Roy-Steiner-equation analysis of pion-nucleon scattering

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In collaboration with:
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Motivation: Why $\pi N$ scattering?

- **Low energies**: test chiral dynamics in the baryon sector
  $\Rightarrow$ low-energy theorems e.g. for the scattering lengths

- **Higher energies**: resonances, baryon spectrum

- **Input for $NN$ scattering**: LECs $c_i$, $\pi NN$ coupling

- **Crossed channel $\pi\pi \rightarrow \bar{NN}$**: nucleon form factors
  $\Rightarrow$ probe the structure of the nucleon
  - scalar form factors (S-wave)
  - electromagnetic form factors (P-waves)
  - generalized PDF (D-waves)
The pion-nucleon $\sigma$-term

**Scalar form factor** of the nucleon:

$$\sigma(t) = \langle N(p')|\hat{m}(\bar{u}u + \bar{d}d)|N(p)\rangle \quad t = (p' - p)^2 \quad \sigma_{\pi N} = \sigma(0)$$

- $\sigma_{\pi N}$ measures the **light-quark contribution** to the nucleon mass
- Unfortunately, **no direct experimental access** to it
- Only very recent precise **lattice results**
- Linked to $\pi N$ via the **Cheng-Dashen** theorem

$$F_\pi^2 \bar{D}^+(\nu = 0, t = 2M_\pi^2) = \sigma(2M_\pi^2) + \Delta_R$$

$$F_\pi^2 (a_{00}^+ + 2M_\pi^2 d_{01}^+) + \Delta_D = \sigma_{\pi N} + \Delta_\sigma$$

$$|\Delta_R| \lesssim 2 \text{ MeV}$$

$$\Delta_D - \Delta_\sigma = (-1.8 \pm 0.2) \text{ MeV}$$
Phenomenological status

- **Karlsruhe/Helsinki** partial-wave analysis KH80
  - comprehensive analyticity constraints, old data
  [Höhler et al. 1980s]

- Formalism for the extraction of $\sigma_{\pi N}$ via the **Cheng–Dashen low-energy theorem**
  - “canonical value” $\sigma_{\pi N} \sim 45$ MeV, based on KH80 input
  [Gasser, Leutwyler, Locher, Sainio 1988,1991]

- **GWU/SAID** partial-wave analysis
  - much larger value $\sigma_{\pi N} = (64 \pm 8)$ MeV
  [Pavan, Strakovsky, Workman, Arndt 2002]

- More recently: ChPT in different regularizations (w/ and w/o $\Delta$)
  - fit to PWAs, $\sigma_{\pi N} = 59 \pm 7$ MeV
  [Alarcón et al. 2012]
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  [Alarcón et al. 2012]

- This talk: two new sources of information on low-energy $\pi N$ scattering  
  - Precision extraction of $\pi N$ **scattering lengths** from **hadronic atoms**  
  [Baru et al. 2011]  
  - **Roy-equation** constraints: analyticity, unitarity, crossing symmetry
Hadronic atoms: constraints for $\pi N$

- $\pi H/\pi D$: bound state of $\pi^-$ and p/d spectrum sensitive to threshold $\pi N$ amplitude

- Combined analysis of $\pi H$ and $\pi D$:
  
  $$a_0^+ \equiv a^+ = (7.5 \pm 3.1) \cdot 10^{-3} M_{\pi}^{-1}$$
  
  $$a_0^- \equiv a^- = (86.0 \pm 0.9) \cdot 10^{-3} M_{\pi}^{-1}$$

  $\rightarrow$ Large $a^+$ suggests a large $\sigma_{\pi N}$,

- But: $a^+$ very sensitive to isospin breaking, PWA based on $\pi^\pm p$ channels

  $\rightarrow$ use instead

  $$\frac{a_{\pi^- p} + a_{\pi^+ p}}{2} = (-0.9 \pm 1.4) \cdot 10^{-3} M_{\pi}^{-1}$$

- Isospin breaking in $\sigma_{\pi N}$ could be important

- We revisit the Cheng-Dashen low-energy theorem

\[ [\text{Baru et al. 2011}] \]
Motivation: Why Roy-Steiner equations?

Roy(-Steiner) eqs. = Partial-Wave (Hyberbolic) Dispersion Relations coupled by unitarity and crossing symmetry

- **Respect all symmetries**: analyticity, unitarity, crossing
- **Model independent** ⇒ the actual parametrization of the data is irrelevant once it is used in the integral.
- Framework allows for **systematic improvements** (subtractions, higher partial waves, ...)
- **PW(H)DRs** help to study processes with high precision:
  - \(\pi\pi\)-scattering: [Ananthanarayan et al. (2001), García-Martín et al. (2011)]
  - \(\pi K\)-scattering: [Büttiker et al. (2004)]
  - \(\gamma\gamma \rightarrow \pi\pi\) scattering: [Hoferichter et al. (2011)]
Warm up: Roy-equations for $\pi\pi$

- $\pi\pi \rightarrow \pi\pi \Rightarrow$ fully crossing symmetric in Mandelstam variables $s$, $t$, and $u = 4M_\pi - s - t$

- Start from twice-subtracted **fixed-t** DRs of the generic form

\[
T^I(s, t) = c(t) + \frac{1}{\pi} \int_{4m^2_\pi}^\infty \frac{ds'}{s'^2} \left[ \frac{s^2}{(s' - s)} - \frac{u^2}{(s' - u)} \right] \text{Im} T^I(s', t)
\]

- Subtraction functions $c(t)$ are determined via crossing symmetry

  $\leftrightarrow$ functions of the $l=0,2$ scattering lengths: $a^0_0$ and $a^2_0$

- PW-expansion of these DRs yields the **Roy-equations**

  \[
t^I_J(s) = ST^I_J(s) + \sum_{J'=0}^\infty (2J' + 1) \sum_{l'=0,1,2} \int_{4m^2_\pi}^\infty ds' K^{ll'}_{JJ'}(s', s) \text{Im} t^{ll'}_{JJ'}(s')
  \]

  $K^{ll'}_{JJ'}(s', s) \equiv$ kernels $\Rightarrow$ analytically known
Roy–Steiner equations for $\pi N$: flow of information

**Limited range of validity**

\[ \sqrt{s} \leq \sqrt{s_m} = 1.38 \text{ GeV} \quad \sqrt{t} \leq \sqrt{t_m} = 2.00 \text{ GeV} \]

**Input/Constraints**

- S- and P-waves above matching point $s > s_m \ (t > t_m)$
- Inelasticities
- Higher waves (D-, F-, \ldots)
- Scattering lengths from hadronic atoms
  
  [Baru et al. 2011]

**Output**

- S- and P-wave phase-shifts at low energies $s < s_m \ (t < t_m)$
- Subthreshold parameters
  - Pion-nucleon $\sigma$-term
  - Nucleon form factor spectral functions
  - ChPT LECs
Uncertainties: s-channel pw

\begin{align*}
S_{11} & \\
S_{31} & \\
P_{13} & \\
P_{33} & \\
P_{11} & \\
P_{31} &
\end{align*}

\begin{align*}
W \, [\text{GeV}] & \\
W \, [\text{GeV}] &
\end{align*}
Uncertainties: Imaginary part t-channel pw

\[ \Im f_0(t) \text{ [GeV]} \]
\[ \Im f_1(t) \text{ [GeV}^{-1}] \]
\[ \Im f_2(t) \text{ [GeV}^{-2}] \]
\[ \Im f_3(t) \text{ [GeV}^{-3}] \]
Results for the sigma-term

\[
\sigma_{\pi N} = F_\pi^2 \left( d_{00}^+ + 2M_\pi^2 d_{01}^+ \right) + \Delta_D - \Delta_\sigma - \Delta_R
\]

- subthreshold parameters output of the Roy–Steiner equations

  \[ d_{00}^+ = -1.36(3)M_\pi^{-1} \quad [\text{KH: } -1.46(10)M_\pi^{-1}], \quad d_{01}^+ = 1.16(2)M_\pi^{-3} \quad [\text{KH: } 1.14(2)M_\pi^{-3}] \]

- \[ \Delta_D - \Delta_\sigma = -(1.8 \pm 0.2) \text{ MeV} \quad [\text{Hoferichter at al. 2012}], \quad |\Delta_R| \lesssim 2 \text{ MeV} \quad [\text{Bernard, Kaiser, Mei\ss ner 1996}] \]

- Isospin breaking in the CD theorem shifts \( \sigma_{\pi N} \) by +3.0 MeV

- Final results:

  \[
  \sigma_{\pi N} = (59.1 \pm 1.9_{\text{RS}} \pm 3.0_{\text{LET}}) \text{ MeV} = (59.1 \pm 3.5) \text{ MeV} \quad [\text{MH, JRE, Kubis, Mei\ss ner}] 
  \]

- \( \sigma_{\pi N} \) depends linearly on the scattering lengths:

  \[
  \sigma_{\pi N} = 59.1 + \sum_{I_s} c_{I_s} \Delta a^{I_s}_{0+} 
  \]

- KH input \( \Rightarrow \sigma_{\pi N} = 46 \text{ MeV} \)

  \[ \leftrightarrow \] to be compared with \( \sigma_{\pi N} = 45 \text{ MeV} \quad [\text{Gasser, Leutwyler, Socher, Sainio 1988}] \]
Comparison with lattice $\sigma_{\pi N}$ results

