# chiral MODEL for the $D^+ \to K^+ K^- K^+$ decay amplitude

R. Aoude, P.C.M. Magalhães, A.C. dos Reis
Centro Brasilero de Pesquisas Físicas - Rio de Janeiro

M.R. Robilotta
Instituto de Física - Universidade de São Paulo

Brazil

# Multi-Meson

chiral MODEL for the $D^+ \to K^+ K^- K^+$ decay amplitude

R. Aoude, P.C.M. Magalhães, A.C. dos Reis
Centro Brasilero de Pesquisas Físicas - Rio de Janeiro

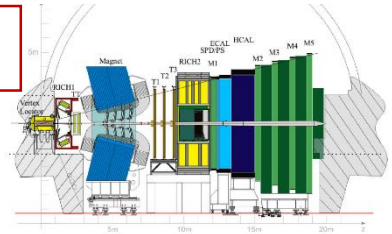
M.R. Robilotta
Instituto de Física - Universidade de São Paulo

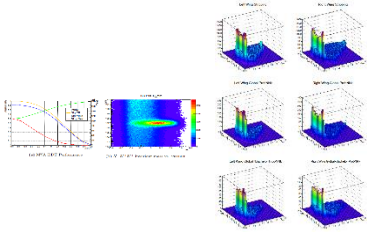
Brazil

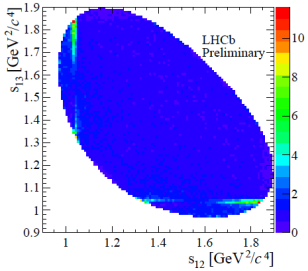
# 3 body decays

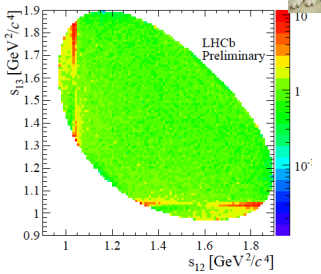
# 3 body decays

## data



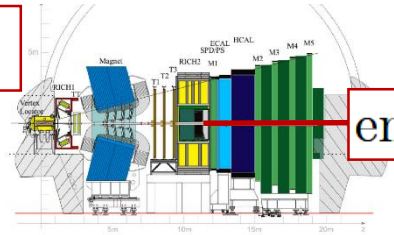





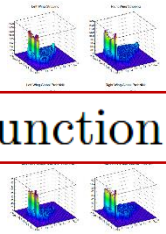



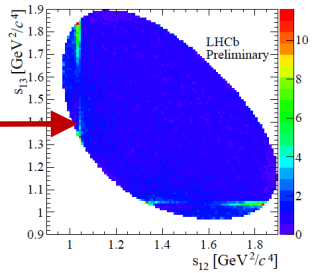
3 body decays

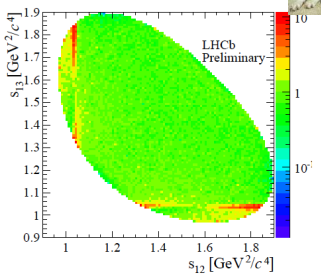
data




empirical function

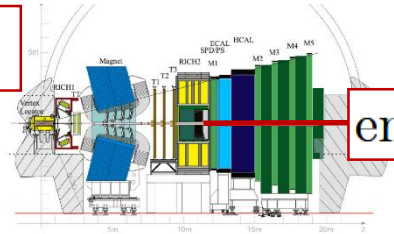




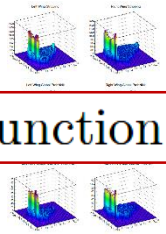



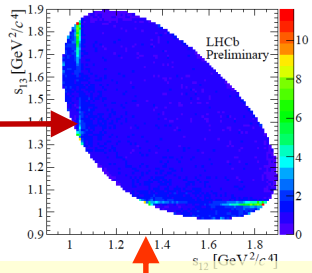
3 body decays

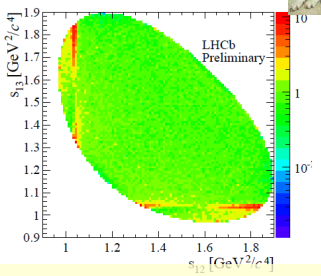
data




empirical function





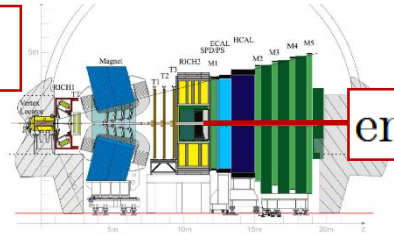

fit

representation

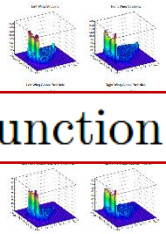
$$F(m_{12}^2, m_{13}^2)$$

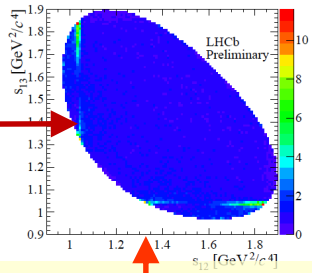
# 3 body decays

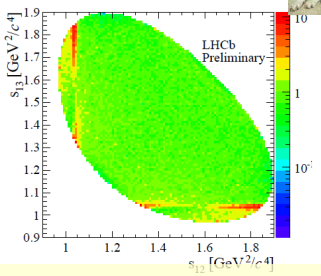
**data**




empirical function






physical picture

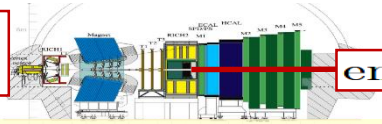
scattering data

$F(m_{12}^2, m_{13}^2)$

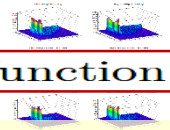
fit

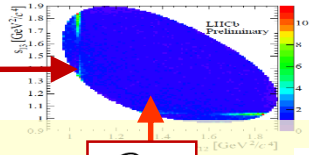
representation

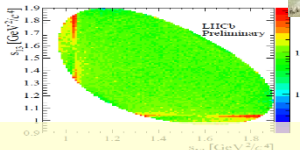
# 3 body decays



empirical function






data

physical picture

scattering data

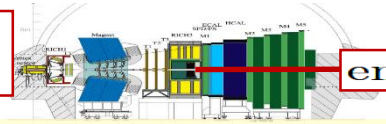
direct ?

$$F(m_{12}^2, m_{13}^2)$$

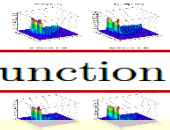
fit

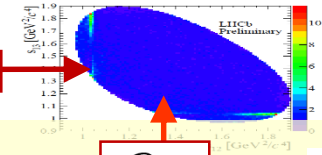
representation

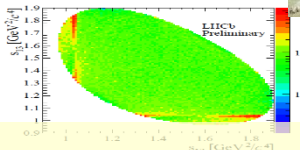
# 3 body decays



empirical function






data

physical picture

scattering data

$F(m_{12}^2, m_{13}^2)$

fit

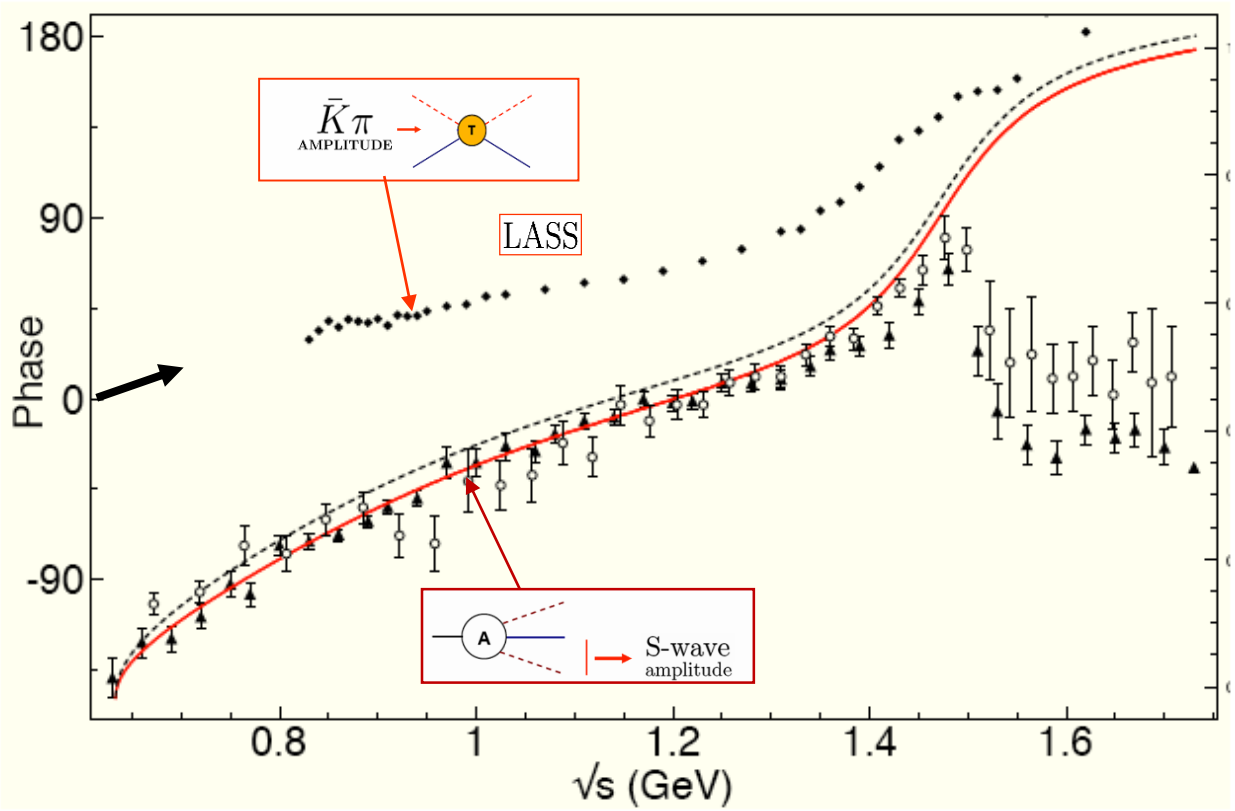
representation

direct ?

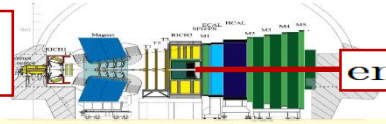
$$D^+ \to K^- \pi^+ \pi^+$$

data

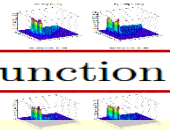
E.M. Aitala *et al.* [E791 Collaboration], Phys. Rev. D**73**, 032004 (2006); Erratum-ibid. D **74**, 059901 (2006)
J.M. Link *et al.* [FOCUS Collaboration], Phys. Lett. B **681**, (2009) 14

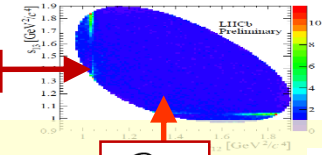


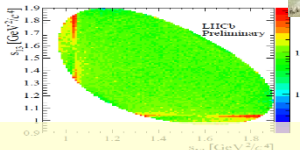
# 3 body decays



empirical function






data

physical picture

scattering data

$F(m_{12}^2, m_{13}^2)$

fit

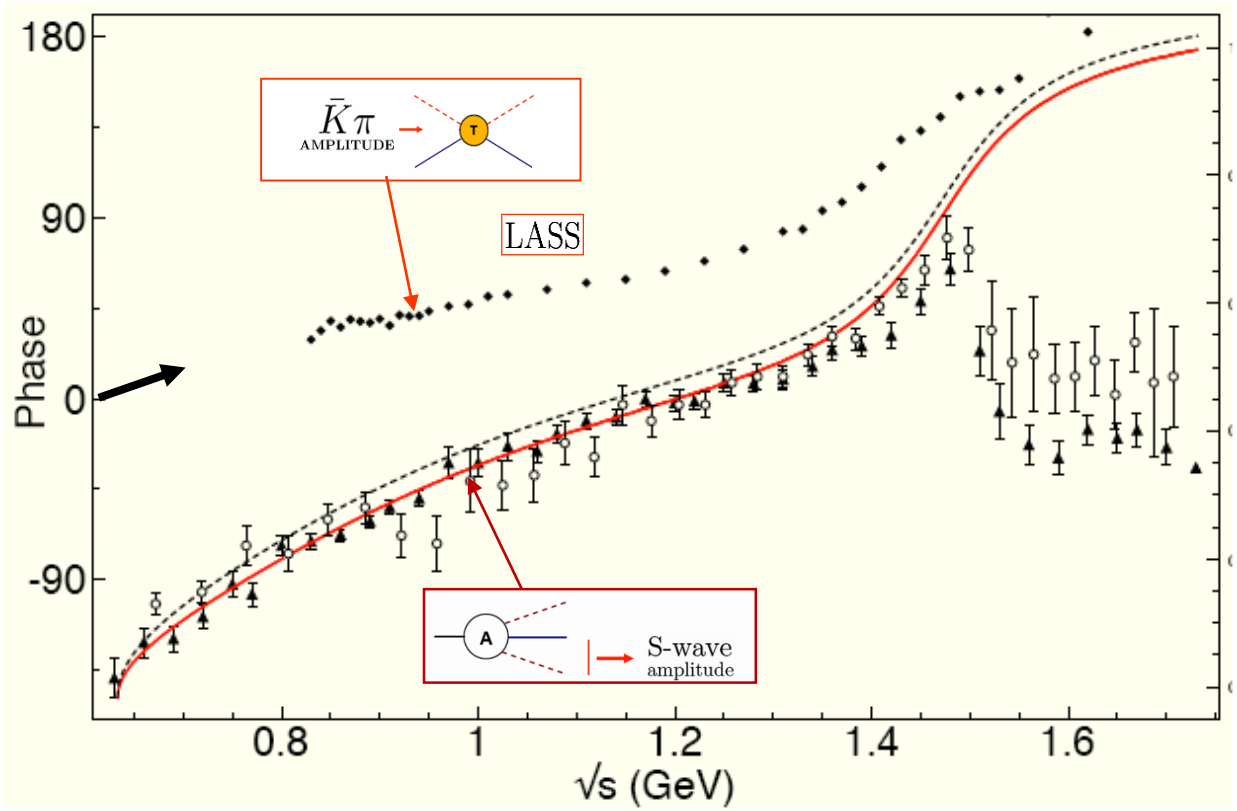
representation

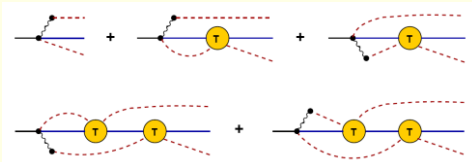
direct ?

$$D^+ \to K^- \pi^+ \pi^+$$

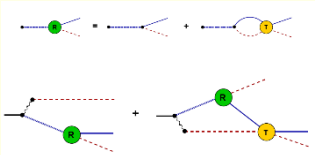
data

E.M. Aitala *et al.* [E791 Collaboration], Phys. Rev. D**73**, 032004 (2006); Erratum-ibid. D **74**, 059901 (2006)
J.M. Link *et al.* [FOCUS Collaboration], Phys. Lett. B **681**, (2009) 14



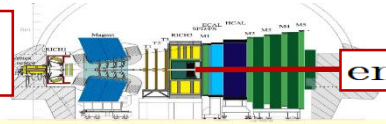
axial



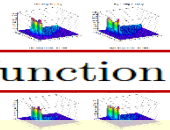
PHYSICAL REVIEW D **84**, 094001 (2011)

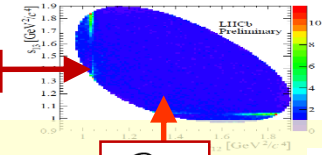
P.C. Magalhães, M.R. R., K.S.F.F. Guimarães, T. Frederico, W.S. de Paula, I. Bediaga, A.C. dos Reis, C.M. Maekawa and G.R.S. Zarnauskas

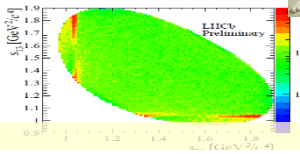
# 3 body decays



empirical function






data

physical picture

scattering data

direct ?

