry
- Still, there are some known properties one can enforce

\[
R_X(m) = B'_{LX} (p, p_0, d) \left( \frac{p}{M_{\Lambda_0^b}} \right)^L_{\Lambda_0^b} X
\]

\[
BW(m| M_{0X}, \Gamma_{0X}) B'_{LX} (q, q_0, d) \left( \frac{q}{M_{0X}} \right)^L_X
\]

- **Kinematical singularities**: e.g. barrier factors (known)
- **Left hand singularities** (need model, e.g. Blatt-Weisskopf)
- **Right hand singularities** = resonant content (Breit Wigner, K-matrix...)

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Kinematical singularities

- Kinematical singularities appear because of the spin of the external particle involved.
- We can write the most general covariant parametrization of the amplitude as a tensor of external polarizations \( \otimes \) scalar amplitudes.
- Scalar amplitudes must be *kinematical singularities free*.
- They can be matched to the helicity amplitudes.
- We can get the minimal energy dependent factor.
- Any other additional energy factor would be model-dependent.
Crossing symmetry in tensor formalisms

- The process $B \rightarrow \bar{D}\pi\pi$ is composed of scalar particles only

LHCb, PRD92, 032002 (2015)

- One defines the helicity angle in the isobar rest frame, then the amplitude is Lorentz Invariant

- Let’s consider the $\rho$ intermediate state, $B \rightarrow \bar{D}\rho(\rightarrow \pi\pi)$

$$A = \frac{m_B^2 + s - m_D^2}{2m_B^2} \cos \theta \times q\rho = \frac{E^{(B)}_{\rho}}{m_B} \cos \theta \times q\rho$$

- The factors $p$ and $q$ are the $L = 1$ expected barrier factors. The additional factor is analytical in $s$, not a kinematical singularity. Why is it there?
The tensor amplitude is given by $p_D^{(B)} \cdot p_\pi^{(\rho)}$, where $p_D^{(B)}$ is the breakup momentum in the B frame, and $p_\pi^{(\rho)}$ the decay momentum in the isobar frame.

$$A = \frac{m^2_{B^0} + s - m^2_{h_3}}{2m^2_{B^0}} pq \cos \theta$$

However, one can consider the scattering process just in the isobar rest frame.

$$A = pq \cos \theta$$

By crossing symmetry the amplitudes must be the same.

The usual implementation fails crossing symmetry.
To consider the effect of spin, let’s consider \( B \rightarrow \psi \pi K \). We focus on the parity violating amplitude for the \( K^* \) isobars, scattering kinematics.

\[
\begin{align*}
\psi (1^-) & \quad \pi (0^-) \\
K^* (0^+, 1^- ...) & \\
B (0^+) & \quad K (0^-)
\end{align*}
\]

\[
p = \frac{\lambda_{12}^{1/2}}{2\sqrt{s}}, \quad q = \frac{\lambda_{34}^{1/2}}{2\sqrt{s}}, \quad z_s = \frac{s(t - u) + (m_1^2 - m_2^2)(m_3^2 - m_4^2)}{\lambda_{12}^{1/2} \lambda_{34}^{1/2}}
\]
Helicity amplitude

\[ A_\lambda = \frac{1}{4\pi} \sum_{j=|\lambda|}^{\infty} (2j + 1) A^i_\lambda(s) \, d^j_{\lambda 0}(z_s) \]

\[ d^j_{\lambda 0}(z_s) = \hat{d}^j_{\lambda 0}(z_s) \xi_{\lambda 0}(z_s), \quad \xi_{\lambda 0}(z_s) = \left( \sqrt{1 - z_s^2} \right)^\lambda \]

\( \hat{d}^j_{\lambda 0}(z_s) \) is a polynomial of order \( j - |\lambda| \)
Helicity amplitude

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\( d^j_{\lambda 0}(z_s) \) is a polynomial of order \( j - |\lambda| \)

The kinematical singularities of \( A^j_\lambda(s) \) can be isolated by writing

\[ A^j_0 = \frac{m_1}{p\sqrt{s}} (pq)^j \hat{A}^j_0 \quad \text{for} \ j \geq 1, \]

\[ A^j_\pm = q (pq)^{j-1} \hat{A}^j_\pm \quad \text{for} \ j \geq 1, \]

\[ A^0_0 = \frac{p\sqrt{s}}{m_1} \hat{A}^0_0 \quad \text{for} \ j = 0, \]
Identify covariants

Two helicity couplings $\rightarrow$ two independent covariant structures

Important: we are not imposing any intermediate isobar

\[ A_\lambda(s, t) = \varepsilon_\mu(\lambda, p_1) \left[ (p_3 - p_4)^\mu - \frac{m_3^2 - m_4^2}{s} (p_3 + p_4)^\mu \right] C(s, t) \]

\[ + \varepsilon_\mu(\lambda, p_1) (p_3 + p_4)^\mu B(s, t) \]
Identify covariants