- Recent lattice determination of $\sigma_{\pi N}$ at (almost) the physical point
  - BMW $\sigma_{\pi N} = 38(3)(3)\text{MeV}$
  - $\chi$QCD $\sigma_{\pi N} = 44.4(3.2)(4.5)\text{MeV}$
  - ETMC $\sigma_{\pi N} = 37.22(2.57)(1)\text{MeV}$
  - RQCD $\sigma_{\pi N} = 35(6)\text{MeV}$

- The linear dependence of $\sigma_{\pi N}$ on the scattering lengths introduces an additional constraint

- Inconsistent with the hadronic atom phenomenology

$\leftrightarrow$ determine the $\pi N$ scattering lengths in the lattice
Comparison with experimental cross-section data

Unravel the tension around the $\sigma$-term comparing with the experimental $\pi N$ data base

- Generate RS differential cross sections
  - RS $S$ and $P$ waves
  - higher partial waves from SAID and KH80
  - EM interactions implemented using Tromborg procedure

- Uncertainties from statistical effects, SL, input variation
  - below $T_\pi = 50$ MeV uncertainties dominated by scattering length errors
  - disentangle RS SL solutions by looking at the data base

- Define:

  $$\chi^2_{a_0^{1/2}} = \sum_{i,j} \left( \frac{\mathcal{O}^{\exp}_{i,j} - \mathcal{O}^{\text{RS}}_{i,j}(a_0^{1/2})}{\Delta \mathcal{O}_{i,j}^{\exp}} \right)^2$$

- Discrepancy concentrated in the $\pi^+ p \rightarrow \pi^+ p$ channel

<table>
<thead>
<tr>
<th></th>
<th>RS</th>
<th>KH80</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_0^{1/2}<em>1$ $[10^{-3} M</em>\pi^{-1}]$</td>
<td>$169.8 \pm 2.0$</td>
<td>$173 \pm 3$</td>
</tr>
<tr>
<td>$a_0^{3/2}<em>0$ $[10^{-3} M</em>\pi^{-1}]$</td>
<td>$-86.3 \pm 1.8$</td>
<td>$-101 \pm 4$</td>
</tr>
</tbody>
</table>
Cross-section data: $\pi^+ p \rightarrow \pi^+ p$ channel

$\pi^+ p \rightarrow \pi^+ p$

$\chi_{HA} = 0.8$
$
\chi_{KH80} = 4.7$

J. Ruiz de Elvira (ITP)
Extracting the $\sigma$-term from experimental cross-section data

- **Linearized** version of RS $d\sigma/d\Omega$ around the HA scattering lengths
- Unbiased fit to the pion-nucleon data base $\Rightarrow$ normalizations constants as fit parameters
- minimize the $\chi^2$-like as a function of $a_{0+}^l$ and $\zeta$

$$
\chi^2(a, a_0, \zeta, \zeta_0, \Delta \zeta_0) = \sum_{k=1}^{N} \chi_k^2(a, a_0, \zeta, \zeta_0, \Delta \zeta_0),
$$

$$
\chi_k^2(a, a_0, \zeta, \zeta_0, \Delta \zeta_0) = \sum_{i,j=1}^{N_k} \left( \zeta_k^{-1}(W_i^k, a) - \sigma_i^k \right) \left( C_k^{-1}(a_0, \zeta_0, \Delta \zeta_0) \right)_{ij} \left( \zeta_k^{-1}(W_j^k, a) - \sigma_j^k \right),
$$

$$
(C_k(a_0, \zeta_0, \Delta \zeta_0))_{ij} = \delta_{ij} (\Delta \sigma_i^k)^2 + \sigma(W_i^k, a_0)\sigma(W_j^k, a_0) \left( \frac{\Delta \zeta_{0,k}}{\zeta_{0,k}^2} \right)^2,
$$

(1)

<table>
<thead>
<tr>
<th>channel</th>
<th>SL combination</th>
<th>result</th>
<th>HA SL</th>
<th>KH80 SL</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi^+ p \rightarrow \pi^+ p$</td>
<td>$a_{0+}^{3/2}$</td>
<td>$-84.4 \pm 1.5$</td>
<td>$-86.3 \pm 1.8$</td>
<td>$-101 \pm 4$</td>
</tr>
<tr>
<td>$\pi^- p \rightarrow \pi^- p$</td>
<td>$(2a_{0+}^{1/2} + a_{0+}^{3/2})/3$</td>
<td>$82.5 \pm 1.5$</td>
<td>$84.4 \pm 1.7$</td>
<td>$81.6 \pm 2.4$</td>
</tr>
<tr>
<td>$\pi^- p \rightarrow \pi^0 n$</td>
<td>$-\sqrt{2}(a_{0+}^{1/2} - a_{0+}^{3/2})/3$</td>
<td>$-122.3 \pm 3.4$</td>
<td>$-120.7 \pm 1.3$</td>
<td>$-129.2 \pm 2.4$</td>
</tr>
</tbody>
</table>
Nucleon form factor spectral functions

- $\pi\pi \rightarrow \bar{NN}$ partial waves + $F^V_\pi$ pion form factor
  - $\pi\pi$ contribution to the isovector spectral functions
- consistent $\pi\pi$ phase shifts in $f^\pm_1$ and $F^V_\pi$
  - $\rightarrow$ Watson theorem is satisfied
- modern pion form factor data
- isospin breaking: $m_p - m_n$ in pole terms, subthreshold parameters, consistent $\rho - \omega$ mixing

\[ \text{[Hoferichter, Kubis, JRE, Hammer, Meißner 2016]} \]

\[ \text{[BaBar 2009, KLOE 2012, BESIII 2015]} \]
sum rules for the isovector radii: \[ \langle r_{E/M}^2 \rangle^v = \frac{6}{\pi} \int_{4M_\pi^2}^{\Lambda} dt' \frac{\text{Im} G_{E/M}^v(t')}{t'^2} \]

<table>
<thead>
<tr>
<th>( \Lambda = 1 \text{ GeV} )</th>
<th>( \Lambda = 2m_N )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \langle r_E^2 \rangle^v ) [fm(^2)]</td>
<td>0.418(32)</td>
</tr>
<tr>
<td>( \langle r_M^2 \rangle^v ) [fm(^2)]</td>
<td>1.83(10)</td>
</tr>
</tbody>
</table>

correcting normalization by single heavier resonance: \( \rho', \rho'' \):
reduces the radii only to: \[ \Delta \langle r_E^2 \rangle^v = -(0.006 \ldots 0.008) \text{ fm}^2 \]
\[ \Delta \langle r_M^2 \rangle^v = -(0.05 \ldots 0.07) \text{ fm}^2 \]
with \( \langle r_E^2 \rangle^n = -0.1161(22) \text{ fm}^2 \) (\( n \) scattering on heavy atoms):
\[ \rightarrow \] proton radius puzzle \( \leftrightarrow \) isovector radius puzzle

\[ \langle r_E^2 \rangle^v = 0.412 \text{ fm}^2 \text{ (} \mu \text{H)} \quad \text{vs.} \quad \langle r_E^2 \rangle^v = 0.442 \text{ fm}^2 \text{ (CODATA)} \]

\( \triangleright \) mild preference for small proton charge radius

[Hoferichter, Kubis, JRE, Hammer, Meißner 2016]
Matching to Chiral Perturbation Theory

Matching to ChPT at the subthreshold point:

- Chiral expansion expected to work best at subthreshold point
  - Maximal distance from threshold singularities
  - $\pi N$ amplitude can be expanded as polynomial

- Preferred choice for $NN$ scattering due to proximity of relevant kinematic regions

Express the subthreshold parameters in terms of the LECs to $O(p^4)$

$$d_{00}^+ = -\frac{2M_\pi^2(2\tilde{c}_1 - \tilde{c}_3)}{F_\pi^2} + \frac{g_\alpha^2(3 + 8g_\alpha^2)M_\pi^3}{64\pi F_\pi^4} + M_\pi^4 \left\{ \frac{16\tilde{e}_{14}}{F_\pi^2} - \frac{2c_1 - c_3}{16\pi^2 F_\pi^4} \right\}$$

- Chiral $\pi N$ amplitude to $O(p^4)$ depends on 13 low-energy constants

- Roy–Steiner system contains 10 subtraction constants

  - Calculate remaining 3 from sum rules
  - Invert the system to solve for LECs
Chiral low-energy constants