$$F(m_{12}^2, m_{13}^2)$$

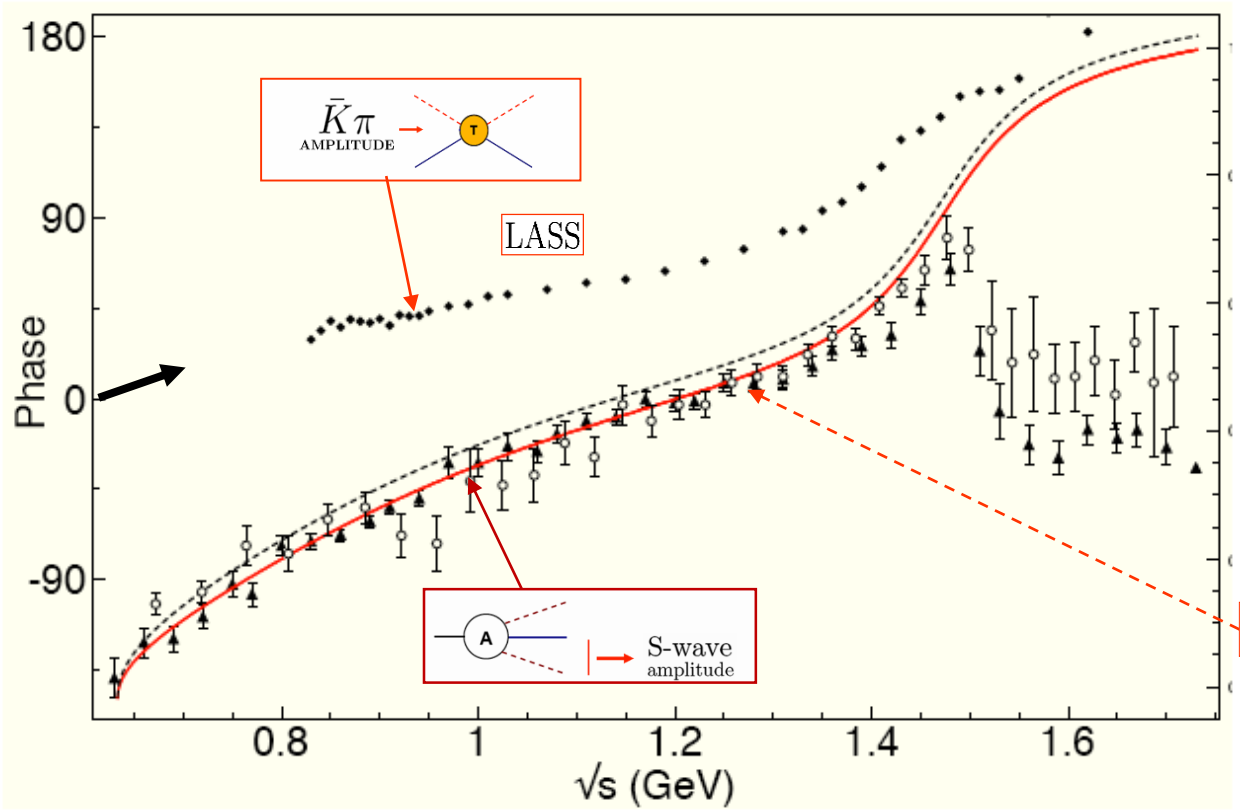
fit

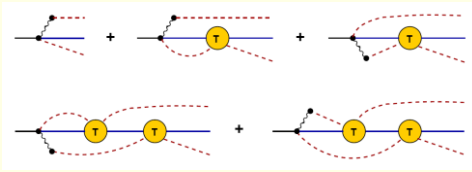
representation

$$D^+ \to K^- \pi^+ \pi^+$$

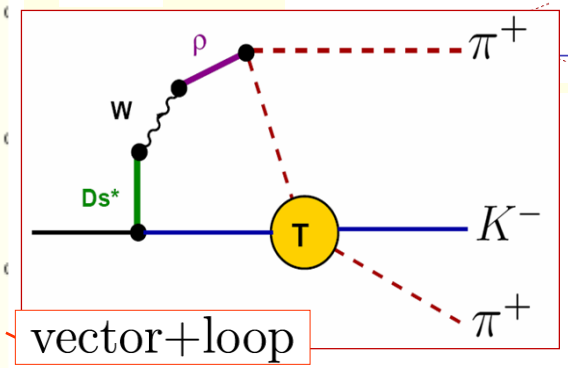
data

E.M. Aitala *et al.* [E791 Collaboration], Phys. Rev. D**73**, 032004 (2006); Erratum-ibid. D **74**, 059901 (2006)
J.M. Link *et al.* [FOCUS Collaboration], Phys. Lett. B **681**, (2009) 14



axial

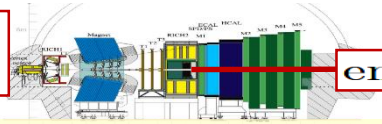



vector+loop

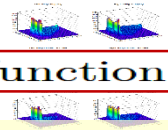
PHYSICAL REVIEW D **84**, 094001 (2011)

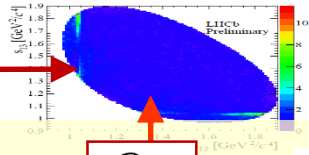
P.C. Magalhães, M.R. R., K.S.F.F. Guimarães, T. Frederico, W.S. de Paula, I. Bediaga, A.C. dos Reis, C.M. Maekawa and G.R.S. Zarnauskas

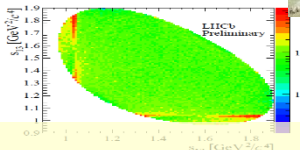
# 3 body decays



empirical function






data

physical picture

scattering data

direct ?

$$F(m_{12}^2, m_{13}^2)$$

fit

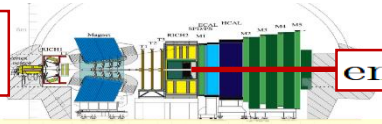
representation

# 3 body decays

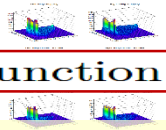
data

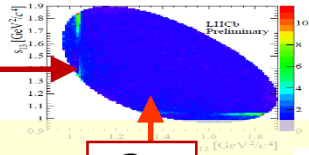
physical picture

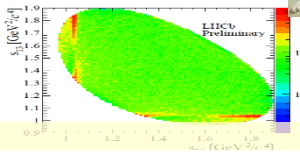
scattering data



empirical function





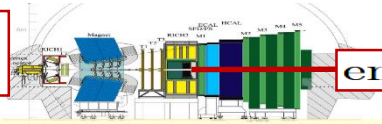

fit

representation

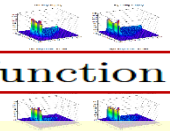
MODEL

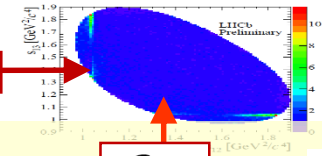
$F(m_{12}^2, m_{13}^2)$

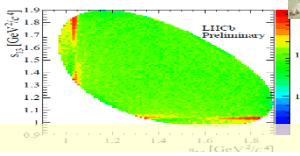
# 3 body decays



empirical function






data

physical picture

scattering data

$F(m_{12}^2, m_{13}^2)$

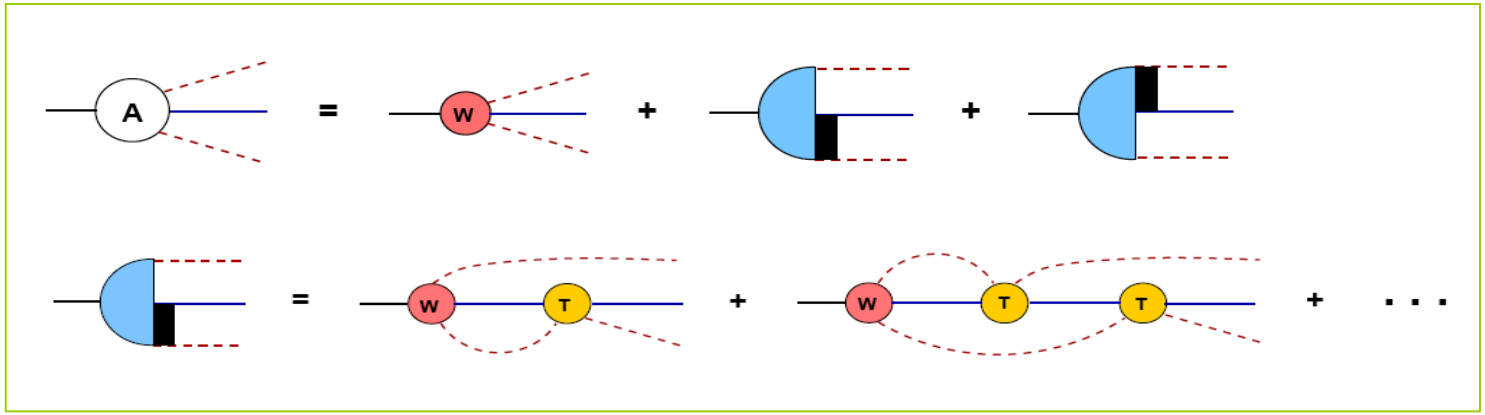
MODEL

fit

representation

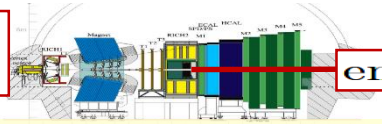
# 3 body unitarity

**Fadeev series**

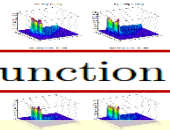


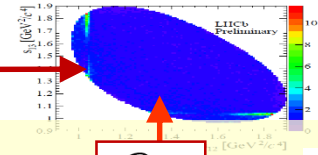
# 3 body decays

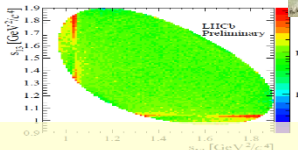
data



empirical function






physical picture

scattering data

$F(m_{12}^2, m_{13}^2)$

MODEL

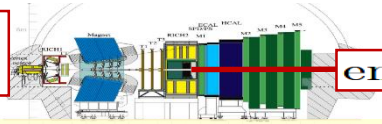
fit

representation

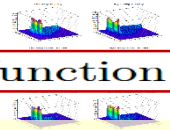
2+1 ~ unitarity

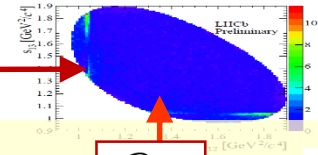
# 3 body decays

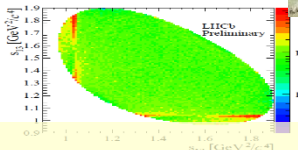
data



empirical function






physical picture

scattering data

$F(m_{12}^2, m_{13}^2)$

MODEL

fit

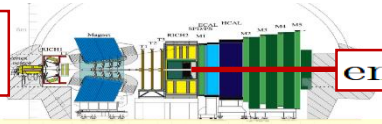
representation

2+1 ~ unitarity

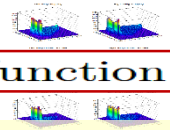
isobar

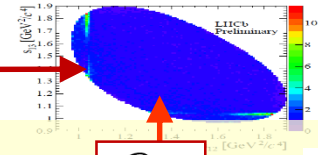
# 3 body decays

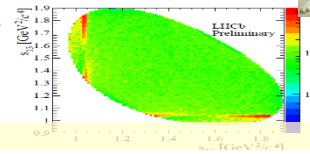
data



empirical function






physical picture

scattering data

$F(m_{12}^2, m_{13}^2)$

MODEL

fit

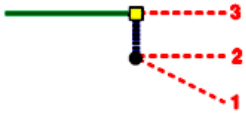
representation

2+1 ~ unitarity

isobar

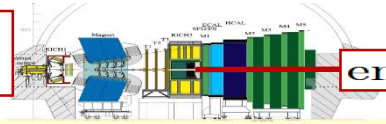
non-resonant background


$$T = c^{NR}\, A^{NR} + \sum c_k\, A_k$$

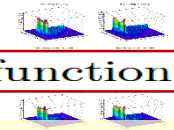


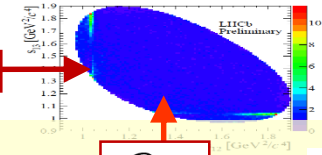

resonant

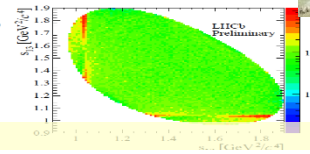
# 3 body decays



empirical function






data

physical picture

scattering data

$F(m_{12}^2, m_{13}^2)$

MODEL

fit

representation

2+1 ~ unitarity

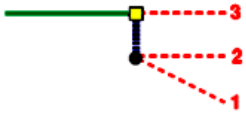
isobar

non-resonant background


constant

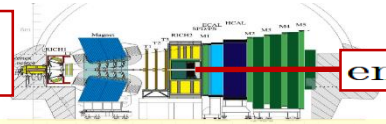
$$T = c^{NR}\, A^{NR} + \sum c_k\, A_k$$

$$T = c^{NR}\, [1] + \sum e^{i\,\delta_k}\, A_k$$

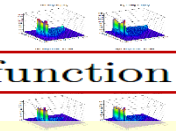


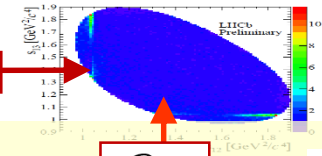

resonant

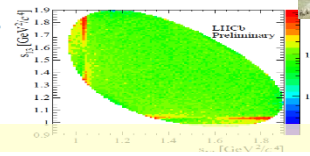
# 3 body decays



empirical function







data

physical picture

scattering data

$F(m_{12}^2, m_{13}^2)$

MODEL

fit

representation

2+1 ~ unitarity

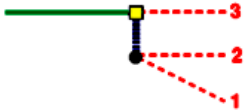
isobar

non-resonant background


constant

$$T = c^{NR}\, A^{NR} + \sum c_k\, A_k$$

$$T = c^{NR}\, [1] + \sum e^{i\,\delta_k}\, A_k$$




resonant

$$A_k(s_{12}, s_{23}) = [\, F\, F\,] \times [\,\text{angular distribution}\,] \times (BW)_k$$

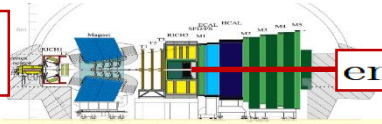
decay vertices

line shape

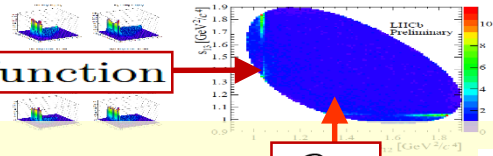
$$(BW)_k = \frac{1}{m_k^2 - s - i\, m_k\, \Gamma_k}$$

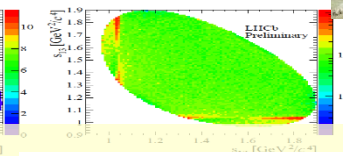
# 3 body decays

data



empirical function





physical picture

scattering data

$F(m_{12}^2, m_{13}^2)$

MODEL

fit

representation

2+1 ~ unitarity

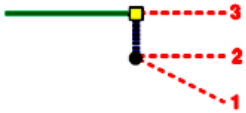
isobar

non-resonant background


constant

$$T = c^{NR}\, A^{NR} + \sum c_k\, A_k$$

$$T = c^{NR}\, [1] + \sum e^{i\,\delta_k}\, A_k$$




resonant

$$A_k(s_{12}, s_{23}) = [\, F\, F\,] \times [\,\text{angular distribution}\,] \times (BW)_k$$

decay vertices

line shape

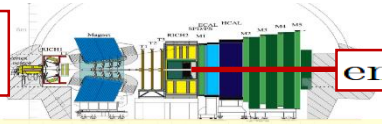
$$(BW)_k = \frac{1}{m_k^2 - s - i\, m_k\, \Gamma_k}$$

sums of BWs do violate unitarity !!!

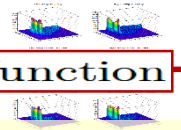
two-body

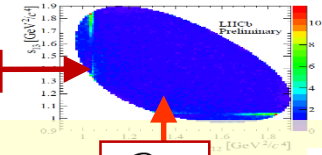
single channel

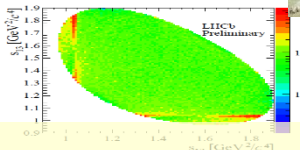
# 3 body decays



empirical function







data

physical picture

scattering data

$F(m_{12}^2, m_{13}^2)$

MODEL

fit

representation

2+1 ~ unitarity

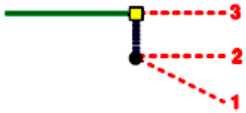
isobar

non-resonant background

constant

$$T = c^{NR}\, A^{NR} + \sum c_k\, A_k$$

$$T = c^{NR}\, [1] + \sum e^{i\,\delta_k}\, A_k$$



resonant

$$A_k(s_{12}, s_{23}) = [\, F\, F\,] \times [\,\text{angular distribution}\,] \times (BW)_k$$

decay vertices

line shape

$$(BW)_k = \frac{1}{m_k^2 - s - i\, m_k\, \Gamma_k}$$

- IM works nevertheless because weights are flexible !!!

