UNIVERSIDAD DE **MURCIA**

$\eta \to 3\pi$ in coupled-channels Khuri-Treimann formalism

Based on: Eur. Phys. J. C**77**, 508 (2017)
[arXiv:1702.04931]

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Outline

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   - Dispersive approaches to $\eta \to 3\pi$
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   - Muskhelisvili-Omnès representation
3.
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   - MO representations
   - Matching with chiral amplitudes
4.
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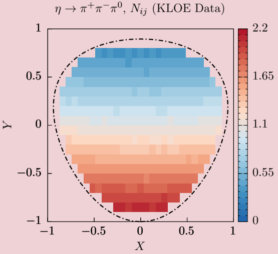
   - Dalitz plot parameters
   - Quark mass ratio Q
5.

Interest of $\eta \to 3\pi$

- In **QCD** isospin-breaking phenomena are driven by

$$\boxed{H_{IB} = -(m_u - m_d)\bar{\psi}\frac{\lambda_3}{2}\psi}$$

- Isospin-breaking induced by EM & strong interactions are **similar** in size, but
- **$\eta \to 3\pi$ is special**, since EM effects are smaller
- $\Gamma_{\eta\to 3\pi} \propto Q^4$, with $\boxed{Q^{-2} = \frac{m_d^2 - m_u^2}{m_s^2 - (m_u+m_d)^2/2}}$



- **Experimental situation:** Several high-statistics studies; $|T|^2$ well known across the Dalitz plot $\Rightarrow$ stringent tests for the amplitudes (before getting Q!)

$\eta \to 3\pi^0$

Crys. Ball, PRL**87**,192001('01)
Crys. Ball@MAMI, A2, PRC**79**,035204('09)
Crys. Ball@MAMI, TAPS, A2, EPJA**39**,169('09)
WASA-at-COSY, PLB**677**,24('09)
KLOE, PLB**694**,16('11)

$\eta \to \pi^+\pi^-\pi^0$

KLOE, JHEP**0805**,006('08)
WASA-at-COSY, PRC**90**,045207('14)
BESIII, PRD**92**,012014('15)
KLOE-2, JHEP**1605**,019('16)

Dispersive approaches to $\eta \to 3\pi$

- **Chiral $\mathcal{O}(p^4)$** amplitude fails in describing experiments.

  Gasser, Leutwyler, Nucl. Phys. B**250**, 539 (1985)

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- $\pi\pi$ scattering **elastic** in the decay region. But **dispersive approaches** require higher energy T-matrix inputs:
  - $\pi\pi$ near 1 GeV rapid energy variation. $f_0(980)$, $(K\bar{K})_0$
  - Double resonance effect $\eta\pi$ ISI, $a_0(980)$, $(K\bar{K})_1$

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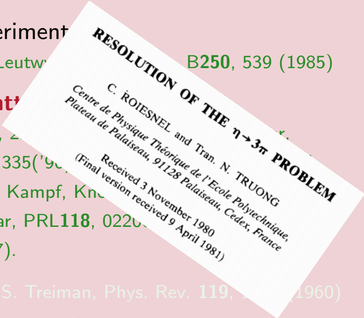
RESOLUTION OF THE $\eta \to 3\pi$ PROBLEM

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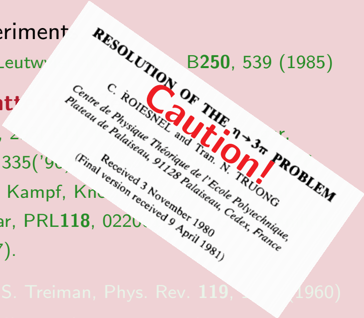

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We propose a **generalization to coupled channels** $[(K\bar{K})_{0,1},\ \eta\pi,\ (\pi\pi)_{0,1,2}]$ of the KT equations, extending their validity up to the physical $\eta\pi \to \pi\pi$ region. Allows for the study of the influence of a_0, f_0 into the decay region.