<table>
<thead>
<tr>
<th></th>
<th>NLO</th>
<th>N^2LO</th>
<th>N^3LO</th>
</tr>
</thead>
<tbody>
<tr>
<td>c_1 [GeV^{-1}]</td>
<td>-0.74 ± 0.02</td>
<td>-1.07 ± 0.02</td>
<td>-1.11 ± 0.03</td>
</tr>
<tr>
<td>c_2 [GeV^{-1}]</td>
<td>1.81 ± 0.03</td>
<td>3.20 ± 0.03</td>
<td>3.13 ± 0.03</td>
</tr>
<tr>
<td>c_3 [GeV^{-1}]</td>
<td>-3.61 ± 0.05</td>
<td>-5.32 ± 0.05</td>
<td>-5.61 ± 0.06</td>
</tr>
<tr>
<td>c_4 [GeV^{-1}]</td>
<td>2.17 ± 0.03</td>
<td>3.56 ± 0.03</td>
<td>4.26 ± 0.04</td>
</tr>
<tr>
<td>\bar d_1 + \bar d_2 [GeV^{-2}]</td>
<td>—</td>
<td>1.04 ± 0.06</td>
<td>7.42 ± 0.08</td>
</tr>
<tr>
<td>\bar d_3 [GeV^{-2}]</td>
<td>—</td>
<td>-0.48 ± 0.02</td>
<td>-10.46 ± 0.10</td>
</tr>
<tr>
<td>\bar d_5 [GeV^{-2}]</td>
<td>—</td>
<td>0.14 ± 0.05</td>
<td>0.59 ± 0.05</td>
</tr>
<tr>
<td>\bar d_{14} - \bar d_{15} [GeV^{-2}]</td>
<td>—</td>
<td>-1.90 ± 0.06</td>
<td>-12.18 ± 0.12</td>
</tr>
<tr>
<td>\bar e_{14} [GeV^{-3}]</td>
<td>—</td>
<td>—</td>
<td>0.89 ± 0.04</td>
</tr>
<tr>
<td>\bar e_{15} [GeV^{-3}]</td>
<td>—</td>
<td>—</td>
<td>-0.97 ± 0.06</td>
</tr>
<tr>
<td>\bar e_{16} [GeV^{-3}]</td>
<td>—</td>
<td>—</td>
<td>-2.61 ± 0.03</td>
</tr>
<tr>
<td>\bar e_{17} [GeV^{-3}]</td>
<td>—</td>
<td>—</td>
<td>0.01 ± 0.06</td>
</tr>
<tr>
<td>\bar e_{18} [GeV^{-3}]</td>
<td>—</td>
<td>—</td>
<td>-4.20 ± 0.05</td>
</tr>
</tbody>
</table>

- Subthreshold errors tiny, chiral expansion dominates uncertainty
- \bar d_i at N^3LO increase by an order of magnitude
  → due to terms proportional to \( g_A^2 (c_3 - c_4) = -16 \text{ GeV}^{-1} \)
  → mimic loop diagrams with \( \Delta \) degrees of freedom
- What's going on with chiral convergence?
  → look at convergence of threshold parameters with LECs fixed at subthreshold point
Convergence of the chiral series

<table>
<thead>
<tr>
<th></th>
<th>NLO</th>
<th>N²LO</th>
<th>N³LO</th>
<th>RS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a^+<em>{0+} \left[10^{-3} M</em>{\pi}^{-1}\right]$</td>
<td>-23.8</td>
<td>0.2</td>
<td>-7.9</td>
<td>-0.9 ± 1.4</td>
</tr>
<tr>
<td>$a^-<em>{0+} \left[10^{-3} M</em>{\pi}^{-1}\right]$</td>
<td>79.4</td>
<td>92.9</td>
<td>59.4</td>
<td>85.4 ± 0.9</td>
</tr>
<tr>
<td>$a^+<em>{1+} \left[10^{-3} M</em>{\pi}^{-3}\right]$</td>
<td>102.6</td>
<td>121.2</td>
<td>131.8</td>
<td>131.2 ± 1.7</td>
</tr>
<tr>
<td>$a^-<em>{1+} \left[10^{-3} M</em>{\pi}^{-3}\right]$</td>
<td>-65.2</td>
<td>-75.3</td>
<td>-89.0</td>
<td>-80.3 ± 1.1</td>
</tr>
<tr>
<td>$a^+<em>{1-} \left[10^{-3} M</em>{\pi}^{-3}\right]$</td>
<td>-45.0</td>
<td>-47.0</td>
<td>-72.7</td>
<td>-50.9 ± 1.9</td>
</tr>
<tr>
<td>$a^-<em>{1-} \left[10^{-3} M</em>{\pi}^{-3}\right]$</td>
<td>-11.2</td>
<td>-2.8</td>
<td>-22.6</td>
<td>-9.9 ± 1.2</td>
</tr>
<tr>
<td>$b^+<em>{0+} \left[10^{-3} M</em>{\pi}^{-3}\right]$</td>
<td>-70.4</td>
<td>-23.3</td>
<td>-44.9</td>
<td>-45.0 ± 1.0</td>
</tr>
<tr>
<td>$b^-<em>{0+} \left[10^{-3} M</em>{\pi}^{-3}\right]$</td>
<td>20.6</td>
<td>23.3</td>
<td>-64.7</td>
<td>4.9 ± 0.8</td>
</tr>
</tbody>
</table>

N³LO results bad due to large Delta loops

matching to ChPT with the explicit $\Delta$’s

improvement of the chiral convergence

Conclusion: lessons for few-nucleon applications

either include the $\Delta$ to reduce the size of the loop corrections

or use LECs from subthreshold kinematics

error estimates: consider chiral convergence of a given observable, difficult to assign a global chiral error to LECs

[Siemens, JRE, Epelbaum, Hoferichter, Krebs, Kubis, Meißer 2016]
The “ruler plot” vs. ChPT

Lattice QCD simulations can be performed at different quark/pion masses

Pion mass dependence of $m_N$ up to NNNLO in ChPT, using

- Input from Roy–Steiner solution

![Graph showing the comparison between various theoretical predictions and lattice data for $m_N$ vs. $M_\pi$.](attachment:graph.png)

thanks to A. Walker-Loud for providing the lattice data

→ range of convergence of the chiral expansion is very limited

→ huge cancellation amongst terms to produce a linear behavior
- Derived a closed system of Roy–Steiner equations for $\pi N$
- Numerical solution and error analysis of the full system of RS eqs.
- Precise determination of the $\sigma_{\pi N}$
  - Roy–Steiner formalism reproduces KH80 result with KH80 input
  - With modern input for scattering lengths and coupling constant $\sigma_{\pi N}$ increases
  - Results from hadronic atom results compatible with low-energy $\pi N$ scattering data
- $t$-channel $\rightarrow$ nucleon form factor spectral functions
  - Sum rules for isovector radii $\rightarrow$ proton radius puzzle
- Extraction of the ChPT LECs
- Study of the chiral convergence
Thank you
Spare slides
Roy-Steiner equations for $\pi N$: flow of information

- Higher partial waves
  $\text{Im} f_{l\pm}^l, \, l \geq 2, \, s \leq s_m$

- $s$-channel partial waves
  solve Roy–Steiner equations for $s \leq s_m$

- Inelasticities
  $\eta_{l\pm}^l, \, l \leq 1, \, s \leq s_m$

- High-energy region
  $\text{Im} f_{l\pm}^l, \, s \geq s_m$

- $t$-channel partial waves
  solve Roy–Steiner equations for $t \leq t_m$

- $\pi\pi$ scattering phases $\delta_J^l$

- High-energy region
  $\text{Im} f_{l\pm}^l, \, t \geq t_m$

- Subtraction constants

- $\pi N$ coupling constant
Roy–Steiner equations for $\pi N$: differences to $\pi\pi$ Roy equations

Key differences compared to $\pi\pi$ Roy equations

- **Crossing**: coupling between $\pi N \rightarrow \pi N$ (s-channel) and $\pi\pi \rightarrow \bar{N}N$ (t-channel)
  
  ⇒ need a different kind of dispersion relations

- **Unitarity** in t-channel, e.g. in single-channel approximation
  
  \[
  \operatorname{Im} f^I_{\pm}(t) = \sigma^\pi_I f^J_{\pm}(t) t^J_J(t)^* \]

  ⇒ **Watson’s theorem**: phase of $f^J_{\pm}(t)$ equals $\delta_{IJ}$

  ⇐ solution in terms of Omnès function

- **Large pseudo-physical region** in t-channel

  ⇐ $\bar{K}K$ intermediate states for s-wave in the region of the $f_0(980)$

[Hite, Steiner 1973, Büttiker et al. 2004]

[Watson 1954]

[Muskhelishvili 1953, Omnès 1958]
Solving Roy-equations: flow information

- **Roy-equations** rigorously valid for a finite energy range
  - ⇒ introduce a matching point $s_m$
- only partial waves with $J \leq J_{\text{max}}$ are solved
- assume isospin limit

**Input**
- High-energy region: $\text{Im} t_{IJ}(s)$ for $s \geq s_m$ and for all $J$
- Higher partial waves: $\text{Im} t_{IJ}(s)$ for $J > J_{\text{max}}$ and for all $s$

**Output**
- Self-consistent solution for $\delta_{IJ}(s)$ for $J \leq J_{\text{max}}$ and $s_{\text{th}} \leq s \leq s_m$
- Constraints on subtraction constants
\[ \pi^a(q) + N(p) \rightarrow \pi^b(q') + N(p') \]

- **Isospin Structure:**
  \[ T^{ba} = \delta^{ba} T^+ + \epsilon^{ab} T^- \]

- **Lorentz Structure:** \[ I \in \{+,-\} \]
  \[ T^I = \bar{u}(p') \left( A^I + \frac{q^I + q'^I}{2} B^I \right) u(p) \]
  \[ D^I = A^I + \nu B^I, \quad \nu = \frac{s-u}{4m} \]

- **Isospin basis:** \[ I_s \in \{1/2, 3/2\} \]
  \[ \{ T^+, T^- \} \leftrightarrow T^{1/2}, T^{3/2} \]

- **PW projection:**
  s-channel pw: \[ f^I_{l\pm} \]
  t-channel pw: \[ f^J_{l\pm} \]

**Bose symmetry** \[ \Rightarrow \text{even/odd } J \leftrightarrow I = +/– \]
\[ f_{l\pm}^l(W) = \frac{1}{16\pi W_1} \left\{ (E + m) [A^l(s) + (W - m)B^l(s)] + (E - m) [-A^l_{l+1}(s) + (W + m)B^l_{l+1}(s)] \right\} \]

\[ X^l_i(s) = \int_{-1}^1 dz_s P_i(z_s) X^l(s, t) \bigg|_{t=t(s,z_s)=-2q^2(1-z_s)} \text{ for } X \in \{A, B\} \text{ and } W = \sqrt{s} \]