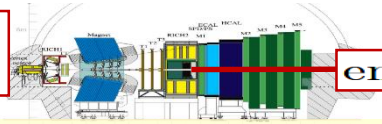
sums of BWs do violate unitarity !!!

two-body

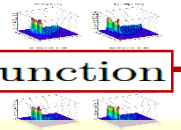
single channel

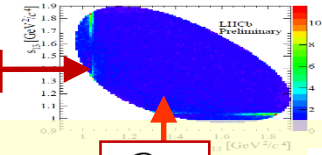
3 body decays

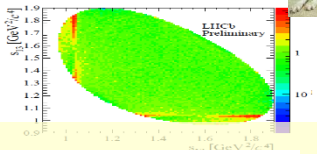
data



empirical function







physical picture

scattering data

?

$F(m_{12}^2, m_{13}^2)$

MODEL

fit

representation

2+1 ~ unitarity

isobar

| Resonance | Magnitude | Phase (°) | Fraction (%) |
|---|---|---|---|
| $\rho(770)$ | 1 [fixed] | 0 [fixed] | 24.1±0.3 |
| $f_0(980)$ | 3.9±0.02 | -157.9±0.5 | 8.1±0.2 |
| $f_2(1270)$ | 1.1±0.01 | 89±0.5 | 14.5±0.2 |
| $\rho(1450)$ | 0.7±0.03 | -80.2±2.5 | 0.4±0.1 |
| $f_0(1370)$ | 0.9±0.05 | 184.4±3.3 | 0.4±0.1 |
| $\sigma(500)$ | 23.2±0.2 | -88.1±0.4 | 58.2±1.5 |
| NR | 10.1±0.2 | -148.3±1.1 | 7.5±0.6 |
| $f_0(1500)$ | 2.1±0.04 | 180.5±1.1 | 2.9±0.2 |
| Total | | | 116.02 |

- IM works nevertheless because weights are flexible !!!

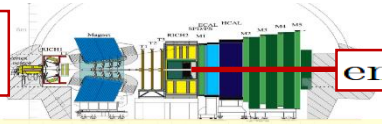
sums of BWs do violate unitarity !!!

two-body

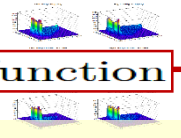
single channel

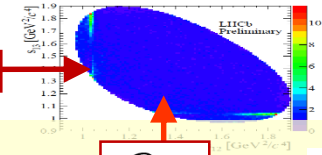
3 body decays

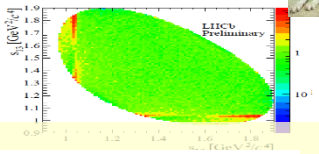
data



empirical function






physical picture

scattering data

?

$F(m_{12}^2, m_{13}^2)$

MODEL

fit

representation

2+1 ~ unitarity

isobar

data summary

| Resonance | Magnitude | Phase (°) | Fraction (%) |
|---|---|---|---|
| $\rho(770)$ | 1 [fixed] | 0 [fixed] | 24.1±0.3 |
| $f_0(980)$ | 3.9±0.02 | -157.9±0.5 | 8.1±0.2 |
| $f_2(1270)$ | 1.1±0.01 | 89±0.5 | 14.5±0.2 |
| $\rho(1450)$ | 0.7±0.03 | -80.2±2.5 | 0.4±0.1 |
| $f_0(1370)$ | 0.9±0.05 | 184.4±3.3 | 0.4±0.1 |
| $\sigma(500)$ | 23.2±0.2 | -88.1±0.4 | 58.2±1.5 |
| NR | 10.1±0.2 | -148.3±1.1 | 7.5±0.6 |
| $f_0(1500)$ | 2.1±0.04 | 180.5±1.1 | 2.9±0.2 |
| Total | | | 116.02 |

- IM works nevertheless because weights are flexible !!!

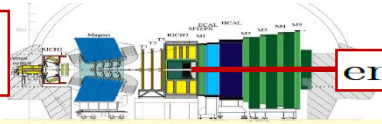
sums of BWs do violate unitarity !!!

two-body

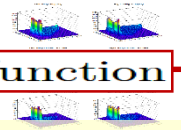
single channel

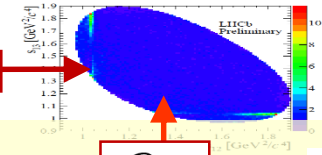
# 3 body decays

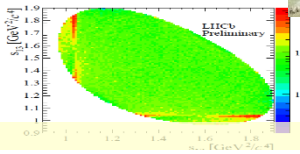
data



empirical function






physical picture

scattering data

?

$F(m_{12}^2, m_{13}^2)$

MODEL

fit

representation

2+1 ~ unitarity

isobar

data summary

| Resonance | Magnitude | Phase (°) | Fraction (%) |
|---|---|---|---|
| $\rho(770)$ | 1 [fixed] | 0 [fixed] | 24.1±0.3 |
| $f_0(980)$ | 3.9±0.02 | -157.9±0.5 | 8.1±0.2 |
| $f_2(1270)$ | 1.1±0.01 | 89±0.5 | 14.5±0.2 |
| $\rho(1450)$ | 0.7±0.03 | -80.2±2.5 | 0.4±0.1 |
| | 0.9±0.05 | 184.4±3.3 | 0.4±0.1 |
| | 23.2±0.2 | -88.1±0.4 | 58.2±1.5 |
| | 10.1±0.2 | -148.3±1.1 | 7.5±0.6 |
| | 2.1±0.04 | 180.5±1.1 | 2.9±0.2 |
| Total | | | 116.02 |

meaning?

- IM works nevertheless because weights are flexible !!!

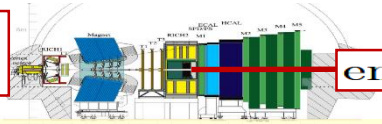
sums of BWs do violate unitarity !!!

two-body

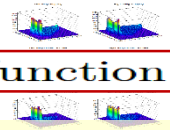
single channel

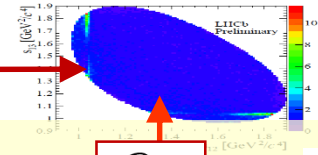
3 body decays

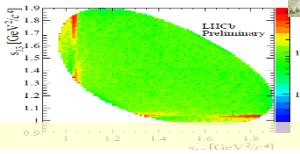
data



empirical function






physical picture

scattering data

$F(m_{12}^2, m_{13}^2)$

MODEL

fit

representation

2+1 ~ unitarity

?