Outline

Khuri-Treiman: elastic channels. Isospin amplitudes

- Start with well-defined **isospin amplitudes:**

$$\mathcal{M}^{I,I_z}(s,t,u) = \langle \eta\pi; 1, I_z | \hat{T}_0^{(1)} | \pi\pi; I, I_z \rangle = \langle I, I_z; 1, 0 | \, 10 \rangle \, \langle \eta\pi \| \hat{T}^{(1)} \| \pi\pi; I \rangle$$

- They can be written in terms of a **single amplitude** ($\eta\pi^0 \to \pi^+\pi^-$), $A(s,t,u)$ (like in $\pi\pi$ scattering):

$$\begin{bmatrix} -\sqrt{3}\mathcal{M}^0(s,t,u) \\ \sqrt{2}\mathcal{M}^1(s,t,u) \\ \sqrt{2}\mathcal{M}^2(s,t,u) \end{bmatrix} = \begin{bmatrix} -\sqrt{3}\mathcal{M}^{0,0}(s,t,u) \\ \sqrt{2}\mathcal{M}^{1,1}(s,t,u) \\ \sqrt{2}\mathcal{M}^{2,1}(s,t,u) \end{bmatrix} = \begin{bmatrix} 3 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} A(s,t,u) \\ A(t,s,u) \\ A(u,t,s) \end{bmatrix}$$

- Reconstruction theorem (for Goldstone bosons):

J. Stern, H. Sazdjian, N. Fuchs, Phys. Rev. D**47**, 3814 (1993)

$$A(s,t,u) = -\epsilon_L \Big[M_0(s) - \frac{2}{3} M_2(s) + M_2(t) + M_2(u) \qquad \epsilon_L = \frac{1}{Q^2} \frac{m_K^2 - m_\pi^2}{3\sqrt{3} f_\pi^2} \frac{m_K^2}{m_\pi^2}$$
$$+ (s-u) M_1(t) + (s-t) M_1(u) \Big]$$

- Or in general, "the" KT approximation:

Infinite sum of s-channel PW $\to$ Truncated sums of s-, t-, and u-channels PWs

- **Single variable functions:** amenable for dispersion relations.

Khuri-Treiman: elastic channels. Partial wave amplitudes

- Summary of previous slide: $\mathcal{M}^I(s,t,u)$ is written in terms of $A(s,t,u)$ (and permutations), and $A(s,t,u)$ is written in terms of $M_I(w)$.
- Now, define **partial waves:** $\mathcal{M}^I(s,t,u) = 16\pi\sqrt{2}\sum_j (2j+1)\mathcal{M}^I_j(s)P_j(z)$

$$\mathcal{M}_0^0(s) = \epsilon_L \frac{\sqrt{6}}{32\pi}[M_0(s)+\hat{M}_0(s)]\ , \quad \mathcal{M}_0^2(s) = \epsilon_L \frac{-1}{32\pi}[M_2(s)+\hat{M}_2(s)]\ ,$$

$$\mathcal{M}_1^1(s) = \epsilon_L \frac{\kappa(s)}{32\pi}[M_1(s)+\hat{M}_1(s)]\ ,$$

LHC $[\hat{M}_I(s)]$

$\hat{M}_I(s)$ written as angular averages. Take $M_0(s)$ as an example:

$$\hat{M}_0(s) = \frac{2}{3}\langle M_0\rangle + \frac{20}{9}\langle M_2\rangle$$

$$+\,2(s-s_0)\langle M_1\rangle + \frac{2}{3}\kappa(s)\langle zM_1\rangle$$

$$\langle z^n M_I\rangle(s) = \frac{1}{2}\int_{-1}^{1} \mathrm{d}z\ z^n M_I(t(s,z))$$

$$\kappa(s) = \sqrt{(1-4m_\pi^2/s)\lambda(s,m_\eta^2,m_\pi^2)}$$

RHC $[M_I(s)]$

$\hat{M}(s)$ no discontinuity along the RHC:

$$\text{disc } M_I(s) = \text{disc}\mathcal{M}^I_j(s) =$$

$$= \sigma_\pi(s)t^I(s)^*\mathcal{M}^I_j(s)$$

$$= \sigma_\pi(s)t^I(s)^*\left(M_I(s)+\hat{M}_I(s)\right)$$

$$\sigma_\pi(s) = \sqrt{1-4m_\pi^2/s}$$

$$\sigma_\pi(s)t^I(s) = \sin\delta_I(s)\, e^{i\delta_I(s)}$$

Khuri-Treiman: elastic channels. Muskhelisvili-Omnès representation

$$\boxed{\text{disc}M_I(s) = \sigma_\pi(s) t_I^*(s) \left[M_I(s) + \hat{M}_I(s) \right]}$$

- MO (dispersive) representation of $M_I(s)$:

- $m_\eta^2 + i\varepsilon$ prescription needed.
- Integral equations solved iteratively.