**McDowell symmetry:** \( f_{l+}^l(W) = -f_{l(l+1)-}^l(-W) \quad \forall \ l \geq 0 \)

**t-channel projection:**

\[ f^J_+ (t) = -\frac{1}{4\pi} \int_0^1 dz_t \ P^J(z_t) \left\{ \frac{p^2_t}{(p_t q_t)^J} A^l(s, t) \bigg|_{s=s(t,z_t)} - \frac{m}{(p_t q_t)^{J-1}} z_t B^l(s, t) \bigg|_{s=s(t,z_t)} \right\} \quad \forall \ J \geq 0 \]

\[ f^J_-(t) = \frac{1}{4\pi} \sqrt{J(J+1)} \frac{1}{2J+1} \int_0^1 dz_t \left[ P_{J-1}(z_t) - P_{J+1}(z_t) \right] B^l(s, t) \bigg|_{s=s(t,z_t)} \quad \forall \ J \geq 1 \]

**Bose symmetry** \( \Rightarrow \text{even/odd } J \Leftrightarrow l = +/− \)
**s-channel** unitarity relations \((l_\pm \in \{1/2, 3/2\})\):

\[
\text{Im} \ f_{l_\pm}^I (W) = q \left| f_{l_\pm}^I (W) \right|^2 \theta (W - W_+) + \frac{1 - (\eta_{l_\pm}^I (W))^2}{4q} \theta (W - W_{\text{inel}})
\]

**t-channel** unitarity relations: 2-body intermediate states: \(\pi\pi + \bar{K}K + \cdots\)

\[
\text{Im} \ f_{l_\pm}^J (t) = \frac{\pi}{2} \left( t_{l_\pm}^J (t) \right)^* f_{l_\pm}^J (t) \theta (t - t_\pi) + 2c_J \sqrt{2} k_t^{2J} \sigma_t^K \left( g_{l_\pm}^J (t) \right)^* h_{l_\pm}^J (t) \theta (t - t_K)
\]

Only linear in \(f_{l_\pm}^J (t)\) \(\Rightarrow\) less restrictive
Roy-Steiner equations for $\pi N$: HDR’s

- **Hyperbolic DRs:** $(s - a)(u - a) = b = (s' - a)(u' - a)$ with $a, b \in \mathbb{R}$

  $$A^+(s, t; a) = \frac{1}{\pi} \int_{s_+}^\infty ds' \left[ \frac{1}{s' - s} + \frac{1}{s' - u} - \frac{1}{s' - a} \right] \text{Im} A^+(s', t') + \frac{1}{\pi} \int_{t\pi}^\infty dt' \frac{\text{Im} A^+(s', t')}{t' - t}$$

  $$B^+(s, t; a) = N^+(s, t) + \frac{1}{\pi} \int_{s_+}^\infty ds' \left[ \frac{1}{s' - s} - \frac{1}{s' - u} \right] \text{Im} B^+(s', t') + \frac{1}{\pi} \int_{t\pi}^\infty dt' \frac{\nu}{\nu'} \frac{\text{Im} B^+(s', t')}{t' - t}$$

  $$N^+(s, t) = g^2 \left( \frac{1}{m^2 - s} - \frac{1}{m^2 - u} \right)$$ similar for $A^-, B^-$ and $N^-$ [Hite/Steiner (1973)]

- **Why HDR?**
  - Combine all physical regions ⇒ crucial for t-channel projection
  - Evade double-spectral regions ⇒ the PW decompositions converge
  - Range of convergence can be maximized by tuning the free hyperbola parameter $a$
  - No kinematical cuts, manageable kernel functions
Recipe to derive Roy-Steiner equations:

- Expand imaginary parts in terms of s- and t-channel partial waves
- Project onto s- and t-channel partial waves
- Combine the resulting equations using s- and t-channel PW unitarity relations

Similar structure to \( \pi \pi \) Roy equations

Validity: assuming Mandelstam analyticity

- s-channel \( \Rightarrow \) optimal for \( a = -23.2 M_\pi^2 \)
  \[ s \in [s_+ = (m + M_\pi)^2, 97.30 M_\pi^2] \Leftrightarrow W \in [W_+ = 1.08 \text{ GeV}, 1.38 \text{ GeV}] \]

- t-channel \( \Rightarrow \) optimal for \( a = -2.71 M_\pi^2 \)
  \[ t \in [t_\pi = 4M_\pi^2, 205.45 M_\pi^2] \Leftrightarrow \sqrt{t} \in [\sqrt{t_\pi} = 0.28 \text{ GeV}, 2.00 \text{ GeV}] \]
**Subtractions** are necessary to ensure the convergence of DR integrals
⇒ asymptotic behavior

Can be introduced to lessen the dependence of the low-energy solution on the high-energy behavior

Parametrize high-energy information in (a priori unknown) subtraction constants
⇒ matching to ChPT

Subthreshold expansion around \( \nu = t = 0 \)

\[
\bar{A}^+ (\nu, t) = \sum_{m,n=0}^{\infty} a^+_{mn} \nu^{2m} t^n \\
\bar{B}^+ (\nu, t) = \sum_{m,n=0}^{\infty} b^+_{mn} \nu^{2m+1} t^n , \\
\bar{A}^- (\nu, t) = \sum_{m,n=0}^{\infty} a^-_{mn} \nu^{2m+1} t^n \\
\bar{B}^- (\nu, t) = \sum_{m,n=0}^{\infty} b^-_{mn} \nu^{2m} t^n ,
\]

where

\[
\bar{A}^+ (s, t) = A^+ (s, t) - \frac{g^2}{m} \\
\bar{B}^+ (s, t) = B^+ (s, t) - g^2 \left[ \frac{1}{m^2 - s} - \frac{1}{m^2 - u} \right] , \\
\bar{A}^- (s, t) = A^- (s, t) , \\
\bar{B}^- (s, t) = B^- (s, t) - g^2 \left[ \frac{1}{m^2 - s} + \frac{1}{m^2 - u} \right] + \frac{g^2}{2m^2} ,
\]
RS-eqs for $\pi N$: subthreshold expansion

- Subthreshold expansion around $\nu = t = 0$

\[ A^+(\nu, t) = \frac{g^2}{m} + d^+_{00} + d^+_{01} t + a^+_{10} \nu^2 + \mathcal{O}(\nu^2 t, t^2) \]

\[ A^-(\nu, t) = \nu a^-_{00} + a^-_{01} \nu t + a^-_{10} \nu^3 + \mathcal{O}(\nu^5, \nu t^2, \nu^3 t) \]

\[ B^+(\nu, t) = g^2 \frac{4m\nu}{(m^2 - s_0)^2} + \nu b^+_{00} + \mathcal{O}(\nu^3, \nu t) \]

\[ B^-(\nu, t) = g^2 \left[ \frac{2}{m^2 - s_0} - \frac{t}{(m^2 - s_0)^2} \right] - \frac{g^2}{2m^2} + b^-_{00} + b^-_{01} t + b^-_{10} \nu^2 + \mathcal{O}(\nu^2, \nu^2 t, t^2) \]

- Pseudovector Born terms: $D^l = A^l + \nu B^l$

\[ \bar{D}^+ = d^+_{00} + d^+_{01} t + d^+_{10} \nu^2 \]

\[ d^+_{mn} = a^+_{mn} + b^+_{m-1,n} \]

\[ d^-_{mn} = a^-_{mn} + b^-_{mn} \]

- Sum rules for subthreshold parameters:

\[ d^+_{00} = -\frac{g^2}{m} + \frac{1}{\pi} \int_{s_+}^{\infty} ds' h_0(s') [\text{Im} A^+(s', z'_s)]_{(0, 0)} + \frac{1}{\pi} \int_{t_\pi}^{\infty} dt' [\text{Im} A^+(t', z'_t)]_{(0, 0)} \]

\[ h_0(s') = \frac{2}{s' - s_0} - \frac{1}{s' - a} \]
Solving Roy-Steiner equations for $\pi N$: Recoupling schemes

**s-channel** subproblem:
- Kernels are diagonal for $I \in \{+, -\}$, but unitarity relations are diagonal for $I_s \in \{1/2, 3/2\} \Rightarrow$ all partial-waves are interrelated
- Once the t-channel PWs are known  
  $\Rightarrow$ Structure similar to $\pi \pi$ Roy-equations

**t-channel** subproblem:
- Only higher PWs couple to lower ones
- Only PWs with even or odd J are coupled
- No contribution from $f_J^{+}$ to $f_{J+1}^{+}$  
  $\Rightarrow$ Leads to Muskhelishvili-Omnès problem
Roy-Steiner equations for $\pi N$: s-channel

**s-channel RS equations**

\[
\begin{align*}
\tilde{f}_{i+}^l(W) &= \tilde{N}_{i+}^l(W) + \frac{1}{\pi} \int_{W+}^{\infty} dW' \sum_{l'=0}^{\infty} \left\{ K_{i,l'}^l(W, W') \operatorname{Im} \tilde{f}_{i'+1}^l(W') + K_{i,l'}^l(W, -W') \operatorname{Im} \tilde{f}_{i'+1}^l(-W') \right\} \\
&\quad + \frac{1}{\pi} \int_{t\pi}^{\infty} dt' \sum_j \left\{ G_{i,j}^l(W, t') \operatorname{Im} \tilde{f}_{+}^j(t') + H_{i,j}^l(W, t') \operatorname{Im} \tilde{f}_{-}^j(t') \right\} \\
&= -\tilde{f}_{i'+1}^l(-W) \quad \forall \ l \geq 0, \quad \text{[Hite/Steiner (1973)]}
\end{align*}
\]