isobar

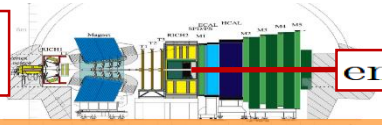
meaning

please wait for

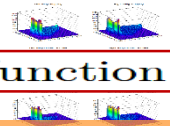
conclusions

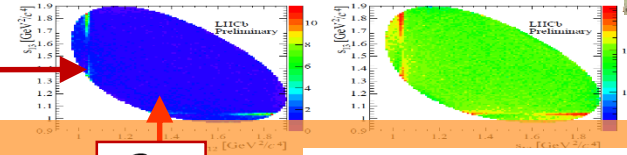
# 3 body decays

data



empirical function






physical picture

scattering data

$F(m_{12}^2, m_{13}^2)$

MODEL

fit

representation

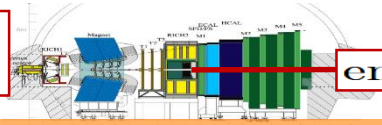
2+1 ~ unitarity

MULTI MESON

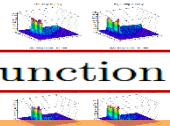
MMM - triple M

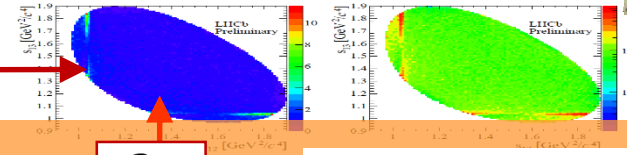
# 3 body decays

data



empirical function






physical picture

scattering data

$F(m_{12}^2, m_{13}^2)$

MODEL

fit

representation

2+1 ~ unitarity

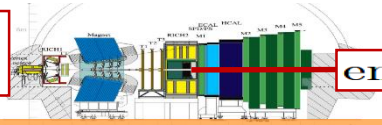
MULTI MESON

MMM - triple M

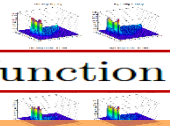
lagrangian

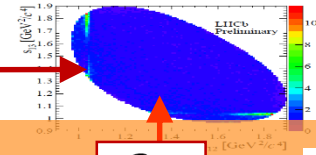
3 body decays

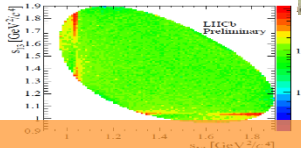
data



empirical function






physical picture

scattering data

$F(m_{12}^2, m_{13}^2)$

MODEL

fit

representation

2+1 ~ unitarity

MULTI MESON

MMM - triple M

lagrangian

many other processes

free parameters

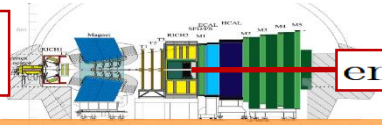
physical meaning

masses

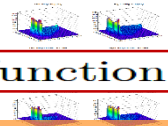
coupling constants

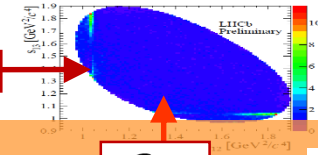
# 3 body decays

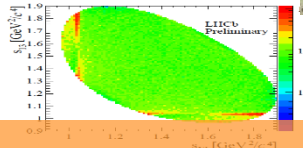
data



empirical function






physical picture

scattering data

$F(m_{12}^2, m_{13}^2)$

MODEL

fit

representation

2+1 ~ unitarity

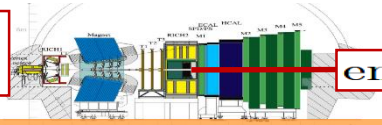
MULTI MESON

MMM - triple M

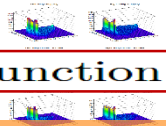
lagrangian

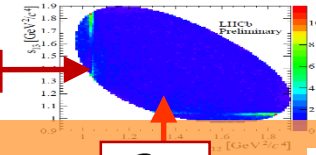
chiral symmetry

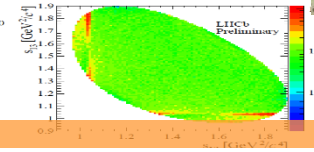
# 3 body decays



empirical function






data

physical picture

scattering data

$F(m_{12}^2, m_{13}^2)$

MODEL

fit

representation

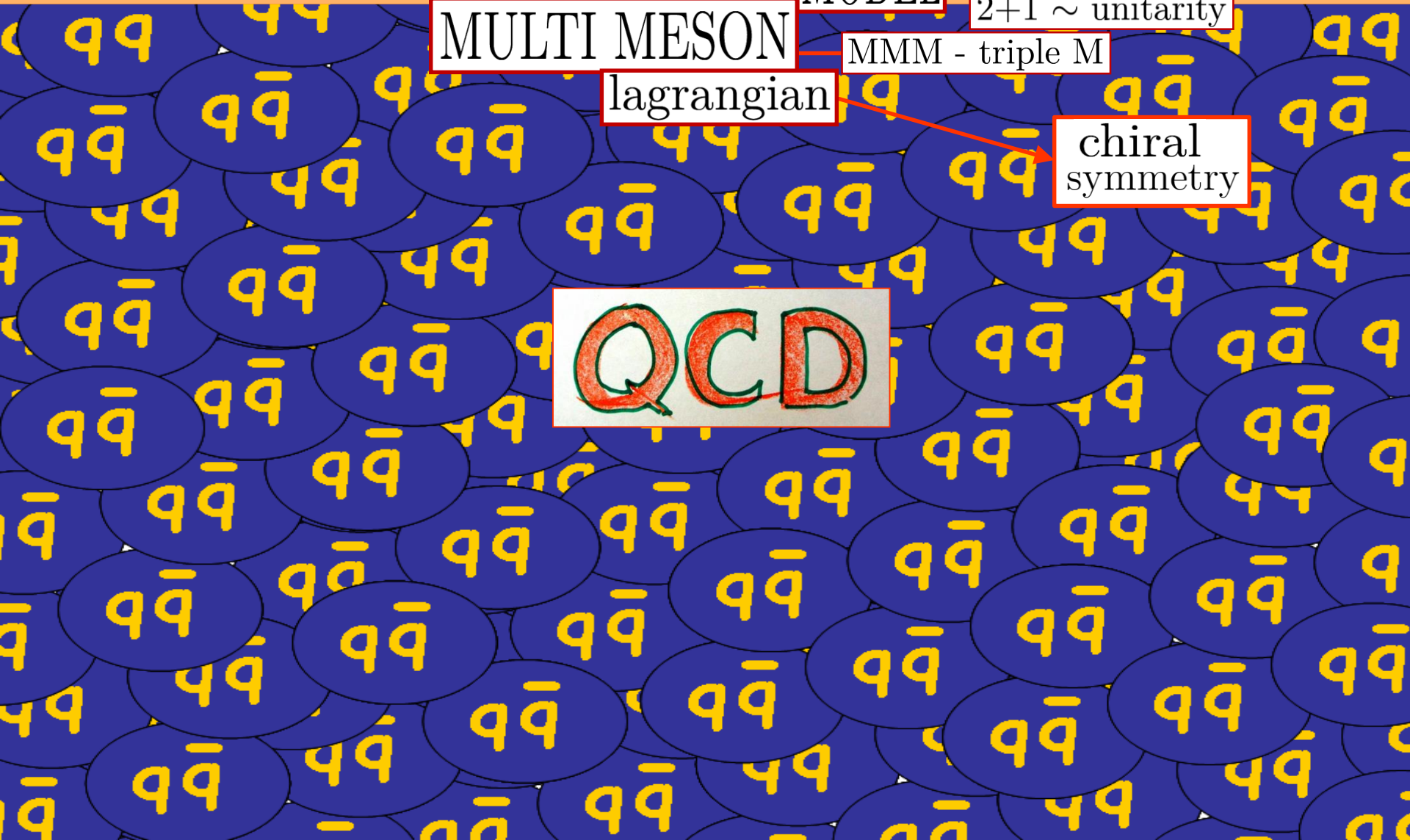
2+1 ~ unitarity

MULTI MESON

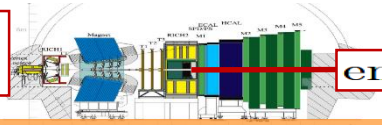
MMM - triple M

lagrangian

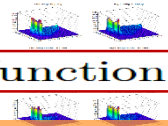
chiral symmetry

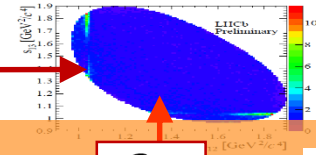


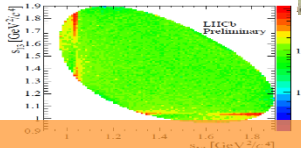
# 3 body decays



empirical function






data

physical picture

scattering data

$F(m_{12}^2, m_{13}^2)$

MODEL

fit

representation

2+1 ~ unitarity

MULTI MESON

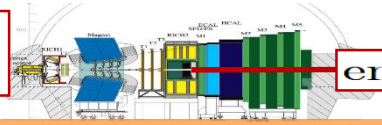
MMM - triple M

lagrangian

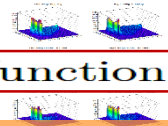
chiral symmetry

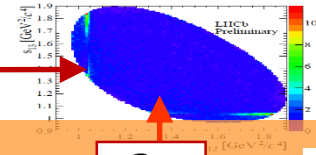
meson-meson interactions

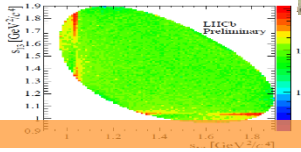
# 3 body decays



empirical function






data

physical picture

scattering data

$F(m_{12}^2, m_{13}^2)$

MODEL

fit

representation

2+1 ~ unitarity

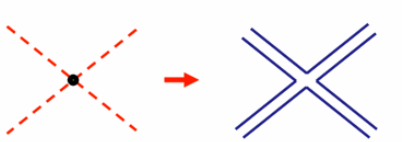
MULTI MESON

MMM - triple M

lagrangian

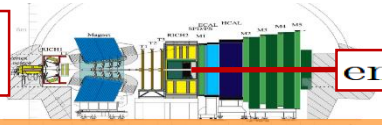
chiral symmetry

meson-meson interactions

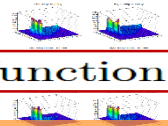


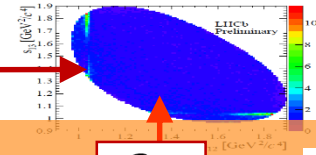
leading term

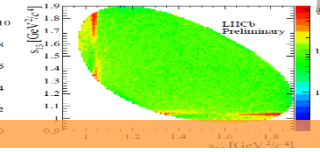
# 3 body decays



empirical function







data

physical picture

scattering data

$F(m_{12}^2, m_{13}^2)$

MODEL

fit

representation

2+1 ~ unitarity

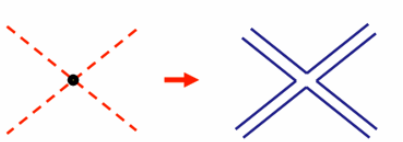
MULTI MESON

MMM - triple M

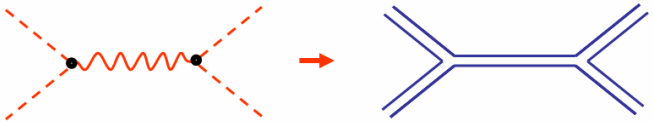
lagrangian

chiral symmetry

meson-meson interactions

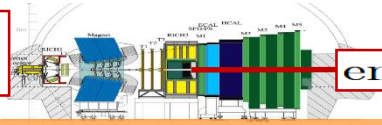


leading term

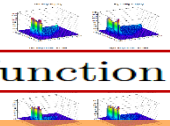


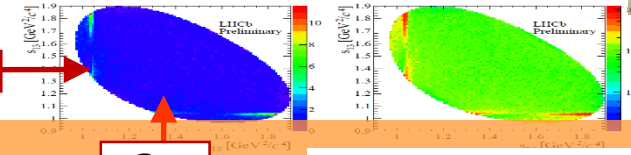
resonance

# 3 body decays



empirical function





data

physical picture

scattering data

$F(m_{12}^2, m_{13}^2)$

MODEL

fit

representation

2+1 ~ unitarity

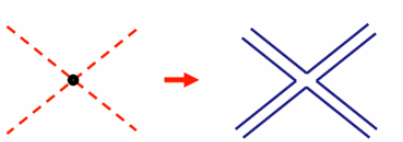
MULTI MESON

MMM - triple M

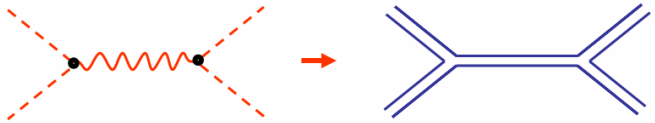
lagrangian

chiral symmetry

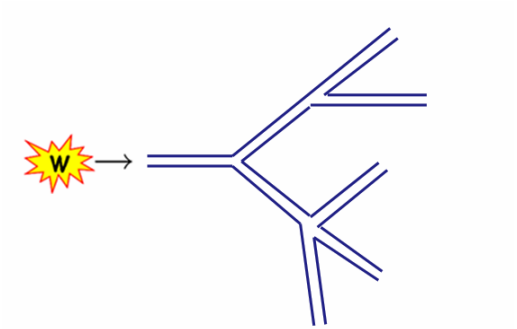
meson-meson interactions



leading term



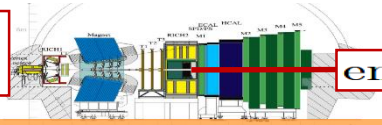
resonance



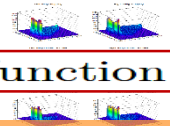

hadronization of WEAK CURRENTS

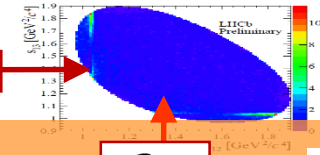
3 body decays

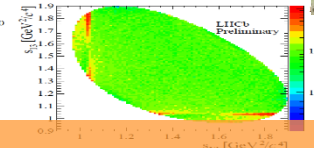
data



empirical function







physical picture

scattering data

$F(m_{12}^2, m_{13}^2)$

MODEL

fit

representation

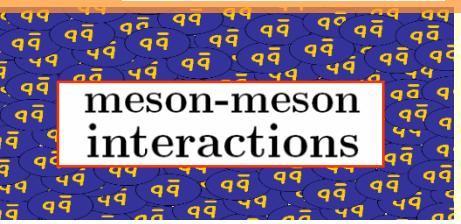
2+1 ~ unitarity

MULTI MESON

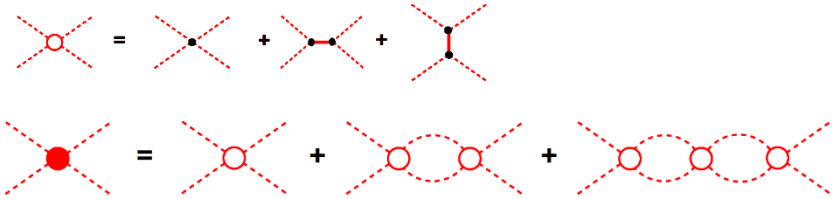
MMM - triple M

lagrangian

chiral symmetry



meson-meson interactions



precision

loops

renormalization

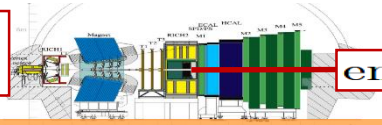
perturbative

low energies

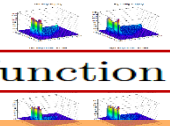
resonances

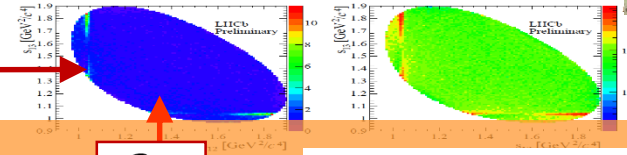
energy

# 3 body decays



empirical function






data

physical picture

scattering data

$F(m_{12}^2, m_{13}^2)$

MODEL

fit

representation

2+1 ~ unitarity

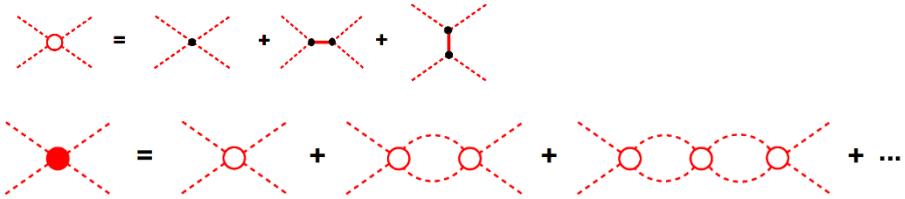
MULTI MESON

MMM - triple M

lagrangian

chiral symmetry

meson-meson interactions



precision

loops

renormalization

perturbative

non-perturbative

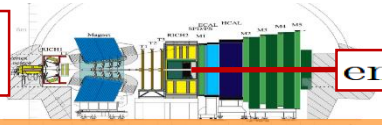
low energies

resonances

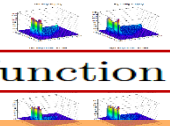
energy

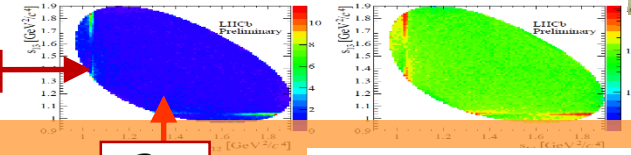
3 body decays

data



empirical function





physical picture

scattering data

$F(m_{12}^2, m_{13}^2)$

MODEL

fit

representation

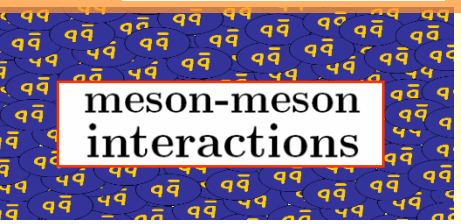
2+1 ~ unitarity

MULTI MESON

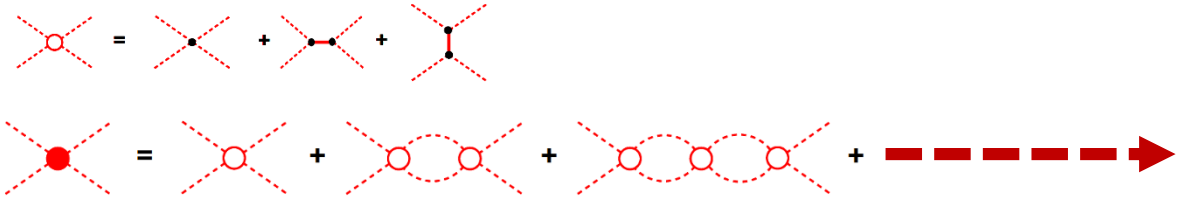
MMM - triple M

lagrangian

chiral symmetry



meson-meson interactions



precision

loops

renormalization

perturbative

resummation

unitarization

non-perturbative

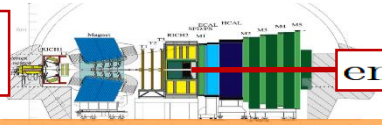
low energies

resonances

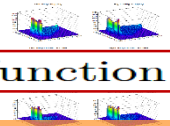
energy

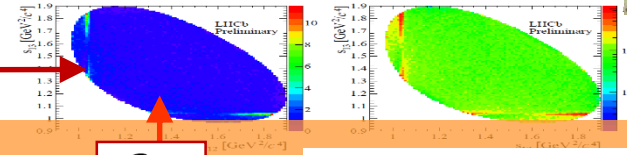
3 body decays

data



empirical function






physical picture