Two helicity couplings $\rightarrow$ two independent covariant structures

**Important**: we are not imposing any intermediate isobar

\[
C(s, t) = \frac{1}{4\pi\sqrt{2}} \sum_{j>0} (2j + 1)(pq)^{j-1} \hat{A}^i_+(s) \hat{d}^j_{10}(z_s)
\]

\[
B(s, t) = \frac{1}{4\pi} \hat{A}^0_0 + \frac{1}{4\pi} \frac{4m_1^2}{\lambda_{12}} \sum_{j>0} (2j + 1)(pq)^j
\]

\[
\times \left[ \hat{A}^j_i(s) \hat{d}^j_{00}(z_s) + \frac{s + m_1^2 - m_2^2}{\sqrt{2}m_1^2} \hat{A}^j_+(s) z_s \hat{d}^j_{10}(z_s) \right].
\]

Everything looks fine **but** the $\lambda_{12}$ in the denominator

The brackets must vanish at $\lambda_{12} = 0 \Rightarrow s = s_{\pm} = (m_1 \pm m_2)^2$, $\hat{A}^j_+$ and $\hat{A}^j_0$ cannot be independent
General form and comparisons

\[ \hat{A}^j_+ = \langle j - 1, 0; 1, 1|j, 1 \rangle g_j(s) + f_j(s) \]

\[ \hat{A}^j_0 = \langle j - 1, 0; 1, 0|j, 0 \rangle \frac{s + m_1^2 - m_2^2}{2m_1^2} g'_j(s) + f'_j(s) \]

\[ g_j(s_{\pm}) = g'_j(s_{\pm}), \text{ and } f_j(s), f'_j(s) \sim O(s - s_{\pm}) \]

All these four functions are free of kinematic singularity.
General form and comparisons

\[ \hat{A}^j_+ = \langle j - 1, 0; 1, 1|j, 1 \rangle g_j(s) + f_j(s) \]

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\[ g_j(s_\pm) = g_j'(s_\pm), \text{ and } f_j(s), f_j'(s) \sim O(s - s_\pm) \]

All these four functions are free of kinematic singularity.

Comparison with ordinary LS

\[ g_j(s) = g_j'(s) \frac{s + m_1^2 - m_2^2}{2m_1 \sqrt{s}} = \left( \frac{2j - 1}{2j + 1} \right)^{1/2} \hat{G}^j_{j-1}(s), \]

\[ f_j(s) = \frac{m_1}{\sqrt{s}} f_j'(s) = p^2 \left( \frac{2j + 3}{2j + 1} \right)^{1/2} \hat{G}^j_{j+1}(s). \]

If the \( \hat{G}^j_L \) are the usual Breit-Wigner, the \( g', f' \) have singularities
General form and comparisons

\[ \hat{A}_+^j = \langle j - 1, 0; 1, 1|j, 1\rangle g_j(s) + f_j(s) \]

\[ \hat{A}_0^j = \langle j - 1, 0; 1, 0|j, 0\rangle \frac{s + m_1^2 - m_2^2}{2m_1^2} g_j'(s) + f_j'(s) \]

\[ g_j(s_{\pm}) = g_j'(s_{\pm}), \text{ and } f_j(s), f_j'(s) \sim O(s - s_{\pm}) \]

All these four functions are free of kinematic singularity.

Comparison with tensor formalisms \((j = 1)\)

\[ A_\lambda = \varepsilon_\mu(\lambda, p_1) \left( -g^{\mu\nu} + \frac{P^\mu P^\nu}{s} \right) X_\nu(q) g_S(s) \]

\[ + \varepsilon^\rho(\lambda, p_1) X_{\rho\mu}(p) \left( -g^{\mu\nu} + \frac{P^\mu P^\nu}{s} \right) X_\nu(q) g_D(s) \]
General form and comparisons

\[ \hat{A}_+^j = \langle j - 1, 0; 1, 1|j, 1 \rangle g_j(s) + f_j(s) \]

\[ \hat{A}_0^j = \langle j - 1, 0; 1, 0|j, 0 \rangle \frac{s + m_1^2 - m_2^2}{2m_1^2} g_j'(s) + f_j'(s) \]

\[ g_j(s_\pm) = g_j'(s_\pm), \text{ and } f_j(s), f_j'(s) \sim O(s - s_\pm) \]

All these four functions are free of kinematic singularity.

Comparison with tensor formalisms \((j = 1)\)

\[ g_1 = g_1' = \frac{4\pi}{3} g_S, \quad f_1 = \frac{2\pi\lambda_{12}}{3s} g_D, \quad f_1' = -\frac{4\pi\lambda_{12}}{3s} \frac{s + m_1^2 - m_2^2}{m_1^2} g_D. \]

If the \(g_S, g_D\) are the usual Breit-Wigner, the \(g', f'\) are fine but for singularities at \(s = 0\)
Few words on form factors and Blatt-Weisskopf

- Left hand singularities are due to dynamics → model-dependent!
- They originate from the cross-channel exchanges in scattering processes
- These singularities are far from the physical region, but modify the high-energy behaviour (e.g. suppress the barrier factors)
- In nonrelativistic potential theory, they can be univocally calculated depending on the potential shape, e.g. Blatt-Weisskopf for square well + centrifugal potential
- **Best practice**: explore different form factor models, check the stability of the resonant parameters
Conclusions

- There is no God-given recipe to build the right amplitude, but one can ensure the right singularities to be respected.
- The LS couplings are perfectly relativistic, although the naive implementation fails to respect the analyticity requirements.
- The tensor formalisms have some of the right features, but explicitly violates crossing symmetry.

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