- **e** $K_{i,l'}^l(W, W')$, $G_{i,j}^l(W, t')$ and $H_{i,j}^l(W, t')$-Kernels: analytically known,
  e.g. $K_{i,l'}^l(W, W') = \frac{\delta_{i,l'}}{W' - W} + \ldots \quad \forall \ l, l' \geq 0$,

- **Validity**: assuming Mandelstam analyticity
  $\Rightarrow$ optimal for $a = -23.2 M^2_\pi$

\[
s \in \left[ s_+ = (m + M_\pi)^2, 97.30 \ M^2_\pi \right] \Leftrightarrow W \in \left[ W_+ = 1.08 \text{ GeV}, 1.38 \text{ GeV} \right]
\]
Roy-Steiner equations for $\pi N$: t-channel

**t-channel RS equations**

\[ f_+^J(t) = \tilde{N}_+^J(t) + \frac{1}{\pi} \int_{W_+}^\infty dW' \sum_{l=0}^\infty \left\{ \tilde{G}_{JI}(t, W') \text{Im} f_+^l(W') + \tilde{G}_{JI}(t, -W') \text{Im} f_{(l+1)-}^l(W') \right\} \]

\[ + \frac{1}{\pi} \int_{t_\pi}^\infty dt' \sum_{J'} \left\{ \tilde{K}_{JJ'}^1(t, t') \text{Im} f_+^{l'}(t') + \tilde{K}_{JJ'}^2(t, t') \text{Im} f_-^{l'}(t') \right\} \quad \forall J \geq 0, \]

\[ f_-^J(t) = \tilde{N}_-^J(t) + \frac{1}{\pi} \int_{W_+}^\infty dW' \sum_{l=0}^\infty \left\{ \tilde{H}_{JI}(t, W') \text{Im} f_+^l(W') + \tilde{H}_{JI}(t, -W') \text{Im} f_{(l+1)-}^l(W') \right\} \]

\[ + \frac{1}{\pi} \int_{t_\pi}^\infty dt' \sum_{J'} \tilde{K}_{JJ'}^3(t, t') \text{Im} f_-^{l'}(t') \quad \forall J \geq 1, \]

**Validity**: assuming Mandelstam analyticity

\[ \Rightarrow \text{optimal for } a = -2.71 M_{\pi}^2 \]

\[ t \in [t_\pi = 4 M_{\pi}^2, 205.45 M_{\pi}^2] \quad \Leftrightarrow \quad \sqrt{t} \in [\sqrt{t_\pi} = 0.28 \text{ GeV}, 2.00 \text{ GeV}]. \]
Solving t-channel: single channel

- Elastic-channel approximation: generic form of the integral equation

\[ f(t) = \Delta(t) + (a + bt)(t - 4m^2) + \frac{t^2(t - 4m^2)}{\pi} \int_{\pi}^{\infty} dt' \frac{\text{Im} f(t')}{t'(t'^2 - 4m^2)(t' - t)} \]

- \( \Delta(t) \): Born terms, s-channel integrals, higher t-channel partial waves
  \( \Rightarrow \) left-hand cut

- Introduce subtractions at \( \nu = t = 0 \Rightarrow \) subthreshold parameters \( a, b \)

- Solution in terms of Omnès function:

\[
f(t) = \Delta(t) + (t - 4m^2)\Omega(t)(1 - t\dot{\Omega}(0))a + t(t - 4m^2)\Omega(t)b - \Omega(t)\frac{t^2(t - 4m^2)}{\pi} \left\{ \int_{4M^2\pi}^{\text{Im}} dt' \frac{\Delta(t')\text{Im} \Omega(t')^{-1}}{t'^2(t' - 4m^2)(t' - t)} + \int_{t_m}^{\infty} dt' \frac{\Omega(t')^{-1}\text{Im} f(t')}{t'(t' - 4m^2)(t' - t)} \right\}
\]

\[
\Omega(t) = \exp \left\{ \frac{t}{\pi} \int_{t_\pi}^{\text{Im}} \frac{dt'}{t'} \frac{\delta(t')}{t' - t} \right\}
\]
Solving t-channel: input and subtractions

- elastic channel approximation: $\sqrt{t_m} = 0.98 - 1.1 \text{ GeV}$, for $t > t_m$ \( \text{Im} f_\pm^J(t) = 0 \)

- First step: check consistency with KH80

- Input needed:
  - $\pi \pi$ phase shifts:
  - $\pi N$ phase shifts: SAID,KH80
  - $\pi N$ at high energies: Regge model
  - $\pi N$ parameters: KH80

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Caprini, Colangelo, Leutwyler, (in preparation), Madrid group
Arndt et al. 2008, Höhler 1983
Huang et al. 2010

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MO solutions in general consistent with KH80 results
Solving t-channel: P, D and F waves up to $\tilde{N}N$
Solving \( t \)-channel: coupled channels

- Generic coupled-channel integral equation

\[
\mathbf{f}(t) = \Delta(t) + \frac{1}{\pi} \int_{\pi}^{m} dt' \frac{T^*(t') \Sigma(t') \mathbf{f}(t')}{t' - t} + \frac{1}{\pi} \int_{m}^{\infty} dt' \text{Im} \mathbf{f}(t') \frac{t'}{t' - t}
\]

- Formal solution as in the single-channel case (now with Omnès matrix \( \Omega(t) \))

\[\Rightarrow \text{Two-channel Muskhelishvili-Omnès problem} \]

\[
\mathbf{f}(t) = \left( \begin{array}{c} f_0^0(t) \\ f_0^+(t) \\ h_0^0(t) \\ h_0^+(t) \end{array} \right) \quad \text{Im} \Omega(t) = (T(t))^* \Sigma(t) \Omega(t)
\]

- Two linearly independent solutions \( \Omega_1, \Omega_2 \)

- In general no analytical solution for the Omnès matrix but for its determinant

\[
\det \Omega(t) = \exp \left\{ \frac{t}{\pi} \int_{\pi}^{m} dt' \frac{\psi(t')}{t'(t' - t)} \right\}.
\]

Muskhelishvili 1953

Moussallam 2000
Solving t-channel S-wave equations: input

- Input needed:
  - $\pi\pi$ s-wave partial waves: Caprini, Colangelo, Leutwyler, (in preparation)
  - $K\bar{K}$ s-wave partial waves: Büttiker. (2004)
  - $\pi N$ and $KN$ s-wave pw: SAID, KH80 Arndt et al. 2008, Höhler 1983,
  - $\pi N$ at high energies: Regge model Huang et al. 2010
  - $\pi N$ parameters: KH80 Jülich model 1989
  - Hyperon couplings
  - $KN$ subthreshold parameters neglected

- Two-channel approximation breaks down at $\sqrt{t_0} = 1.3$ GeV $\Rightarrow 4\pi$ channel

- From $t_0$ to $t = 2$ GeV, different approximations considered
Solving t-channel: S-wave results
Solving t-channel: S-wave results

MO solutions in general consistent with KH80 results
Solving s-channel: S-wave results

- General form of the s-channel integral equation

\[
\begin{align*}
  f_{l+}^I(W) &= \Delta_{l+}^I(W) + \frac{1}{\pi} \int_{W_+}^{\infty} dW' \sum_{l'=0}^{\infty} \left\{ K_{ll'}^I(W, W') \Im f_{l+}^{l'}(W') + K_{ll'}^I(W, -W') \Im f_{(l'+1)-}^{l'}(W') \right\}
\end{align*}
\]

\[\Rightarrow\text{ form of } \pi\pi \text{ Roy-Equations}\]

- \(\Delta_{l+}^I(W) \equiv t\text{-channel contribution and pole term}\)

- valid up to \(W_m = 1.38 \text{ GeV}\)

**Input:**
- RS t-channel solutions for S and P waves
- s-channel partial waves for \(J > 1\)
- s-channel partial waves for \(W_m < W < 2.5 \text{ GeV}\)
- high energy contribution for \(W > 2.5 \text{ GeV}: \text{Regge model}\)

**Output:**
- Self-consistent solution for S and P waves for \(J \leq J_{\text{max}}\) and \(s_{\text{th}} \leq s \leq s_m\)
- Constraints on subtraction constants \(\Rightarrow\) subthreshold parameters
Solving s-channel: subtractions

- **Existence and uniqueness** of solutions
  - no-cusp condition for each pw + 2 additional constraints are needed

- Take advantage of the precise data for pionic atoms
  - Impose as a **constraint** scattering lengths from a combined analysis of pionic hydrogen and deuterium

\[
a_{0+}^{1/2} = (169.8 \pm 2.0) \times 10^{-3} M^{-1}_\pi \quad a_{0+}^{3/2} = (-86.3 \pm 1.8) \times 10^{-3} M^{-1}_\pi
\]

\[
\text{Re } f_{I}^I (s) = q^{2I} \left( a_I^I + b_I^I q^2 + \cdots \right)
\]

**10 subthreshold parameters** are needed to match d.o.f
- **three subtractions**
Parameterize S and P waves up to $W < W_m$

Using SAID partial waves as starting point

Impose as constraints the hadronic atom scattering lengths

Introduce as many subtractions as necessary to match d.o.f

Minimize difference between LHS and the RHS on a grid of points $W_j$

$$\chi^2 = \sum_{l,l_s,\pm} \sum_{j=1}^N \left( \frac{\text{Re} f_{l\pm}^l(W_j) - F[f_{l\pm}^l](W_j)}{\text{Re} f_{l\pm}^l(W_j)} \right)^2$$