scattering data

$F(m_{12}^2, m_{13}^2)$

MODEL

fit

representation

2+1 ~ unitarity

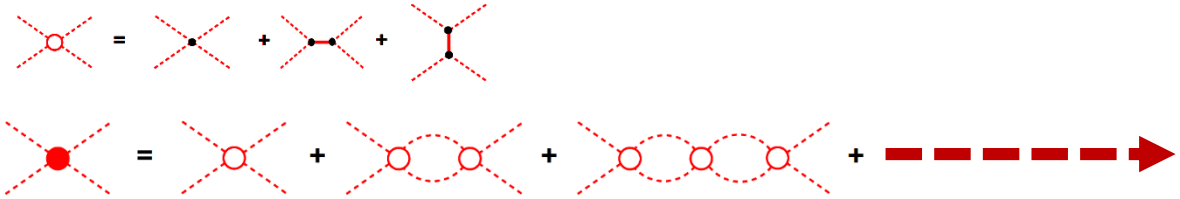
MULTI MESON

MMM - triple M

lagrangian

meson-meson interactions

chiral symmetry



precision

loops

renormalization

perturbative

extension

resummation

unitarization

non-perturbative

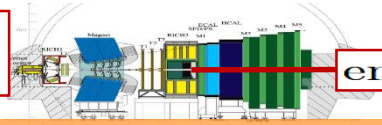
low energies

resonances

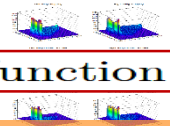
energy

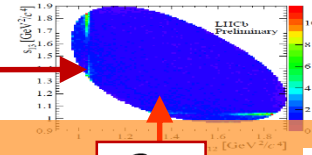
3 body decays

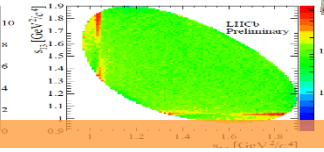
data



empirical function







physical picture

scattering data

$F(m_{12}^2, m_{13}^2)$

MODEL

fit

representation

2+1 ~ unitarity

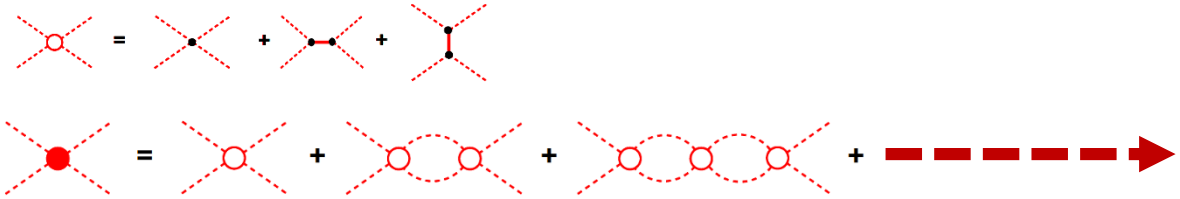
MULTI MESON

MMM - triple M

lagrangian

chiral symmetry

meson-meson interactions



precision

loops

isobar

resummation

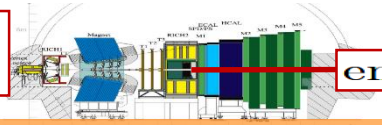
MMM - triple M

low energies

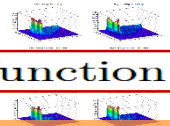
resonances

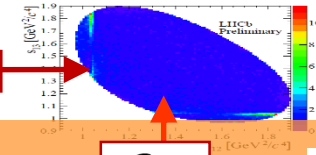
energy

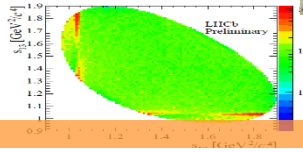
# 3 body decays



empirical function






data

physical picture

scattering data

$F(m_{12}^2, m_{13}^2)$

MODEL

fit

representation

2+1 ~ unitarity

MULTI MESON

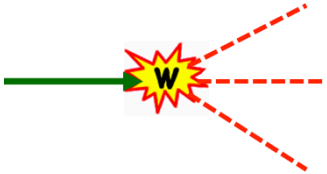
MMM - triple M

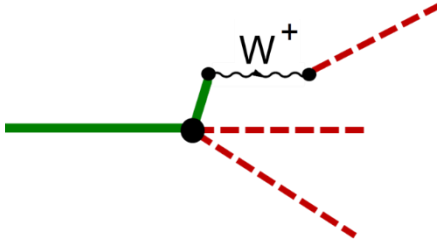
$D^+ \to K^- K^+ K^+$

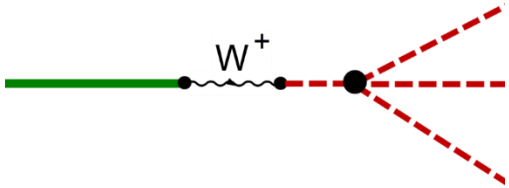
doubly Cabibbo suppressed

topologies

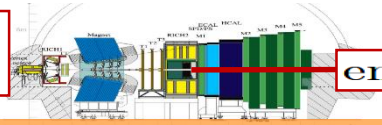
tree level



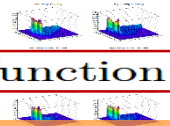



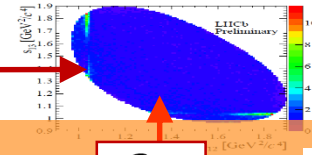



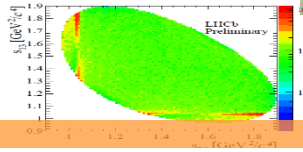
# 3 body decays



empirical function






data

physical picture

scattering data

$F(m_{12}^2, m_{13}^2)$

MODEL

fit

representation

2+1 ~ unitarity

MULTI MESON

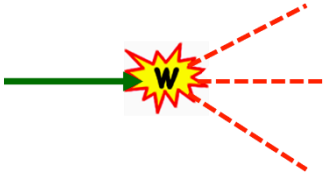
MMM - triple M

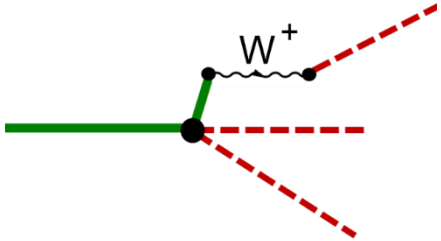
$D^+ \to K^- K^+ K^+$

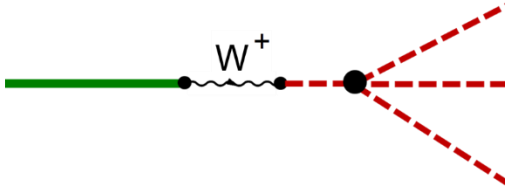
doubly Cabibbo suppressed

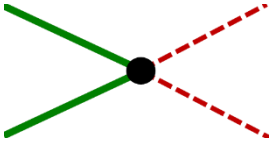
topologies

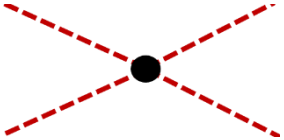
tree level



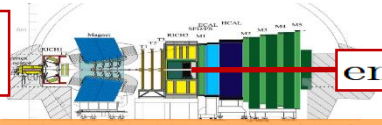



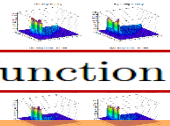



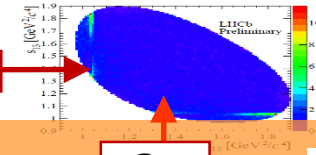


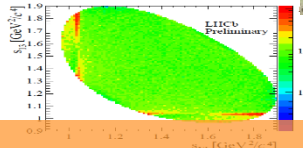
# 3 body decays



empirical function






data

physical picture

scattering data

$F(m_{12}^2, m_{13}^2)$

MODEL

fit

representation

2+1 ~ unitarity

MULTI MESON

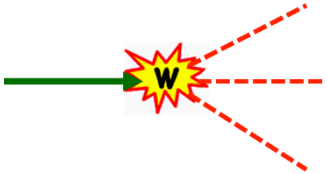
MMM - triple M

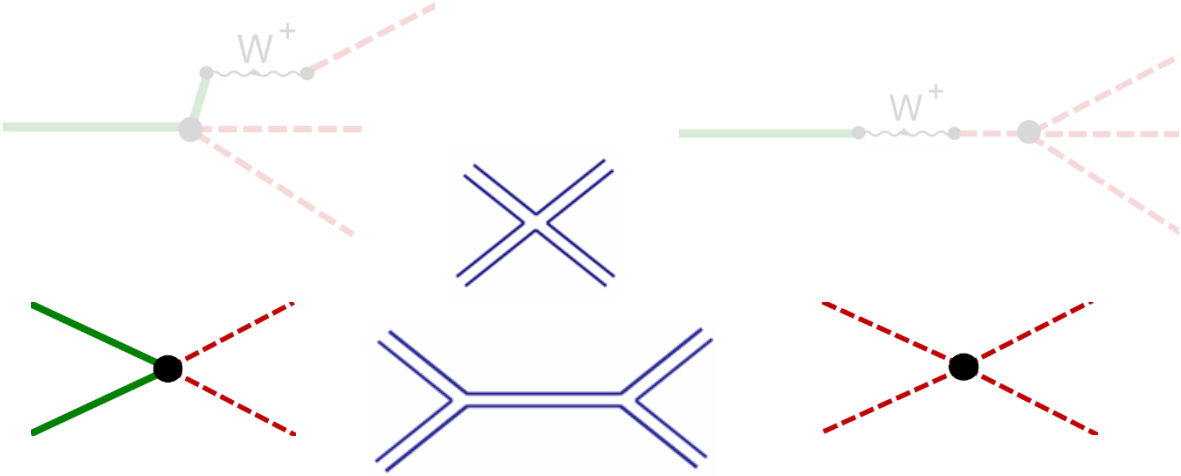
$D^+ \to K^- K^+ K^+$

doubly Cabibbo suppressed

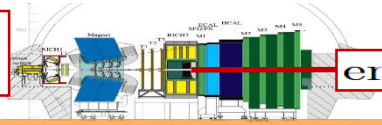
topologies

tree level

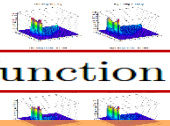


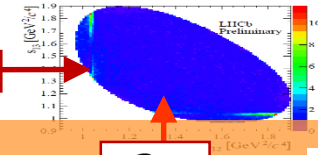



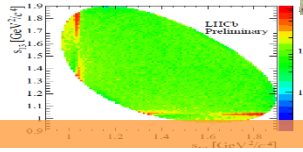
# 3 body decays



empirical function







data

physical picture

scattering data

$F(m_{12}^2, m_{13}^2)$

MODEL

fit

representation

2+1 ~ unitarity

MULTI MESON

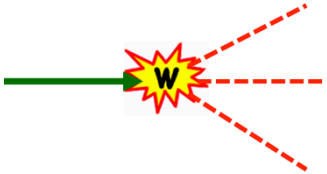
MMM - triple M

$D^+ \to K^- K^+ K^+$

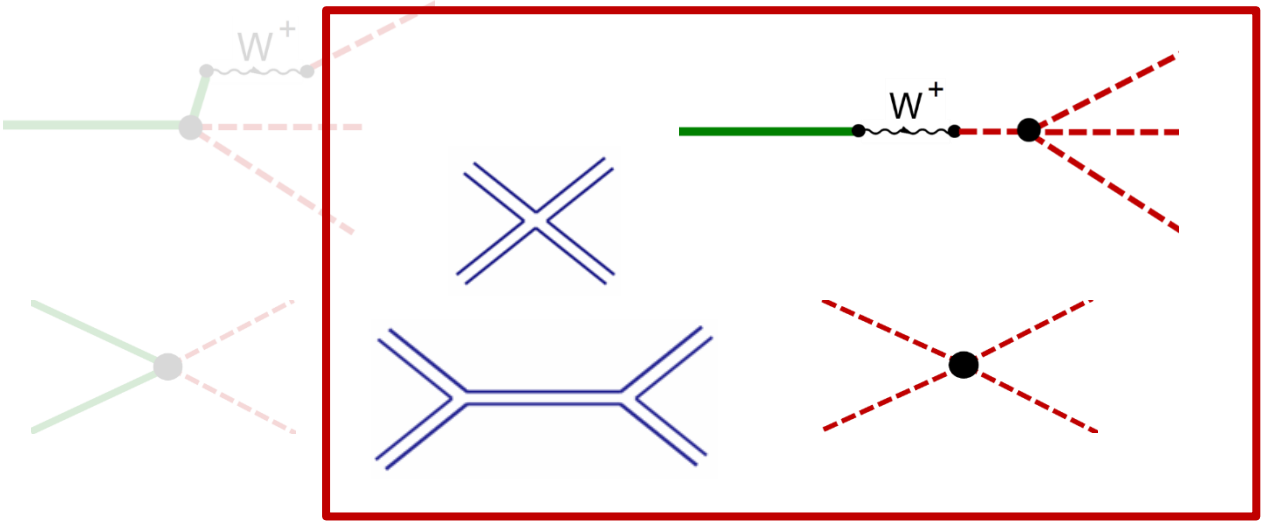
doubly Cabibbo suppressed

topologies

tree level



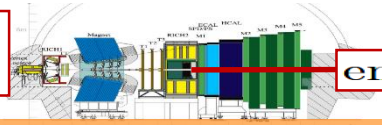

mathematical structures



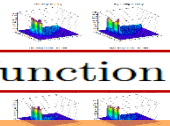

isobar

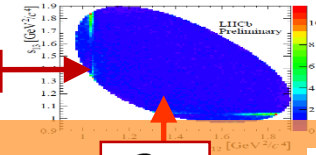
triple M

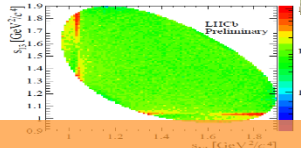
# 3 body decays



empirical function







data

physical picture

scattering data

$F(m_{12}^2, m_{13}^2)$

MODEL

fit

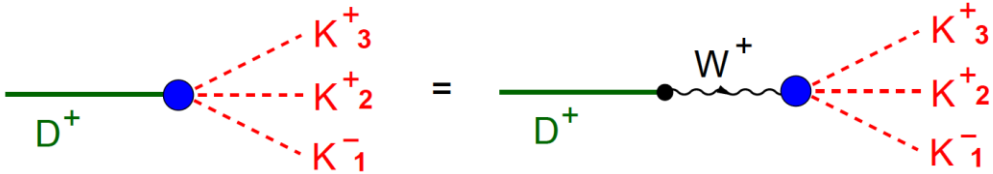
representation

2+1 ~ unitarity

$D^+ \to K^- K^+ K^+$

MULTI MESON

MMM - triple M



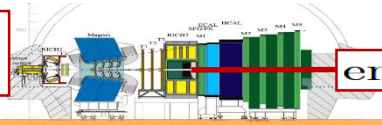

$$T = -\left[\frac{G_F}{\sqrt{2}}\sin^2\theta_C\right]\langle K^-(p_1)\,K^+(p_2)\,K^+(p_3)|A^\mu|\,0\,\rangle\,\langle\,0\,|A_\mu|\,D^+(P)\rangle$$

$$\langle\,0\,|A_\mu|\,D^+(P)\rangle = -i\sqrt{2}\,F_D\,P_\mu$$

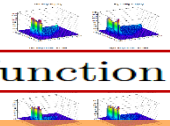
$$T = i\left[\frac{G_F}{\sqrt{2}}\sin^2\theta_C\right]\sqrt{2}\,F_D\,\left[P_\mu\,\langle K^-(p_1)\,K^+(p_2)\,K^+(p_3)|A^\mu|\,0\,\rangle\right]$$

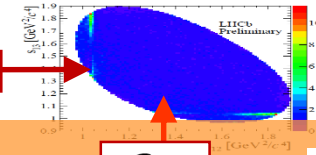
# 3 body decays

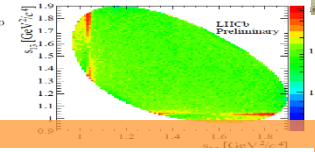
data



empirical function






physical picture

scattering data

$F(m_{12}^2, m_{13}^2)$

MODEL

fit

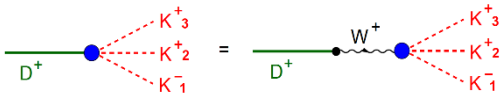
representation

2+1 ~ unitarity

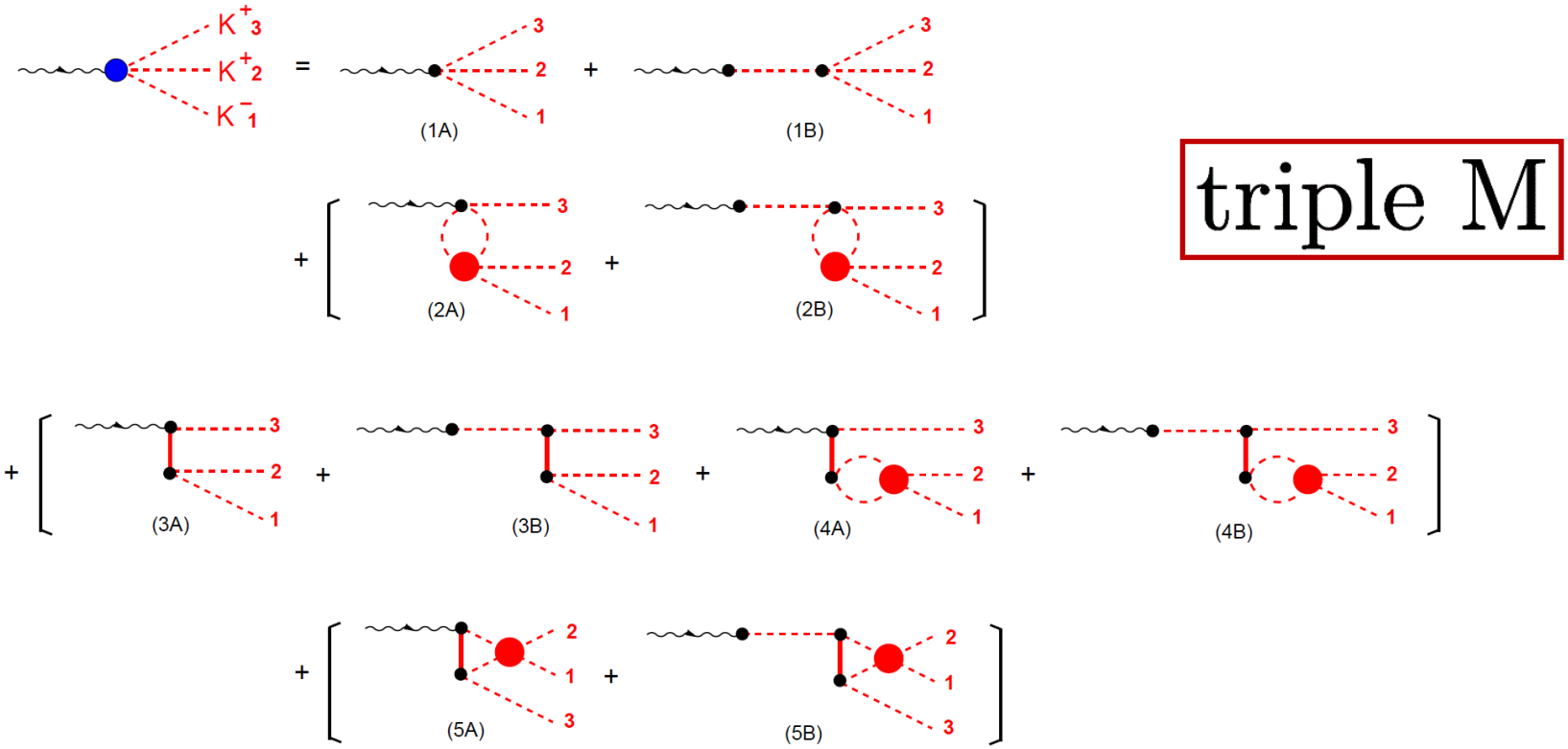
$D^+ \to K^- K^+ K^+$

MULTI MESON

MMM - triple M



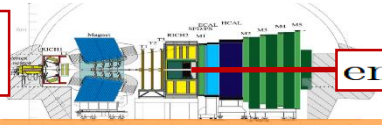

$$T = T_{NR} + \left[ T^{(1,1)} + T^{(1,0)} + T^{(0,1)} + T^{(0,0)} + (2 \leftrightarrow 3) \right]$$



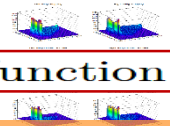

triple M

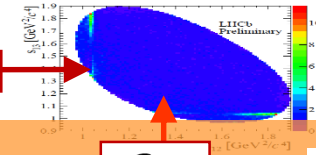
# 3 body decays

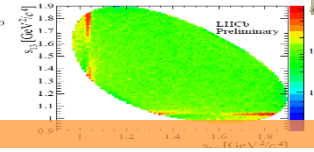
data



empirical function







fit

representation

physical picture

scattering data

$F(m_{12}^2, m_{13}^2)$

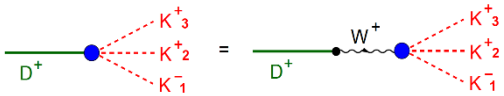
MODEL

2+1 ~ unitarity

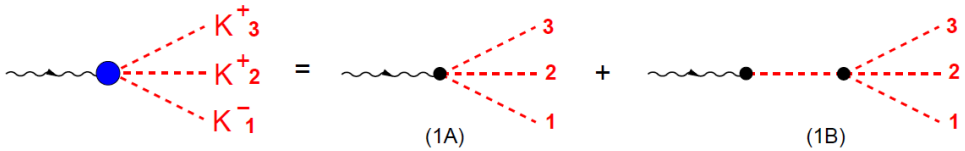
$D^+ \to K^- K^+ K^+$

MULTI MESON

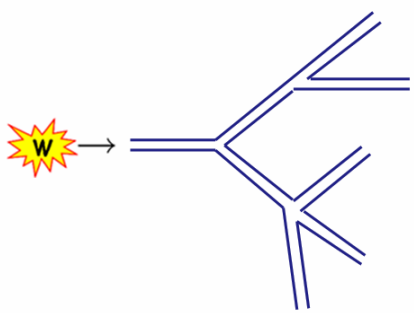
MMM - triple M




$$T = T_{NR} + \left[T^{(1,1)} + T^{(1,0)} + T^{(0,1)} + T^{(0,0)} + (2 \leftrightarrow 3)\right]$$

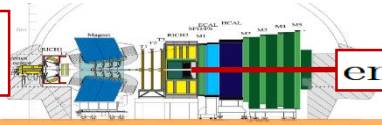



non-resonant

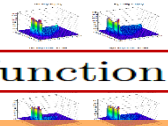


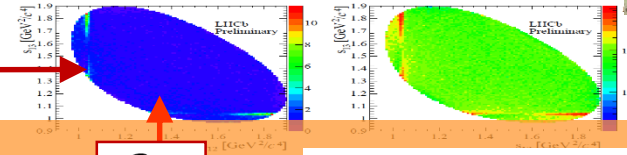

chiral symmetry

# 3 body decays



empirical function





data

physical picture

scattering data

$F(m_{12}^2, m_{13}^2)$

MODEL

fit

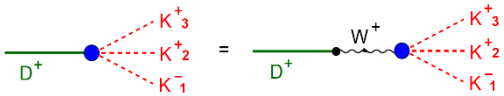
representation

2+1 ~ unitarity

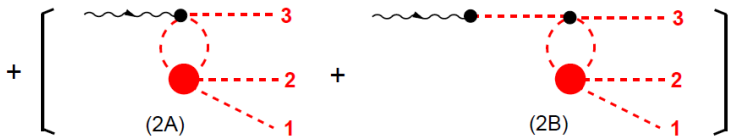
$D^+ \to K^- K^+ K^+$

MULTI MESON

MMM - triple M

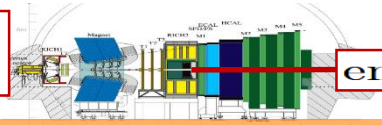



$$T = T_{NR} + \left[ T^{(1,1)} + T^{(1,0)} + T^{(0,1)} + T^{(0,0)} + (2 \leftrightarrow 3) \right]$$