Khuri-Treiman: elastic channels. Muskhelisvili-Omnès representation

$$\boxed{\mathrm{disc} M_I(s) = \sigma_\pi(s) t_I^*(s) \left[M_I(s) \right]}$$

- MO (dispersive) representation of $M_I(s)$:

$$\begin{aligned} M_0(s) &= \Omega_0(s) \left[P_0(s) \right] , \\ M_1(s) &= \Omega_1(s) \left[P_1(s) \right] , \\ M_2(s) &= \Omega_2(s) \left[P_2(s) \right] . \end{aligned}$$

$$\Omega_I(s) = \exp \left[\frac{s}{\pi} \int_{4m_\pi^2}^{\infty} \mathrm{d}s' \, \frac{\delta_I(s')}{s'(s'-s)} \right] \text{ (Omnès function/matrix)}$$

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$$\hat{I}_I(s) = \frac{1}{\pi} \int_{4m_\pi^2}^{\infty} \frac{\sin \delta_I(s') \hat{M}_I(s')}{|\Omega_I(s')| \, (s')^{m_I} (s'-s)} , \quad (m_{0,2} = 2, \; m_1 = 1)$$

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$$M_0(s) = \Omega_0(s)\left[\alpha_0 + \beta_0 s + \gamma_0 s^2 + \hat{I}_0(s) s^2\right] ,$$
$$M_1(s) = \Omega_1(s)\left[\beta_1 s + \hat{I}_1(s) s\right] ,$$
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Unitarity and closed sys. of KT eqs.

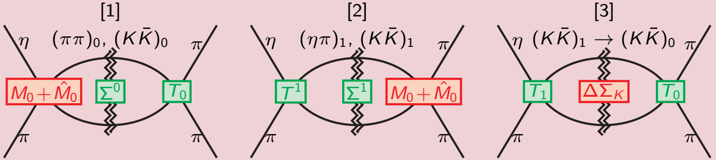
Coupled channels: take into account **intermediate states** other than $(\pi\pi)_I$.

$$\boldsymbol{M}_0 = \begin{bmatrix} M_0 & G_{10} \\ N_0 & H_{10} \end{bmatrix} = \begin{bmatrix} (\eta\pi)_1 \to (\pi\pi)_0 & (K\bar{K})_1 \to (\pi\pi)_0 \\ (\eta\pi)_1 \to (K\bar{K})_0 & (K\bar{K})_1 \to (K\bar{K})_0 \end{bmatrix},$$

$$\boldsymbol{T}_0 = \begin{bmatrix} t_{(\pi\pi)_0\to(\pi\pi)_0} & t_{(\pi\pi)_0\to(K\bar{K})_0} \\ t_{(\pi\pi)_0\to(K\bar{K})_0} & t_{(K\bar{K})_0\to(K\bar{K})_0} \end{bmatrix}, \boldsymbol{T}_1 = \begin{bmatrix} t_{(\eta\pi)_1\to(\eta\pi)_1} & t_{(\eta\pi)_1\to(K\bar{K})_1} \\ t_{(\eta\pi)_1\to(K\bar{K})_1} & t_{(K\bar{K})_1\to(K\bar{K})_1} \end{bmatrix}$$

$$\begin{aligned} \text{disc } \boldsymbol{M}_0(s) &= \boldsymbol{T}^{0*}(s)\Sigma^0(s)\left[\boldsymbol{M}_0(s+i\epsilon)+\hat{\boldsymbol{M}}_0(s)\right] && \to [1] \\ &+ \left[(\boldsymbol{M}_0(s-i\epsilon)+\hat{\boldsymbol{M}}_0(s)\right]\Sigma^1(s)\,\boldsymbol{T}^1(s) && \to [2] \\ &+ \boldsymbol{T}^{0*}(s)\Delta\Sigma_K(s)\boldsymbol{T}^1(s) && \to [3] \end{aligned}$$

Schematically:



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- MO representation for $\boldsymbol{M}_0(s)$:

$$\begin{bmatrix} M_0(s) & G_{10}(s) \\ N_0(s) & H_{10}(s) \end{bmatrix} = \boldsymbol{\Omega}_0(s)\left[\boldsymbol{P}_0(s)+s^2\left(\hat{\boldsymbol{I}}_a(s)+\hat{\boldsymbol{I}}_b(s)\right)\right]{}^t\boldsymbol{\Omega}_1(s)$$