$F[f_{l\pm}^l](W_j) \equiv$ right hand side of RS-equations

Parametrization and subthreshold parameters are the fitting parameters
Solving s-channel: results

\[ \sqrt{s} (\text{GeV}) \]

\[ s_{11}(s) \]

\[ s_{31}(s) \]

\[ p_{13}(s) \]

\[ p_{33}(s) \]

\[ p_{11}(s) \]

\[ p_{31}(s) \]

Notation: \( L_{2I\bar{S}2J} \)

\text{blue/red} \quad \uparrow \quad \text{LHS/RHS}

\text{gray/black} \quad \uparrow \quad \text{LHS/RHS}

after the fit

before the fit
Results: s-channel PWs

\[ s_{11}(s) \]

\[ s_{31}(s) \]

\[ p_{13}(s) \]

\[ p_{33}(s) \]

\[ p_{11}(s) \]

\[ p_{31}(s) \]

\[ \sqrt{s} \text{ (GeV)} \]

\[ \sqrt{s} \text{ (GeV)} \]

blue/red

LHS/RHS

after the fit

gray/black

LHS/RHS

before the fit

Notation: \( L_{21s2J} \)

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Results: s-channel PWs

- $s_{11}(s)$
- $s_{31}(s)$
- $p_{13}(s)$
- $p_{33}(s)$
- $p_{11}(s)$
- $p_{31}(s)$

Notation: $L_{2I_s2J}$

$s_2(1275)$ effect

Gray/black
LHS/RHS before the fit

Blue/red
LHS/RHS after the fit
Results: t-channel PWs

\[ \text{Imf}_0^+ (s) \]

\[ \text{Imf}_1^+ (s) \]

\[ \text{Imf}_1^- (s) \]

\[ \text{Imf}_2^+ (s) \]

\[ \text{Imf}_2^- (s) \]

\[ \sqrt{t} \text{ (GeV)} \]

\text{blue}

\text{KH80}

\text{before the fit}

\text{red}

\text{after the fit}
Solving the full RS system: strategy

- Full solution: self-consistent, iterative solution of the full RS system
  ⇒ consistent set of s- and t-channel PWs & low-energy parameters

- However:
  - t-channel RS eqs. depend only weakly on s-channel PWs
  - resulting s-channel PW change little from SAID

A full solution can be achieved including in the s-channel RS eqs. the t-channel dependence on the subthreshold parameters
Uncertainties

- **Statistical errors** *(at intermediate energies)*
  - important correlations between subthreshold parameters
  - shallow fit minima
  - ⇒ Sum rules for subthreshold parameters become essential to reduce the errors

- **Input variation** *(small)*
  - small effect for considering s-channel KH80 input
  - very small effects from $L > 5$ s-channel PWs
  - small effect from the different S-wave extrapolation for $t > 1.3$ GeV
  - negligible effect of $\rho'$ and $\rho''$
  - very significant effects of the D-waves ($f_2(1275)$)
  - F-waves shown to be negligible

- **matching conditions** *(close to $W_m$)*

- **scattering length (SL) errors** *(on S-waves and subthreshold parameters)*
  - very important for the $\sigma_{\pi N}$
Uncertainties: Real part s-channel pw

Figure: Real parts of the s-channel partial waves calculated from the RHS of the RS equations (dashed lines) in comparison to the SAID results (red dot-dashed lines). The dashed lines indicate the position of the matching point $\sqrt{s} = 1.38$ GeV.
Uncertainties: Real part t-channel pw

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Decomposition of s-channel RS pw

\[ i_s^l \pm (W) = N_s^l \pm (W) + K_s T(W)_s^l \pm + K_t T(W)_t^l \pm + DT(W)_l^l \pm \]

\[ S_{11} \]

\[ S_{31} \]

\[ P_{13} \]

\[ P_{33} \]

\[ P_{11} \]

\[ P_{31} \]
Comparison with KH80

- Karlsruhe-Helsinki analysis **KH80**
  - comprehensive analyticity constraints based on fixed-t dispersion relations
  - old experimental data

- Here, an update of **KH80** results with modern input
  - HDR increase the range of validity of the equations
  - $\pi N$ scattering length extracted from hadronic atoms $\Rightarrow$ essential for the $\sigma_{\pi N}$
  - Goldberger-Miyazawa-Oehme sum rule:
    \[
    g_{\pi N}^2 / 4\pi = 13.7 \pm 0.2
    \]
    compare: $g_{\pi N}^2 / 4\pi = 14.28$

- s-channel PWs from **SAID**
  - $f_2(1275)$ included $\Rightarrow$ sizable effect

- **KH80** is internally consistent $\Rightarrow$ RS reproduces **KH80** results with **KH80** input

Höhler et al. 1980

Baru et al. 2011

Höhler et al. 1983
Results: s-channel PWs with KH80 input

\[ s_{11}(s) \]
\[ s_{31}(s) \]
\[ p_{13}(s) \]
\[ p_{33}(s) \]
\[ p_{11}(s) \]
\[ p_{31}(s) \]

Notation: \( L_{2I\bar{I}2J} \)

\[ \begin{array}{c}
\text{blue/red} \\
\text{LHS/RHS} \\
\text{after the fit}
\end{array} \]

\[ \begin{array}{c}
\text{gray/black} \\
\text{LHS/RHS} \\
\text{before the fit}
\end{array} \]
Results: t-channel PWs with KH80 input

\[ \Im f^0_{+}(s) \]

\[ \Im f^1_{+}(s) \]

\[ \Im f^1_{-}(s) \]

\[ \Im f^2_{+}(s) \]

\[ \Im f^2_{-}(s) \]

before the fit

after the fit
Comparison with KH80

- RS eqs. with KH80 input $\Rightarrow \sigma_{\pi N} = 46 \text{ MeV}$
  $\Leftarrow$ to be compared with $\sigma_{\pi N} = 45 \text{ MeV}$
  Gasser, Leutwyler, Socher, Sainio 1988, Gasser, Leutwyler, Sainio 1991
- KH80 is internally consistent but at odd with the modern SL determinations

How are $d^+_{00}$ and $d^+_{01}$ extracted in KH80 and SAID?

- Standard approach:
  replace $d^+_{00}$ and $d^+_{01}$ in favor of threshold parameters: $a^+_{0+}$ and $a^+_{1+}$
  $\Leftarrow$ corrections from PWA via DRs ($D^+$ and $E^+$)

<table>
<thead>
<tr>
<th></th>
<th>Born</th>
<th>$a^+_{0+}$</th>
<th>$a^+_{1+}$</th>
<th>$D^+$</th>
<th>$E^+$</th>
<th>$\Sigma_d = F^2_{\pi} \left( d^+<em>{00} + 2 M^2</em>{\pi} d^+_{01} \right)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>KH80</td>
<td>-133</td>
<td>-7</td>
<td>+352</td>
<td>-91</td>
<td>-72</td>
<td>50</td>
</tr>
<tr>
<td>SAID</td>
<td>-127</td>
<td>0</td>
<td>+351</td>
<td>-88</td>
<td>-69</td>
<td>67</td>
</tr>
<tr>
<td>diff</td>
<td>+6</td>
<td>7</td>
<td>-1</td>
<td>+3</td>
<td>+3</td>
<td>17</td>
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</table>

- difference with KH80 due the $a^+_{0+}$

- large weight of $a^+_{1+}$ $\Rightarrow$ It has to be known extremely accurately!
  $\Leftarrow$ the difference 132.7 (SAID)/131.2 (RS) translates in 5 MeV in the $\sigma_{\pi N}$
Nucleon strangeness

relate $\sigma_{\pi N}$ to strangeness content of the nucleon:

$$\sigma_{\pi N} = \frac{\hat{m}}{2m_N} \frac{\langle N|\bar{u}u + \bar{d}d - 2\bar{s}s|N\rangle}{1 - y} = \frac{\sigma_0}{1 - y}, \quad y \equiv \frac{2\langle N|\bar{s}s|N\rangle}{\langle N|\bar{u}u + \bar{d}d|N\rangle}$$

$$(m_s - m)(\bar{u}u + \bar{d}d - 2\bar{s}s) \subset \text{LQCD produces SU(3) mass splittings:}$$

$$\sigma_{\pi N} = \frac{\sigma_0}{1 - y}, \quad \sigma_0 = \frac{\hat{m}}{m_s - \hat{m}} (m_\Xi + m_\Sigma - 2m_N) \sim 26 \text{ MeV}$$

higher-order corrections: $\sigma_0 \rightarrow (36 \pm 7) \text{ MeV}$

potentially large effects

▷ from the decuplet
▷ from relativistic corrections (EOMS vs. heavy-baryon)
↩ may increase to $\sigma_0 = (58 \pm 8) \text{ MeV}$

Conclusion:

▷ $\sigma_{\pi N} = (59.1 \pm 3.5) \text{ MeV}$ not incompatible with small $y$
▷ chiral convergence of $\sigma_0$ (hence $\langle N|\bar{s}s|N\rangle$) very doubtful
Threshold parameters

- Threshold parameters defined as: \( \text{Re } f_{l\pm}^l(s) = q^2 \{ a_{l\pm}^l + b_{l\pm}^l q^2 + \cdots \} \)
- Extracted from hyperbolic sum rules