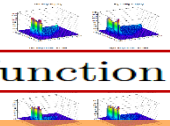


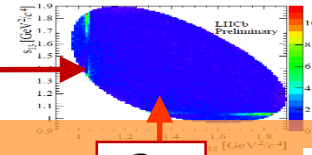
# 3 body decays

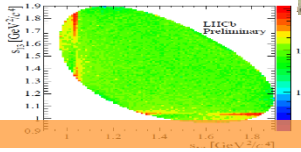
data



empirical function






physical picture

scattering data

$F(m_{12}^2, m_{13}^2)$

MODEL

fit

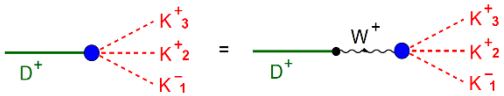
representation

2+1 ~ unitarity

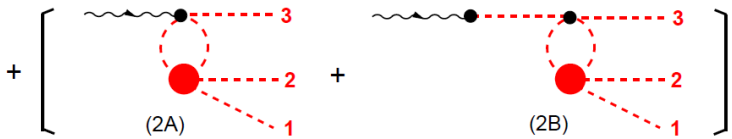
$D^+ \to K^- K^+ K^+$

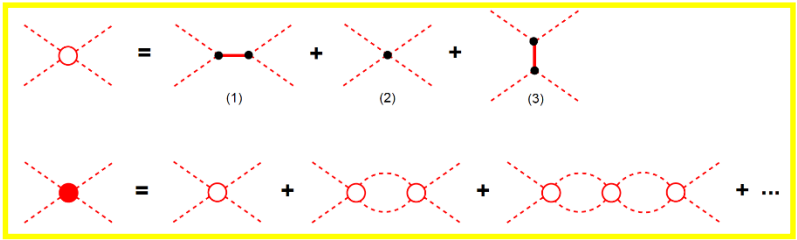
MULTI MESON

MMM - triple M



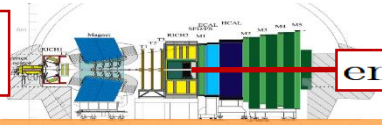

$$T = T_{NR} + \left[ T^{(1,1)} + T^{(1,0)} + T^{(0,1)} + T^{(0,0)} + (2 \leftrightarrow 3) \right]$$



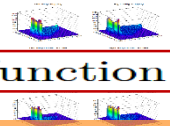



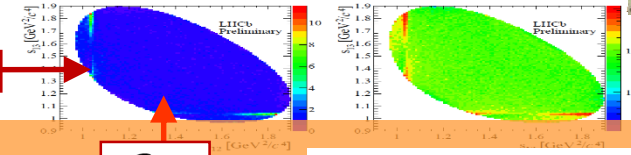
# 3 body decays

data



empirical function






physical picture

scattering data

$F(m_{12}^2, m_{13}^2)$

MODEL

fit

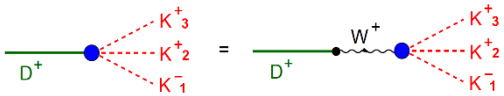
representation

2+1 ~ unitarity

$D^+ \to K^- K^+ K^+$

MULTI MESON

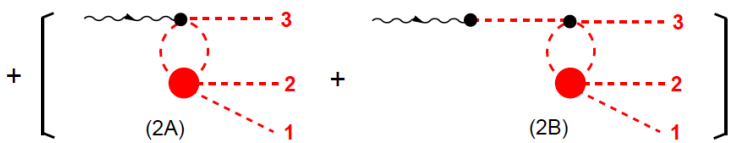
MMM - triple M

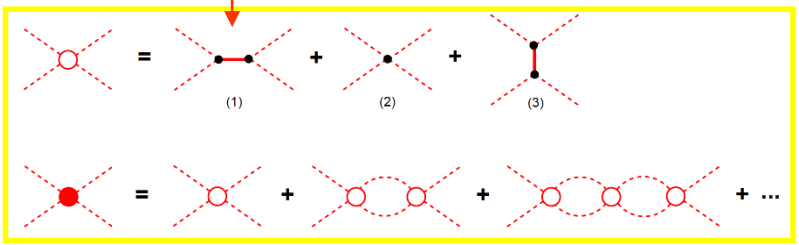



$$T = T_{NR} + \left[ T^{(1,1)} + T^{(1,0)} + T^{(0,1)} + T^{(0,0)} + (2 \leftrightarrow 3) \right]$$

$\rho \quad \phi \quad a_0 \quad f_0$

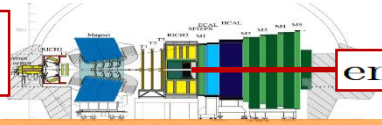
$SU(3)$ singlet $S_1$ octet $S_o$



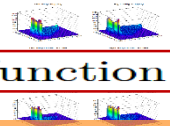



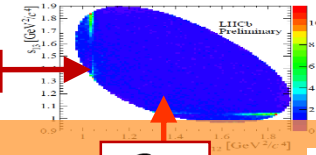
# 3 body decays

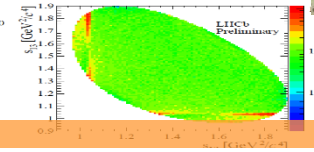
data



empirical function






physical picture

scattering data

$F(m_{12}^2, m_{13}^2)$

MODEL

fit

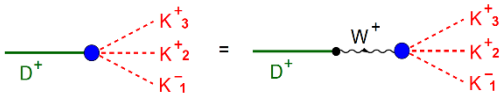
representation

2+1 ~ unitarity

$D^+ \to K^- K^+ K^+$

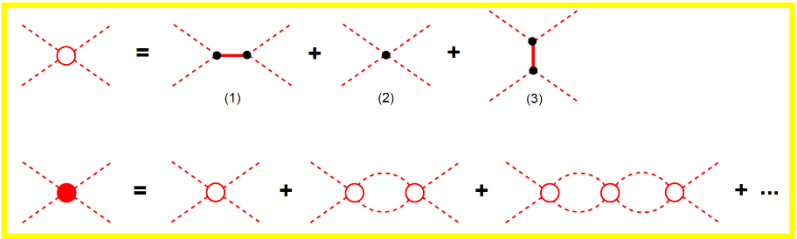
MULTI MESON

MMM - triple M



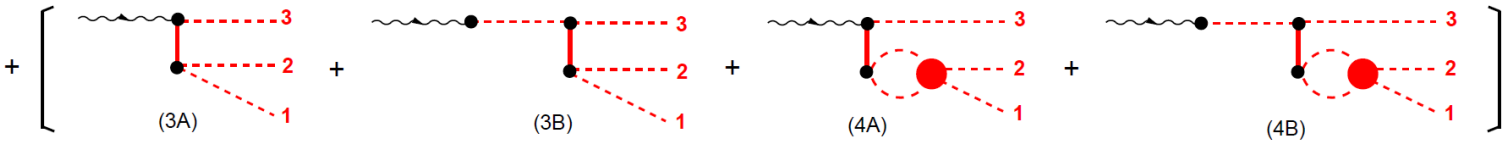

$$T = T_{NR} + \left[ T^{(1,1)} + T^{(1,0)} + T^{(0,1)} + T^{(0,0)} + (2 \leftrightarrow 3) \right]$$

$\rho \quad \phi \quad a_0 \quad f_0$



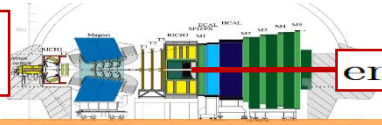

$SU(3)$ singlet $S_1$ octet $S_o$

dynamical width

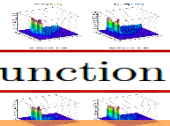


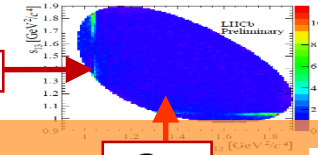
# 3 body decays

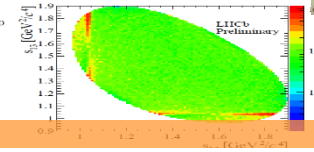
data



empirical function







fit

representation

physical picture

scattering data

$F(m_{12}^2, m_{13}^2)$

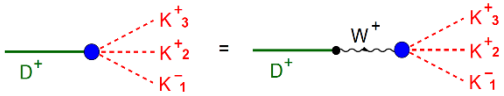
MODEL

2+1 ~ unitarity

$D^+ \to K^- K^+ K^+$

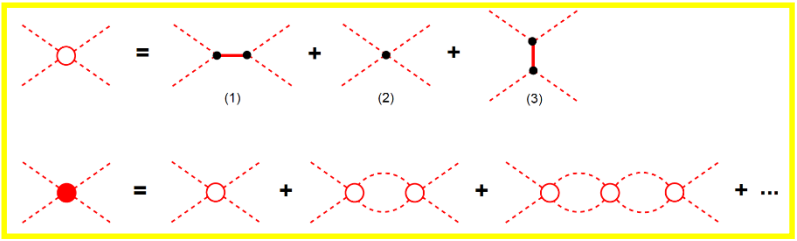
MULTI MESON

MMM - triple M



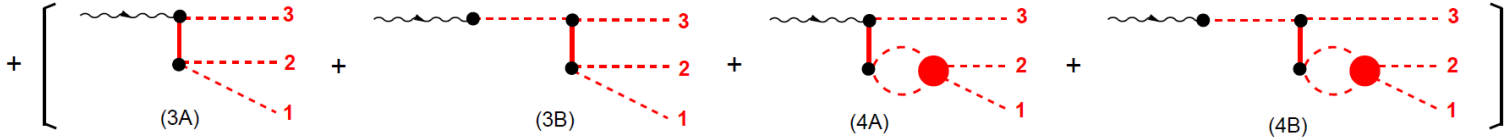

$$T = T_{NR} + \left[ T^{(1,1)} + T^{(1,0)} + T^{(0,1)} + T^{(0,0)} + (2 \leftrightarrow 3) \right]$$

$\rho \quad \phi \quad a_0 \quad f_0$




$SU(3)$ singlet $S_1$ octet $S_o$

dynamical width

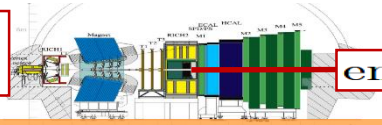



isobar

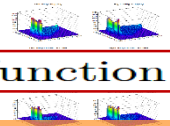
$$(BW)_k = \frac{1}{m_k^2 - s - i\, m_k\, \Gamma_k}$$

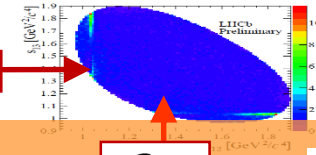
# 3 body decays

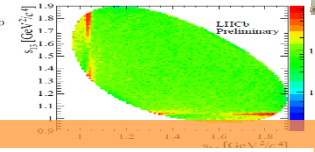
data



empirical function






physical picture

scattering data

$F(m_{12}^2, m_{13}^2)$

MODEL

fit

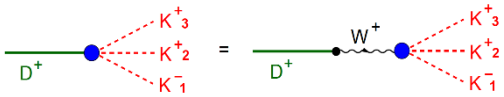
representation

2+1 ~ unitarity

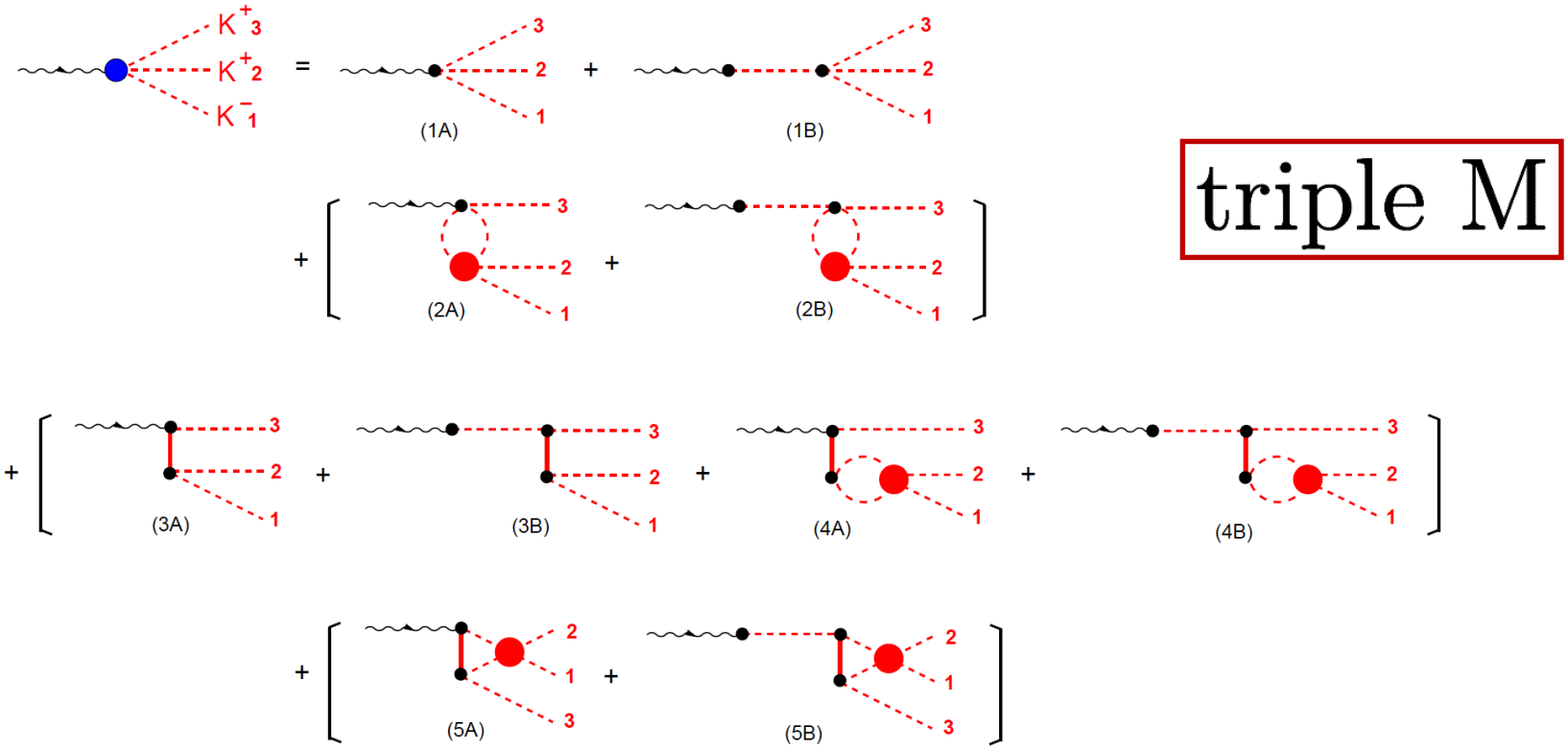
$D^+ \to K^- K^+ K^+$

MULTI MESON

MMM - triple M



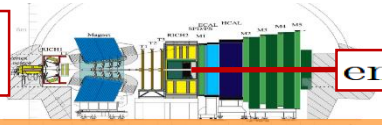

$$T = T_{NR} + \left[ T^{(1,1)} + T^{(1,0)} + T^{(0,1)} + T^{(0,0)} + (2 \leftrightarrow 3) \right]$$



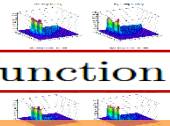

triple M

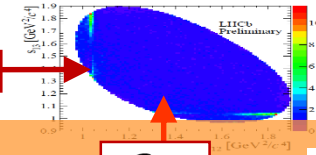
# 3 body decays

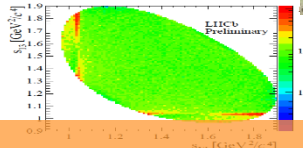
data



empirical function







physical picture

scattering data

$F(m_{12}^2, m_{13}^2)$

MODEL

fit

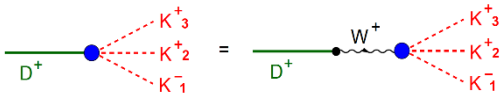
representation

2+1 ~ unitarity

MULTI MESON

MMM - triple M

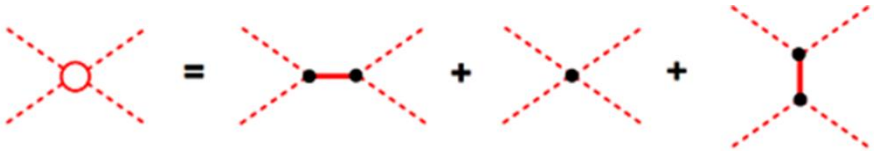
$D^+ \to K^- K^+ K^+$




$$T = T_{NR} + \left[T^{(1,1)} + T^{(1,0)} + T^{(0,1)} + T^{(0,0)} + (2 \leftrightarrow 3)\right]$$