<table>
<thead>
<tr>
<th></th>
<th>RS</th>
<th>KH80</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_{0+}^+ [10^{-3} M_\pi^{-1}] )</td>
<td>-0.9 ± 1.4</td>
<td>-9.7 ± 1.7</td>
</tr>
<tr>
<td>( a_{0+}^- [10^{-3} M_\pi^{-1}] )</td>
<td>85.4 ± 0.9</td>
<td>91.3 ± 1.7</td>
</tr>
<tr>
<td>( a_{1+}^+ [10^{-3} M_\pi^{-3}] )</td>
<td>131.2 ± 1.7</td>
<td>132.7 ± 1.3</td>
</tr>
<tr>
<td>( a_{1+}^- [10^{-3} M_\pi^{-3}] )</td>
<td>-80.3 ± 1.1</td>
<td>-81.3 ± 1.0</td>
</tr>
<tr>
<td>( a_{1-}^+ [10^{-3} M_\pi^{-3}] )</td>
<td>-50.9 ± 1.9</td>
<td>-56.7 ± 1.3</td>
</tr>
<tr>
<td>( a_{1-}^- [10^{-3} M_\pi^{-3}] )</td>
<td>-9.9 ± 1.2</td>
<td>-11.7 ± 1.0</td>
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<tr>
<td>( b_{0+}^+ [10^{-3} M_\pi^{-3}] )</td>
<td>-45.0 ± 1.0</td>
<td>-44.3 ± 6.7</td>
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<tr>
<td>( b_{0+}^- [10^{-3} M_\pi^{-3}] )</td>
<td>4.9 ± 0.8</td>
<td>13.3 ± 6.0</td>
</tr>
</tbody>
</table>

- Reasonable agreement with KH80 but for the scattering lengths
- Disagreement in the scattering lengths in \( \sim 4\sigma \)
Threshold parameters

- Threshold parameters defined as: \( \text{Re } f_i^\pm (s) = q^{2l} \{ a_i^\pm + b_i^\pm q^2 + \cdots \} \)

- Extracted from hyperbolic sum rules

<table>
<thead>
<tr>
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<th>RS</th>
<th>KH80</th>
</tr>
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<tbody>
<tr>
<td>(a_0^{1/2} [10^{-3} M^{-1}_\pi])</td>
<td>169.8 ± 2.0</td>
<td>173 ± 3</td>
</tr>
<tr>
<td>(a_0^{3/2} [10^{-3} M^{-1}_\pi])</td>
<td>-86.3 ± 1.8</td>
<td>-101 ± 4</td>
</tr>
<tr>
<td>(a_1^{1/2} [10^{-3} M^{-3}_\pi])</td>
<td>-29.4 ± 1.0</td>
<td>-30 ± 2</td>
</tr>
<tr>
<td>(a_1^{3/2} [10^{-3} M^{-3}_\pi])</td>
<td>211.5 ± 2.8</td>
<td>214 ± 2</td>
</tr>
<tr>
<td>(a_1^{-1/2} [10^{-3} M^{-3}_\pi])</td>
<td>-70.7 ± 4.1</td>
<td>-81 ± 2</td>
</tr>
<tr>
<td>(a_1^{-3/2} [10^{-3} M^{-3}_\pi])</td>
<td>-41.0 ± 1.1</td>
<td>-45 ± 2</td>
</tr>
<tr>
<td>(b_0^{1/2} [10^{-3} M^{-3}_\pi])</td>
<td>-35.2 ± 2.2</td>
<td>-18 ± 12</td>
</tr>
<tr>
<td>(b_0^{3/2} [10^{-3} M^{-3}_\pi])</td>
<td>-49.8 ± 1.1</td>
<td>-58 ± 9</td>
</tr>
</tbody>
</table>

- Reasonable agreement with KH80 but for the scattering lengths
- Disagreement in the \(a_0^{3/2}\) scattering length in \(\sim 4\sigma\)
RS-eqs for $\pi N$: Range of convergence

- **Assumption:** Mandelstam analyticity
  
  \[ \Rightarrow \ T(s, t) \text{ can be written in terms double spectral densities: } \rho_{st}, \rho_{su}, \rho_{ut} \]

  \[ \begin{align*}
  T(s, t) &= \frac{1}{\pi^2} \int \int ds' du' \frac{\rho_{su}(s', u')}{(s' - s)(u' - u)} + \frac{1}{\pi^2} \int \int dt' du' \frac{\rho_{tu}(t', u')}{(t' - t)(u' - u)} + \frac{1}{\pi^2} \int \int ds' dt' \frac{\rho_{st}(s', t')}{(s' - s)(t' - t)}
  \end{align*} \]

  \[ \hookrightarrow \text{ integration ranges defined by the support of the double spectral densities } \rho \]

- **Boundaries of } \rho \text{ are given lowest lying intermediate states}

- **They limit the range of validity of the HDRS:**
  - Pw expansion converge
    \[ \Rightarrow z = \cos \theta \in \text{Lehmann ellipses} \]
  - the hyperbolae \((s - a)(u - a) = b\) does not enter any double spectral region
    \[ \Rightarrow \text{for a value of } a, \text{ constraints on } b \text{ yield ranges in } s \text{ & } t \]
Dispersion relation for the scalar form factor of the nucleon

- Unitarity relation: \( \text{Im} \sigma(t) = \frac{2}{4m^2-t} \left\{ \frac{3}{4} \sigma^\pi_t \left( F^S_\pi(t) \right)^* f_0^+(t) + \sigma^K_t \left( F^K_\pi(t) \right)^* h_0^+(t) \right\} \)

- Once subtracted dispersion relation: \( \sigma(t) = \sigma^\pi N + \frac{t}{t_\pi} \int_{t_\pi}^\infty dt' \frac{\text{Im} \sigma(t')}{t'(t'-t)} \)

- \( \Delta \sigma = \sigma(2M^2_\pi) - \sigma^\pi N \)
Dispersion relation for the $\pi N$ amplitude

- t-channel expansion of the subtracted pseudo-Born amplitude

$$\bar{D}(\nu = 0, t) = 4\pi \left\{ \frac{1}{p_t^2} f_0^+(t) + \frac{5}{2} q_t^2 f_2^+(t) + \frac{27}{8} p_t^2 q_t^4 f_4^+(t) + \frac{56}{16} p_t^4 q_t^6 f_6^+(t) + \cdots \right\}$$

- Insert t-channel RS equations for Born-term-subtracted amplitudes $f_j^+(t)$

$$\bar{D}(\nu = 0, t) = d_{00}^+ + d_{01}^+ t - 16t^2 \int_{t/\pi}^{\infty} dt' \frac{\text{Im}\bar{f}_0^+(t')}{t'^2(t' - 4m^2)(t' - t)} + \{J \geq 2\} + \{s\text{-channel integral}\}$$

- $\Delta_D = F_\pi^2 (\bar{D}(\nu = 0, t) - d_{00}^+ + d_{01}^+ t)$ from evaluation at $t = 2M_\pi^2$
Summary: $\sigma$-term corrections

- Nucleon scalar form factor

$$\Delta_{\sigma} = (13.9 \pm 0.3) \text{ MeV}$$

$$+ Z_1 \left( \frac{g^2}{4\pi} - 14.28 \right) + Z_2 \left( d_{00}^+ M_\pi + 1.46 \right) + Z_3 \left( d_{01}^+ M_\pi^3 - 1.14 \right) + Z_4 \left( b_{00}^+ M_\pi^3 + 3.54 \right)$$

$$Z_1 = 0.36 \text{ MeV} , \quad Z_2 = 0.57 \text{ MeV} , \quad Z_3 = 12.0 \text{ MeV} , \quad Z_4 = -0.81 \text{ MeV}$$

- $\pi N$ amplitude

$$\Delta_D = (12.1 \pm 0.3) \text{ MeV}$$

$$+ \hat{Z}_1 \left( \frac{g^2}{4\pi} - 14.28 \right) + \hat{Z}_2 \left( d_{00}^+ M_\pi + 1.46 \right) + \hat{Z}_3 \left( d_{01}^+ M_\pi^3 - 1.14 \right) + \hat{Z}_4 \left( b_{00}^+ M_\pi^3 + 3.54 \right)$$

$$\hat{Z}_1 = 0.42 \text{ MeV} , \quad \hat{Z}_2 = 0.67 \text{ MeV} , \quad \hat{Z}_3 = 12.0 \text{ MeV} , \quad \hat{Z}_4 = -0.77 \text{ MeV}$$

$\leftrightarrow$ most of the dependence on the $\pi N$ parameters cancels in the difference

Full Correction

$$\Delta_D - \Delta_{\sigma} = (-1.8 \pm 0.2) \text{ MeV}$$
Cheng-Dashen theorem in the presence of isospin breaking

- Define as isoscalar as 
\[ X^+ \to X^p = \frac{1}{2}(X^+_{\pi\pi} + X^+_{p\pi} + X^-_{\pi\pi} - X^-_{p\pi}), \quad X \in \{D, d_{00}, d_{01}, a_{0+} \ldots\} \]
and “isospin limit” by proton and charged pion

- Assume virtual photons to be removed

- Calculate IV corrections in SU(2) ChPT, mainly due to \( \Delta_\pi = M^2 - M^2_\pi \)
  - For the \( \sigma \) term no differences at \( \mathcal{O}(p^3) \)
    \[ \sigma_{\pi N} = \sigma_p = \sigma_N = -4c_1M^2_\pi \frac{3g^2AM^2_\pi}{64\pi F^2_\pi} (2M_\pi + M^2_\pi) + \mathcal{O}(M^4_\pi) \]
  - Slope of the scalar form factor
    \[ \Delta^p_\sigma = \sigma_p(2M^2_\pi) - \sigma_p = \frac{3g^2AM^3_\pi}{64\pi F^2_\pi} + \frac{g^2AM_\pi \Delta_\pi}{128\pi F^2_\pi} \left(-7 + \sqrt{2} \log(3 + 2\sqrt{2})\right) + \mathcal{O}(M^4_\pi) \]
  - Similarly for \( \Delta^0_\pi \)
    \[ \Delta^0_D = F^2_\pi \left\{ \tilde{D}_p(0, 2M^2_\pi) - d^p_{00} - 2M^2_\pi d^p_{01} \right\} = \frac{23g^2AM^3_\pi}{384\pi F^2_\pi} + \frac{g^2AM_\pi \Delta_\pi}{256\pi F^2_\pi} \left(3 + 4\sqrt{2} \log \left(1 + \sqrt{2}\right)\right) + \mathcal{O}(M^4_\pi) \]
Taking everything together

\[
\sigma_{\pi N} = F_{\pi}^2 (d_{00}^p + 2M_{\pi}^2 d_{01}^p) - \Delta_R + \Delta_D - \Delta_{\sigma} + (\Delta_D^p - \Delta_D) - (\Delta_{\sigma}^p - \Delta_{\sigma}) \\
+ \sigma_p(2M_{\pi}^2) + F_{\pi}^2 \bar{D}(0,2M_{\pi}^2)
\]

\[
= F_{\pi}^2 (d_{00}^p + 2M_{\pi}^2 d_{01}^p) - \underbrace{\Delta_R}_{\lesssim 2 \text{MeV}} + \underbrace{\Delta_D - \Delta_{\sigma}}_{(-1.8 \pm 0.2) \text{MeV}} + \underbrace{\frac{81g_a^2 M_{\pi} \Delta_{\pi}}{256\pi F_{\pi}^2}}_{3.4 \text{MeV}} + \underbrace{\frac{e^2}{2} F_{\pi}^2 (4f_1 + f_2)}_{(-0.4 \pm 2.2) \text{MeV}}
\]

\(\hookrightarrow\) sizable corrections from \(\Delta_{\pi}\) increasing the value of the \(\sigma_{\pi N}\)
Electromagnetic nucleon form factor: \[
\langle N(p')|j_{em}^\mu|N(p)\rangle = \bar{u}(p') \left[ F_1^N(t) \gamma^\mu + \frac{i \sigma^\mu \nu q_\nu}{2m_N} F_2^N(t) \right] u(p),
\]
\[
G_E^N(t) = F_1^N(t) + \frac{t}{4m_N^2} F_2^N(t), \quad G_M^N(t) = F_1^N(t) + F_2^N(t).
\]

first inelastic correction from $\pi\pi$ continuum
\[
\text{Im } G_E^V(t) = \frac{q_t^3}{m_N \sqrt{t}} (F_V^\pi(t))^\ast f_+^1(t) \theta(t - t_\pi)
\]
\[
\text{Im } G_M^V(t) = \frac{q_t^3}{\sqrt{2t}} (F_V^\pi(t))^\ast f_-^1(t) \theta(t - t_\pi)
\]

↕ rigorous constraint fixed from:

▷ RS t-channel partial waves
▷ pion form factor

update of Höhler spectral functions, including also isospin breaking
\pi\pi\text{-continuum: } \rho - \omega \text{ mixing}

- Isovector and isoscalar nucleon form factor

\[ F_i^s(t) = \frac{1}{2} (F_i^p(t) + F_i^n(t)), \quad F_i^v(t) = \frac{1}{2} (F_i^p(t) - F_i^n(t)) \]

\[ \text{Im } G_E^v(t) = \frac{q_i^3}{m_N \sqrt{t}} |\Omega^1_1(t)||f^1_+(t)| \theta (t - t_\pi) \]
\[ \times \left( 1 + \alpha t + \frac{\varepsilon t}{M_\omega^2 + i M_\omega \Gamma_\omega - t} \right) \]
\[ + \varepsilon \text{ Im} \left( \frac{t}{M_\omega^2 - i M_\omega \Gamma_\omega - t} \right) \]
\[ \times \frac{1}{\pi} \int_{t_\pi}^{\infty} \frac{q_i^3}{m_N \sqrt{t'}} |\Omega^1_1(t')||f^1_+(t')| \]
\[ \times \frac{1}{t' - t - i \varepsilon} \frac{m_N \sqrt{t'}}{t'} \]

\[ \text{Im } G_M^v(t) = \frac{q_i^3}{\sqrt{2} t} |\Omega^1_1(t)||f^1_-(t)| \theta (t - t_\pi) \]
\[ \times \left( 1 + \alpha t + \frac{\varepsilon t}{M_\omega^2 + i M_\omega \Gamma_\omega - t} \right) \]
\[ + \varepsilon \text{ Im} \left( \frac{t}{M_\omega^2 - i M_\omega \Gamma_\omega - t} \right) \]
\[ \times \frac{1}{\pi} \int_{t_\pi}^{\infty} \frac{q_i^3}{\sqrt{2} t'} |\Omega^1_1(t')||f^1_-(t')| \]
\[ \times \frac{1}{t' - t - i \varepsilon} \frac{\sqrt{2} t'}{t'} \]
Chiral Low Energy Constants with $\Delta$'s

<table>
<thead>
<tr>
<th>$N^2$ LO</th>
<th>$Q^3$</th>
<th>$\varepsilon^3$</th>
<th>$Q^3$</th>
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| J. Ruiz de Elvira (ITP) | Roy-Steiner-equation analysis | $\pi N$ scattering | HADRON 2017 | 74 |
## Threshold kinematics from subthreshold with Δ’s

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<tr>
<th></th>
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<td>$Q^3$</td>
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<td>$10^{-3} M_{\pi^{-}}^{-1}$</td>
<td>0.5</td>
<td>−9.8(10.9)</td>
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<td>−14.1</td>
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<td>26.4(1.0)</td>
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<td>17.6(8)</td>
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<td>58.6</td>
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<td>$a_{1+}^+$</td>
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<td>134.3</td>
<td>136.0(9.7)</td>
<td>132.1</td>
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<td>−73.7</td>
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<td>−23.7</td>
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<td>$b_{0+}^+$</td>
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<td>−44.5</td>
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<td>−31.6</td>
<td>7.1(2.3)</td>
<td>−65.2</td>
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</table>
Goldberger-Miyazawa-Oehme sum rule

- Fixed-$t$ dispersion relations at threshold $\leftrightarrow$ GMO sum rule

$$
\frac{g^2}{4\pi} = \left( \left( \frac{m_p + m_n}{M_\pi} \right)^2 - 1 \right) \left\{ \left( 1 + \frac{M_\pi}{m_p} \right) \frac{M_\pi}{4} \left( a_{\pi^-p} - a_{\pi^+p} \right) - \frac{M_\pi^2}{2} J^- \right\} 
$$

$$
= 13.69 \pm 0.12 \pm 0.15
$$

$$
J^- = \frac{1}{4\pi^2} \int_0^\infty dk \frac{\sigma_{\pi^-p}^{\text{tot}}(k) - \sigma_{\pi^+p}^{\text{tot}}(k)}{\sqrt{M_\pi^2 + k^2}}
$$

- $J^-$ known very accurately

- other determinations

<table>
<thead>
<tr>
<th></th>
<th>de Swart et al. 97</th>
<th>Arndt et al. 94</th>
<th>Ericson et al. 02</th>
<th>Bugg et al. 73</th>
<th>KH80</th>
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<td>method</td>
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<td>$\pi N$</td>
<td>GM0</td>
<td>$\pi N$</td>
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<td>$g^2/4\pi$</td>
<td>$13.54 \pm 0.05$</td>
<td>$13.75 \pm 0.15$</td>
<td>$14.11 \pm 0.20$</td>
<td>$14.30 \pm 0.18$</td>
<td>$14.28$</td>
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</table>

- With KH80 scattering lengths $g^2/4\pi = 14.28$ is reproduced exactly

$\leftrightarrow$ discrepancy related to old scattering length values
Effective Lagrangian

\[ \mathcal{L} = \frac{C^{SS}_{qq}}{\Lambda^3} \bar{\chi} \chi \bar{q} q + \frac{C^{VV}_{qq}}{\Lambda^2} \bar{\chi} \gamma^\mu \chi \bar{q} q + \frac{C^{S}_{gg}}{\Lambda^3} \bar{\chi} \chi G_{\mu\nu} G^{\mu\nu} \]

WIMP $\chi$ Dirac fermion and SM singlet

Spin–independent cross section at vanishing momentum transfer

\[ \sigma_{SI}^N = \frac{\mu_\chi^2}{\Lambda^4} \left| \left( \frac{m_N}{\Lambda} \right) C^{SS}_{qq} f_q^N - 12\pi C^{S}_{gg} f_Q^N \right| + C^{VV}_{qq} f_V^N \]

\[ \mu_\chi = \frac{m_\chi m_N}{m_\chi + m_N} \quad f_q^N = 2 \quad f_q^N = \frac{\sigma_{\pi N}(1 - \xi)}{m_N} + \Delta f_q^N \]

nucleon-matrix elements dominated by $\sigma_{\pi N}$
Roy-equations: $\pi\pi$ results

Solution for the $\pi\pi$ S0-wave

Garcia-Martin, Kaminski, Pelaez, JRE (2011)