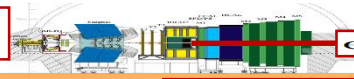
## free parameters

## triple M

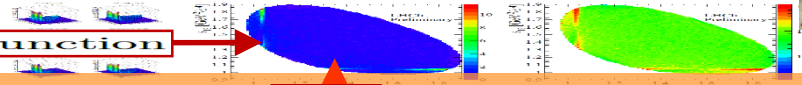



- masses: $\rho,\ \phi \qquad a_0,\ S_1,\ S_o$
- coupling constants: $F \qquad g_\rho,\ g_\phi \qquad c_d,\ c_m,\ \tilde{c}_d,\ \tilde{c}_m$

3 body decays



empirical function



physical picture

scattering data

$F(m_{12}^2, m_{13}^2)$

MODEL

fit

representation

MULTI MESON

2+1 ~ unitarity

MMM - triple M

$D^+ \to K^- K^+ K^+$

# loops

renormalization

real+imaginary

$$\{I_{ab};\, I_{ab}^{\mu\nu}\} = \int \frac{d^4\ell}{(2\pi)^4} \frac{\{1;\, \ell^\mu\, \ell^\nu\}}{D_a\, D_b}$$

$$D_a = (\ell + p/2)^2 - M_a^2 \qquad D_b = (\ell - p/2)^2 - M_b^2$$

$$\bar{\Omega}_{ab}^S = -\frac{i}{8\pi}\, \frac{Q_{ab}}{\sqrt{s}}\, \theta(s - (M_a + M_b)^2)$$

$$\bar{\Omega}_{aa}^P = -\frac{i}{6\pi}\, \frac{Q_{aa}^3}{\sqrt{s}}\, \theta(s - 4\, M_a^2)$$

$$Q_{ab} = \frac{1}{2} \sqrt{s - 2\,(M_a^2 + M_b^2) + (M_a^2 - M_b^2)^2/s}$$

3 body decays

data

physical picture

scattering data

$D^+ \to K^- K^+ K^+$


$F(m_{12}^2, m_{13}^2)$

MODEL

MULTI MESON

2+1 ~ unitarity

MMM - triple M

# loops

$$\{I_{ab};\, I_{ab}^{\mu\nu}\} = \int \frac{d^4\ell}{(2\pi)^4} \frac{\{1;\, \ell^\mu \ell^\nu\}}{D_a\, D_b}$$

$$D_a = (\ell + p/2)^2 - M_a^2 \qquad D_b = (\ell - p/2)^2 - M_b^2$$

~~real~~+imaginary

# K-matrix approximation

$$\bar{\Omega}_{ab}^S = -\frac{i}{8\pi} \frac{Q_{ab}}{\sqrt{s}}\, \theta(s - (M_a + M_b)^2)$$

$$\bar{\Omega}_{aa}^P = -\frac{i}{6\pi} \frac{Q_{aa}^3}{\sqrt{s}}\, \theta(s - 4\, M_a^2)$$

$$Q_{ab} = \frac{1}{2} \sqrt{s - 2\,(M_a^2 + M_b^2) + (M_a^2 - M_b^2)^2/s}$$

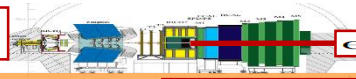
3 body decays

data

physical picture

scattering data

$D^+ \to K^- K^+ K^+$



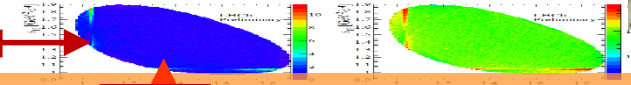
empirical function

$F(m_{12}^2, m_{13}^2)$

MULTI MESON

MODEL

fit



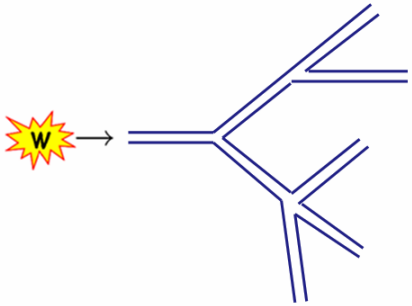
representation

2+1 ~ unitarity

MMM - triple M

$$T_{NR} = C \left\{ \left[ (m_{12}^2 - M_K^2) + (m_{13}^2 - M_K^2) \right] \right\}$$

real polynomial



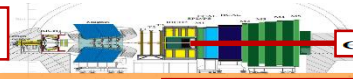

chiral symmetry

3 body decays

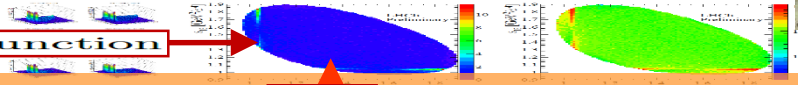
data

physical picture

scattering data



empirical function



$F(m_{12}^2, m_{13}^2)$

MODEL

fit

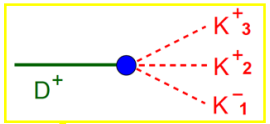
representation

2+1 ~ unitarity

MULTI MESON

MMM - triple M

$D^+ \to K^- K^+ K^+$




$$T^{(1,1)} = -\frac{1}{4}\left[\bar{\Gamma}^{(1,1)}_{KK} - \Gamma^{(1,1)}_{c|KK}\right](m_{13}^2 - m_{23}^2)$$

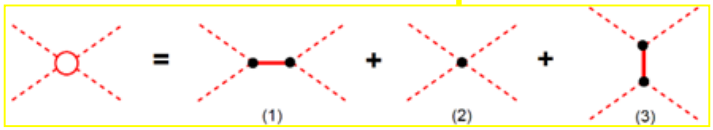
$$\bar{\Gamma}^{(1,1)}_{KK} = \frac{m_{12}^2 - m_\rho^2}{D_\rho(m_{12}^2)}\left[M^{(1,1)}_{21}\,\Gamma^{(1,1)}_{(0)\,\pi\pi} + \left(1 - M^{(1,1)}_{11}\right)\Gamma^{(1,1)}_{(0)\,KK}\right]$$

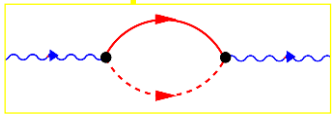
$$D_\rho = (m_{12}^2 - m_\rho^2)\left[\left(1 - M^{(1,1)}_{11}\right)\left(1 - M^{(1,1)}_{22}\right) - M^{(1,1)}_{12}M^{(1,1)}_{21}\right]$$

$\rho$

$$M^{(1,1)}_{11} = -\mathcal{K}^{(1,1)}_{\pi\pi|\pi\pi}\,[\bar{\Omega}^P_{\pi\pi}/2] \qquad M^{(1,1)}_{12} = -\mathcal{K}^{(1,1)}_{\pi\pi|KK}\,[\bar{\Omega}^P_{KK}/2]$$

$$M^{(1,1)}_{21} = -\mathcal{K}^{(1,1)}_{\pi\pi|KK}\,[\bar{\Omega}^P_{\pi\pi}/2] \qquad M^{(1,1)}_{22} = -\mathcal{K}^{(1,1)}_{KK|KK}\,[\bar{\Omega}^P_{KK}/2]$$



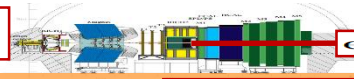



3 body decays

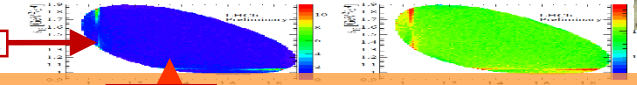
data

physical picture

scattering data



empirical function



$F(m_{12}^2, m_{13}^2)$

MODEL

fit

representation

2+1 ~ unitarity

MULTI MESON

MMM - triple M

$D^+ \to K^- K^+ K^+$

$\rho$

$$T^{(1,1)} = -\frac{1}{4}\left[\bar{\Gamma}^{(1,1)}_{KK} - \Gamma^{(1,1)}_{c|KK}\right](m_{13}^2 - m_{23}^2)$$

$$\bar{\Gamma}^{(1,1)}_{KK} = \frac{m_{12}^2 - m_\rho^2}{D_\rho(m_{12}^2)}\left[M^{(1,1)}_{21}\,\Gamma^{(1,1)}_{(0)\,\pi\pi} + \left(1 - M^{(1,1)}_{11}\right)\Gamma^{(1,1)}_{(0)\,KK}\right]$$

$$D_\rho = (m_{12}^2 - m_\rho^2)\left[\left(1 - M^{(1,1)}_{11}\right)\left(1 - M^{(1,1)}_{22}\right) - M^{(1,1)}_{12}M^{(1,1)}_{21}\right]$$

$$M^{(1,1)}_{11} = -\mathcal{K}^{(1,1)}_{\pi\pi|\pi\pi}\,[\bar{\Omega}^P_{\pi\pi}/2] \qquad M^{(1,1)}_{12} = -\mathcal{K}^{(1,1)}_{\pi\pi|KK}\,[\bar{\Omega}^P_{KK}/2]$$

$$M^{(1,1)}_{21} = -\mathcal{K}^{(1,1)}_{\pi\pi|KK}\,[\bar{\Omega}^P_{\pi\pi}/2] \qquad M^{(1,1)}_{22} = -\mathcal{K}^{(1,1)}_{KK|KK}\,[\bar{\Omega}^P_{KK}/2]$$

just resonance

real numerator

$$\bar{\Gamma}^{(1,1)}_{KK} = \frac{m_{12}^2 - m_\rho^2}{D_\rho(m_{12}^2)}\,\Gamma^{(1,1)}_{(0)\,KK}$$

Flatté

$$D_\rho(s) = (s - m_\rho^2) + i\,\frac{1}{12\pi\sqrt{s}}\left\{\frac{G_V^2}{F^4}\,s\,\left[2\,Q^3_{\pi\pi} + Q^3_{KK}\right]\right\}$$

isobar

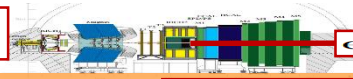
$$\frac{\text{an imaginary coefficient}}{m_k^2 - s - i\,m_k\,\Gamma_k}$$

3 body decays

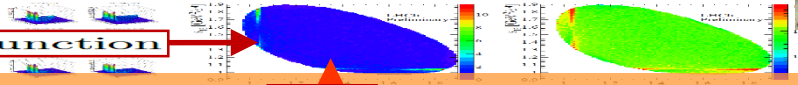
data

physical picture

scattering data



empirical function



$F(m_{12}^2, m_{13}^2)$

MODEL

fit

representation

2+1 ~ unitarity

MULTI MESON

MMM - triple M

$D^+ \to K^- K^+ K^+$

$\phi$

$$T^{(1,0)} = -\frac{1}{4}\left[\bar{\Gamma}^{(1,0)}_{KK} - \Gamma^{(1,0)}_{c|KK}\right](m_{13}^2 - m_{23}^2)$$

$$\bar{\Gamma}^{(1,0)}_{KK} = \frac{D_\phi^{\pi\rho}}{D_\phi(m_{12}^2)}\,\Gamma^{(1,0)}_{(0)\,KK}$$

$$D_\phi(m_{12}^2) = (m_{12}^2 - m_\phi^2) + i\, m_\phi\, \Gamma_\phi(m_{12}^2)$$

$$m_\phi\, \Gamma_\phi(m_{12}^2) = \Gamma_{\pi\rho}\, \frac{m_{12}^3}{m_\phi^2}\, \frac{Q_{\pi\rho}^3}{\tilde{Q}_{\pi\rho}^3}$$

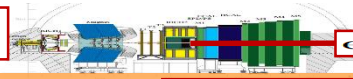
$$+\frac{1}{12\pi\, m_{12}}\left[\frac{3\, G_V^2\, \sin^2\theta}{F^4}\, m_{12}^2 - (m_{12}^2 - m_\phi^2)\, P^{(1,0)}_{(KK|KK)}\right] Q_{KK}^3$$

3 body decays

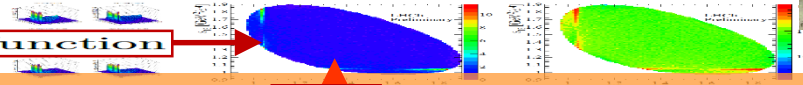
data

physical picture

scattering data



empirical function



$F(m_{12}^2, m_{13}^2)$

MODEL

fit

representation

MULTI MESON

2+1 ~ unitarity

MMM - triple M

$D^+ \to K^- K^+ K^+$

$a_0$

$$T^{(0,1)} = -\frac{1}{2}\left[\bar{\Gamma}^{(0,1)}_{KK} - \Gamma^{(0,1)}_{c|KK}\right]$$

$$\bar{\Gamma}^{(0,1)}_{KK} = \frac{(m_{12}^2 - m_{a_0}^2)}{D_{a_0}(m_{12}^2)}\left[M^{(0,1)}_{21}\,\Gamma^{(0,1)}_{(0)\,\pi 8} + \left(1 - M^{(0,1)}_{11}\right)\,\Gamma^{(0,1)}_{(0)\,KK}\right]$$

$$D_{a_0} = (m_{12}^2 - m_{a_0}^2)\,\left[\left(1 - M^{(0,1)}_{11}\right)\,\left(1 - M^{(0,1)}_{22}\right) - M^{(0,1)}_{12}\,M^{(0,1)}_{21}\right]$$

just resonance

$$M^{(0,1)}_{11} = -\mathcal{K}^{(0,1)}_{\pi 8|\pi 8}\,[\bar{\Omega}^S_{\pi 8}] \qquad M^{(0,1)}_{12} = -\mathcal{K}^{(0,1)}_{\pi 8|KK}\,[\bar{\Omega}^S_{KK}/2]$$

$$M^{(0,1)}_{21} = -\mathcal{K}^{(0,1)}_{\pi 8|KK}\,[\bar{\Omega}^S_{\pi 8}] \qquad M^{(0,1)}_{22} = -\mathcal{K}^{(0,1)}_{KK|KK}\,[\bar{\Omega}^S_{KK}/2]$$

real numerator

$$\bar{\Gamma}^{(0,1)}_{KK} = \frac{(m_{12}^2 - m_{a_0}^2)}{D_{a_0}(m_{12}^2)}\,\Gamma^{(0,1)}_{(0)\,KK}$$

Flatté

$$D_{a_0}(s) = (s - m_{a_0}^2) + i\,m_{a_0}\,\Gamma_{a_0}(s)$$

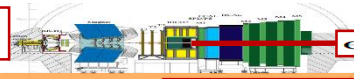
$$m_{a_0}\Gamma_{a_0}(s) = \frac{1}{8\pi\sqrt{s}}\left\{\left[\frac{4}{3\,F^4}\right]\,\left[c_d\,(s - M_\pi^2 - M_8^2) + 2\,c_m\,M_\pi^2\right]^2\,Q_{\pi 8} + \left[\frac{1}{F^4}\right]\,\left[c_d\,(s - 2\,M_K^2) + 2\,c_m\,M_K^2\right]^2\,Q_{KK}\right\}$$

3 body decays

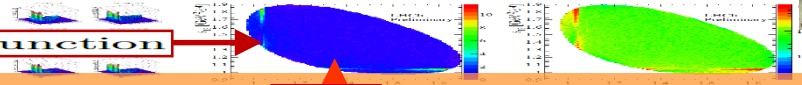
data

physical picture

scattering data



empirical function



$F(m_{12}^2, m_{13}^2)$

MODEL

fit

representation

MULTI MESON

2+1 ~ unitarity

MMM - triple M

$D^+ \to K^- K^+ K^+$

$a_0$

$$T^{(0,1)} = -\frac{1}{2}\left[\bar{\Gamma}^{(0,1)}_{KK} - \Gamma^{(0,1)}_{c|KK}\right]$$

$$\bar{\Gamma}^{(0,1)}_{KK} = \frac{(m_{12}^2 - m_{a_0}^2)}{D_{a_0}(m_{12}^2)}\left[M^{(0,1)}_{21}\,\Gamma^{(0,1)}_{(0)\,\pi 8} + \left(1 - M^{(0,1)}_{11}\right)\,\Gamma^{(0,1)}_{(0)\,KK}\right]$$

$$D_{a_0} = (m_{12}^2 - m_{a_0}^2)\,\left[\left(1 - M^{(0,1)}_{11}\right)\,\left(1 - M^{(0,1)}_{22}\right) - M^{(0,1)}_{12}\,M^{(0,1)}_{21}\right]$$

just resonance

$$M^{(0,1)}_{11} = -\mathcal{K}^{(0,1)}_{\pi 8|\pi 8}\,[\bar{\Omega}^S_{\pi 8}] \qquad M^{(0,1)}_{12} = -\mathcal{K}^{(0,1)}_{\pi 8|KK}\,[\bar{\Omega}^S_{KK}/2]$$

$$M^{(0,1)}_{21} = -\mathcal{K}^{(0,1)}_{\pi 8|KK}\,[\bar{\Omega}^S_{\pi 8}] \qquad M^{(0,1)}_{22} = -\mathcal{K}^{(0,1)}_{KK|KK}\,[\bar{\Omega}^S_{KK}/2]$$

real numerator

$$\bar{\Gamma}^{(0,1)}_{KK} = \frac{(m_{12}^2 - m_{a_0}^2)}{D_{a_0}(m_{12}^2)}\,\Gamma^{(0,1)}_{(0)\,KK}$$

Flatté

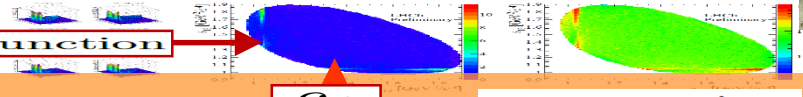
$$D_{a_0}(s) = (s - m_{a_0}^2) + i\,m_{a_0}\,\Gamma_{a_0}(s)$$

$$m_{a_0}\Gamma_{a_0}(s) = \frac{1}{8\pi\sqrt{s}}\left\{\left[\frac{4}{3\,F^4}\right]\left[c_d\,(s - M_\pi^2 - M_8^2) + 2\,c_m\,M_\pi^2\right]^2 Q_{\pi 8} + \left[\frac{1}{F^4}\right]\left[c_d\,(s - 2\,M_K^2) + 2\,c_m\,M_K^2\right]^2 Q_{KK}\right\}$$

rich dynamics

3 body decays

empirical function



physical picture

scattering data

$F(m_{12}^2, m_{13}^2)$

MODEL

fit

representation

2+1 ~ unitarity

MULTI MESON

MMM - triple M

$D^+ \to K^- K^+ K^+$

$f_0$

$SU(3)$ singlet $S_1$ octet $S_o$

$$T^{(0,0)} = -\frac{1}{2}\left[\bar{\Gamma}^{(0,0)}_{KK} - \Gamma^{(0,0)}_{c|KK}\right]$$

$$\bar{\Gamma}^{(0,0)}_{KK} = \frac{1}{\det\left[1 - M^{(0,0)}\right]}\left\{\left[M^{(0,0)}_{21}\left(1 - M^{(0,0)}_{33}\right) + M^{(0,0)}_{23} M^{(0,0)}_{31}\right]\Gamma^{(0,0)}_{(0)\,\pi\pi}\right.$$

$$+\left[\left(1 - M^{(0,0)}_{11}\right)\left(1 - M^{(0,0)}_{33}\right) - M^{(0,0)}_{13} M^{(0,0)}_{31}\right]\Gamma^{(0,0)}_{(0)\,KK}$$

$$\left. + \left[M^{(0,0)}_{23}\left(1 - M^{(0,0)}_{11}\right) + M^{(0,0)}_{13} M^{(0,0)}_{21}\right]\Gamma^{(0,0)}_{(0)\,88}\right\}$$

$$\det(1 - M) = [1 - M_{11}][1 - M_{22}][1 - M_{33}]$$

$$- [1 - M_{11}] M_{23} M_{32} - [1 - M_{22}] M_{13} M_{31} - [1 - M_{33}] M_{12} M_{21}$$

$$- M_{12} M_{23} M_{31} - M_{21} M_{32} M_{13}$$

$$M^{(0,0)}_{11} = -\mathcal{K}^{(0,0)}_{\pi\pi|\pi\pi}\left[\bar{\Omega}^S_{\pi\pi}/2\right] \qquad M^{(0,0)}_{12} = -\mathcal{K}^{(0,0)}_{\pi\pi|KK}\left[\bar{\Omega}^S_{KK}/2\right] \qquad M^{(0,0)}_{13} = -\mathcal{K}^{(0,0)}_{\pi\pi|88}\left[\bar{\Omega}^S_{88}/2\right]$$

$$M^{(0,0)}_{21} = -\mathcal{K}^{(0,0)}_{\pi\pi|KK}\left[\bar{\Omega}^S_{\pi\pi}/2\right] \qquad M^{(0,0)}_{22} = -\mathcal{K}^{(0,0)}_{KK|KK}\left[\bar{\Omega}^S_{KK}/2\right] \qquad M^{(0,0)}_{23} = -\mathcal{K}^{(0,0)}_{KK|88}\left[\bar{\Omega}^S_{88}/2\right]$$

$$M^{(0,0)}_{31} = -\mathcal{K}^{(0,0)}_{\pi\pi|88}\left[\bar{\Omega}^S_{\pi\pi}/2\right] \qquad M^{(0,0)}_{32} = -\mathcal{K}^{(0,0)}_{KK|88}\left[\bar{\Omega}^S_{KK}/2\right] \qquad M^{(0,0)}_{33} = -\mathcal{K}^{(0,0)}_{88|88}\left[\bar{\Omega}^S_{88}/2\right]$$

3 body decays

data

physical picture

scattering data

empirical function

$F(m_{12}^2, m_{13}^2)$

MODEL

fit

representation

MULTI MESON

2+1 ~ unitarity

MMM - triple M

$D^+ \to K^- K^+ K^+$

$f_0$

$SU(3)$

singlet $S_1$

octet $S_o$

just resonance

just resonance

$$T^{(0,0)} = -\frac{1}{2}\left[\bar{\Gamma}^{(0,0)}_{KK} - \Gamma^{(0,0)}_{c|KK}\right]$$

$$\bar{\Gamma}^{(0,0)}_{KK} = \frac{(m_{12}^2 - m_{S1}^2)\,(m_{12}^2 - m_{So}^2)}{D^2_{f_0}(m_{12}^2)}\,\Gamma^{(0,0)}_{(0)\,KK}$$

$$D^2_{f_0}(s) = (s - m_{S1}^2)\,(s - m_{So}^2) + i\,D_{IM}(s)$$

$$\begin{aligned}
D_{IM}(s) = \frac{1}{8\pi\sqrt{s}}\Bigg\{ & (s - m_{So}^2)\left[\frac{1}{F^4}\right]\Big[6\,\left[\tilde{c}_d\,s - (\tilde{c}_d - \tilde{c}_m)\,2M_\pi^2\right]^2 Q_{\pi\pi} \\
& + 8\,\left[\tilde{c}_d\,s - (\tilde{c}_d - \tilde{c}_m)\,2M_K^2\right]^2 Q_{KK} + 2\,\left[\tilde{c}_d\,s - (\tilde{c}_d - \tilde{c}_m)\,2M_8^2\right]^2 Q_{88}\Big] \\
& + (s - m_{S1}^2)\left[\frac{1}{F^4}\right]\Big[\left[c_d\,s - (c_d - c_m)\,2M_\pi^2\right]^2 Q_{\pi\pi} \\
& \quad + \frac{1}{3}\,\left[c_d\,s - (c_d - c_m)\,2M_K^2\right]^2 Q_{KK} \\
& \quad + \frac{1}{3}\,\left[c_d\,(s - 2M_8^2) + c_m\,(16M_K^2 - 10M_\pi^2)/3\right]^2 Q_{88}\Big]\Bigg\}
\end{aligned}$$

3 body decays

data

physical picture

scattering data

empirical function

$F(m_{12}^2, m_{13}^2)$

MODEL

fit

representation

2+1 ~ unitarity

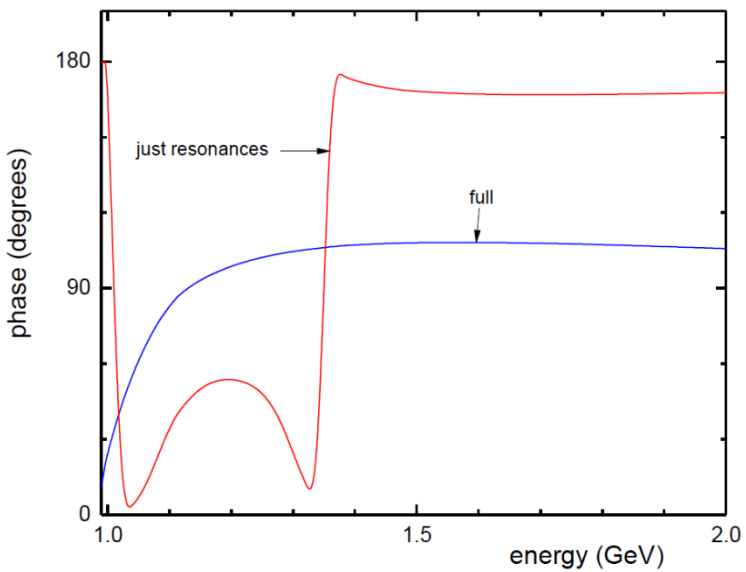
MULTI MESON

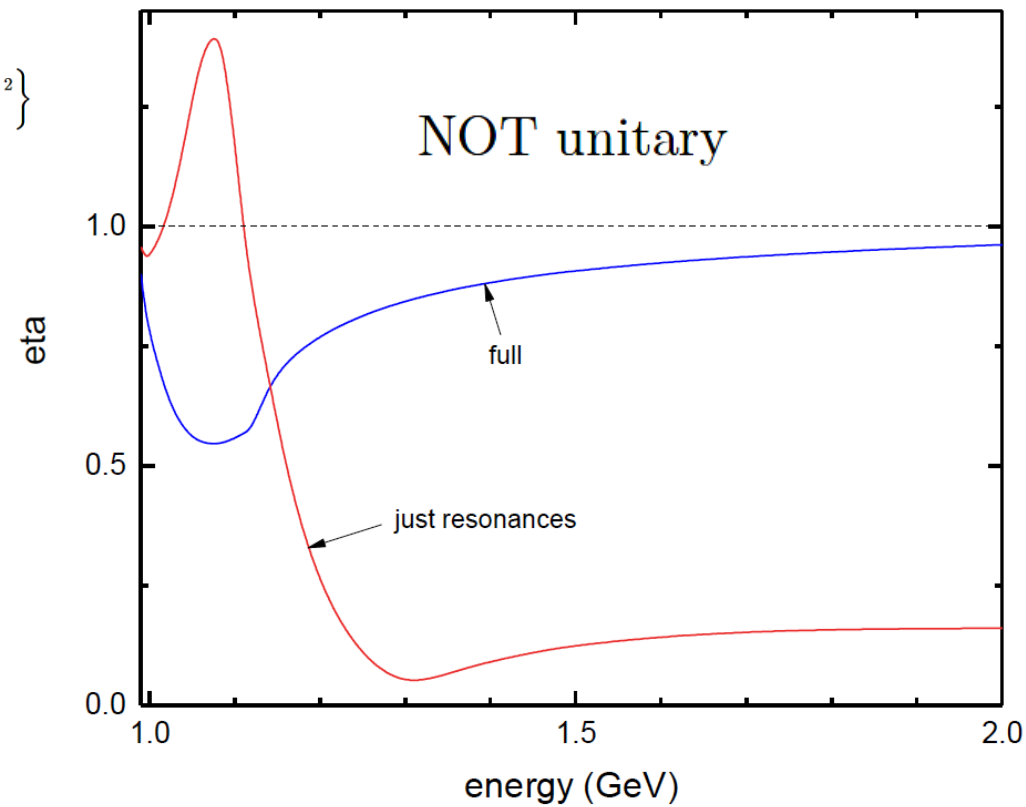
MMM - triple M

$$D^+ \to K^- K^+ K^+$$

scattering

$$A^{(0,0)}_{KK|KK} = -\frac{1}{D^2_{f_0}(m^2_{12})} \left\{ -(s - m^2_{So}) \left[\frac{16}{F^4}\right] \left[\tilde{c}_d \left(s - 2M_K^2\right) + 2\,\tilde{c}_m\, M_K^2\right]^2 \right.$$
$$\left. -(s - m^2_{S1}) \left[\frac{2}{3\,F^4}\right] \left[c_d \left(s - 2M_K^2\right) + 2\,c_m\, M_K^2\right]^2 \right\}$$



# conclusions

- lagrangian → meaning to parameters
- chiral symmetry → extended energy range

- MMM

→ K-matrix

→ real non-resonant background

→ resonances: dynamical widths

→ $f_0$: $SU(3)$ singlet + octet

→ coupled channel srtucture: cannot be ignored

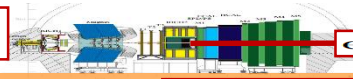
→ allows extraction of scattering amplitudes

3 body decays

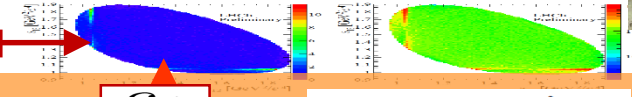
data

physical picture

scattering data



empirical function



$F(m_{12}^2, m_{13}^2)$

MODEL

fit

representation

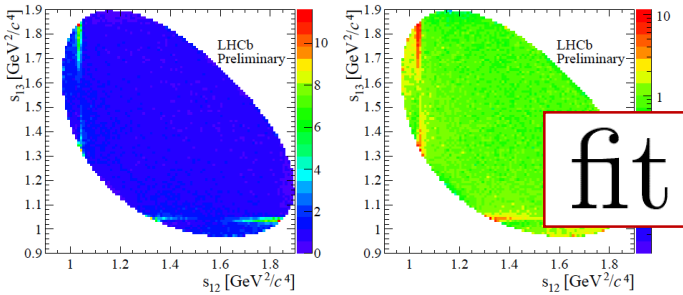
MULTI MESON

2+1 ~ unitarity

MMM - triple M

$D^+ \to K^- K^+ K^+$

# conclusions




fit

$$T = -\frac{1}{2}\left[\bar{\Gamma}_{KK} - \Gamma_{c|KK}\right]$$

decay

$$\bar{\Gamma}_{KK} = \frac{(m_{12}^2 - m_{a_0}^2)}{D_{a_0}(m_{12}^2)}\left[M_{21}\,\Gamma_{(0)\,\pi 8} + (1-M_{11})\,\Gamma_{(0)\,KK}\right]$$

$\neq$

$$A_{KK|KK} = \frac{(m_{12}^2 - m_{a_0}^2)}{D_{a_0}(m_{12}^2)}\left[M_{21}\,\mathcal{K}_{\pi 8|KK} + (1-M_{11})\,\mathcal{K}_{KK|KK}\right]$$

scattering

$$\bar{\Gamma}_{KK} = H_{KK|KK} \times A_{KK|KK}$$

$$H_{KK|KK} = \frac{\left[M_{21}\,\Gamma_{(0)\,\pi 8} + (1-M_{11})\,\Gamma_{(0)\,KK}\right]}{\left[M_{21}\,\mathcal{K}_{\pi 8|KK} + (1-M_{11})\,\mathcal{K}_{KK|KK}\right]}$$

3 body decays

data

physical picture

scattering data

empirical function

$F(m_{12}^2, m_{13}^2)$

MODEL

fit

representation

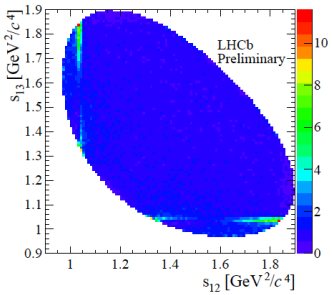
MULTI MESON

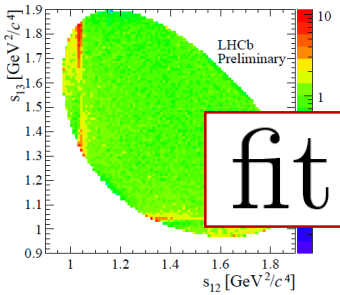
2+1 ~ unitarity

MMM - triple M

$D^+ \to K^- K^+ K^+$

# conclusions




fit

$$T = -\frac{1}{2}\left[\bar{\Gamma}_{KK} - \Gamma_{c|KK}\right]$$

decay

$$\bar{\Gamma}_{KK} = \frac{(m_{12}^2 - m_{a_0}^2)}{D_{a_0}(m_{12}^2)}\left[M_{21}\,\Gamma_{(0)\,\pi 8} + (1-M_{11})\,\Gamma_{(0)\,KK}\right]$$

free parameters
physical meaning

coupling constants
masses

$\neq$

$$A_{KK|KK} = \frac{(m_{12}^2 - m_{a_0}^2)}{D_{a_0}(m_{12}^2)}\left[M_{21}\,\mathcal{K}_{\pi 8|KK} + (1-M_{11})\,\mathcal{K}_{KK|KK}\right]$$

scattering

$$\bar{\Gamma}_{KK} = H_{KK|KK} \times A_{KK|KK}$$

$$H_{KK|KK} = \frac{\left[M_{21}\,\Gamma_{(0)\,\pi 8} + (1-M_{11})\,\Gamma_{(0)\,KK}\right]}{\left[M_{21}\,\mathcal{K}_{\pi 8|KK} + (1-M_{11})\,\mathcal{K}_{KK|KK}\right]}$$