- $\boldsymbol{P}_0(s)$ is a matrix of polynomials.
- The $\hat{\boldsymbol{I}}(s)$ functions are:

$$\hat{\boldsymbol{I}}_{a,b}(s) = \frac{1}{\pi}\int_{4m_\pi^2}^{\infty}\frac{ds'}{(s')^2(s'-s)}\,\Delta\boldsymbol{X}_{a,b}(s')\ ,$$

$$\Delta\boldsymbol{X}_a = \boldsymbol{\Omega}_0^{-1}(s-i\epsilon)\left[\underbrace{\boldsymbol{T}^{0*}(s)\,\Sigma^0(s)\,\hat{\boldsymbol{M}}_0(s)}_{[1]}+\underbrace{\hat{\boldsymbol{M}}_0(s)\,\Sigma^1(s)\,\boldsymbol{T}^1(s)}_{[2]}\right]{}^t\boldsymbol{\Omega}_1^{-1}(s+i\epsilon)\ ,$$

$$\Delta\boldsymbol{X}_b = \underbrace{\boldsymbol{\Omega}_0^{-1}(s-i\epsilon)\boldsymbol{T}^{0*}(s)\,\Delta\Sigma_K(s)\,\boldsymbol{T}^1(s)\,{}^t\boldsymbol{\Omega}_1^{-1}(s+i\epsilon)}_{[3]}$$

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$$\Delta\boldsymbol{X}_b = \underbrace{\boldsymbol{\Omega}_0^{-1}(s-i\epsilon)\boldsymbol{T}^{0*}(s)\,\Delta\Sigma_K(s)\,\boldsymbol{T}^1(s)\,{}^t\boldsymbol{\Omega}_1^{-1}(s+i\epsilon)}_{[3]}$$

Khuri-Treiman: coupled channels. Amplitudes M_1 and M_2

An analogous analysis can be done with $M_1(s)$ and $M_2(s)$ amplitudes:

$M_1(s)$ [P-wave]

$$\boldsymbol{M}_1(s) = \begin{bmatrix} M_1 \\ N_1 \end{bmatrix} = \begin{bmatrix} (\eta\pi)_{1-} \to (\pi\pi)_{1+} \\ (\eta\pi)_{1-} \to (K\bar{K})_{1+} \end{bmatrix}$$

$$\boldsymbol{T}_1^1(s) = \begin{bmatrix} (\pi\pi)_1 \to (\pi\pi)_1 & (\pi\pi)_1 \to (K\bar{K})_1 \\ (\pi\pi)_1 \to (K\bar{K})_1 & (K\bar{K})_1 \to (K\bar{K})_1 \end{bmatrix}$$

$$\Delta \boldsymbol{M}_1(s) = \boldsymbol{T}_1^{1*}(s)\Sigma^0(s) \times \left[\boldsymbol{M}_1(s+i\epsilon) + \hat{\boldsymbol{M}}_1(s)\right]$$

$M_2(s)$ [S-wave]

$$\boldsymbol{M}_2(s) = \begin{bmatrix} M_2 \\ G_{12} \end{bmatrix} = \begin{bmatrix} (\eta\pi)_1 \to (\pi\pi)_2 \\ (K\bar{K})_1 \to (\pi\pi)_2 \end{bmatrix}$$

$$t_0^2(s) = t_{(\pi\pi)_2 \to (\pi\pi)_2}$$

$$\text{disc}\, \boldsymbol{M}_2(s) = \boldsymbol{T}^1(s)\Sigma^1(s) \times (\boldsymbol{M}_2(s-i\epsilon) + \hat{\boldsymbol{M}}_2(s)) + \sigma_\pi(s)(t_0^2(s))^*(\boldsymbol{M}_2(s+i\epsilon) + \hat{\boldsymbol{M}}_2(s))$$

- **Consistent approximation:** $\hat{N}_0(s)$, $\hat{G}_{10}(s)$, $\hat{H}_{10}(s)$, $\hat{G}_{12}(s)$: we neglect these LHC functions (would require all the related cross channels amplitudes. . .).
- Further approximation: For $I = J = 1$, we consider elastic $\pi\pi$.

Khuri-Treiman: coupled channels. Matching with chiral amplitudes

- **Elastic case:** $M_0(s) = \Omega_0(s)\left(\alpha_0 + \beta_0 s + s^2(\gamma_0 + \hat{I}_0(s))\right)$
- Subtraction constants: Most natural way is to match with **ChPT:**

Descotes-Genon, Moussallam, Eur. Phys. J. C**74**, 2946 (2014)

$$\boxed{\mathcal{M}(s,t,u) - \overline{\mathcal{M}}_\chi(s,t,u) = \mathcal{O}(p^6)}$$

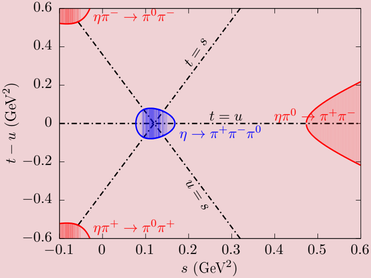
- The difference of the discontinuities start already at $\mathcal{O}(p^6)$, so the polynomial expansion of $M_I(s) - \bar{M}_I(s)$ can be inserted above. **Matching conditions:**

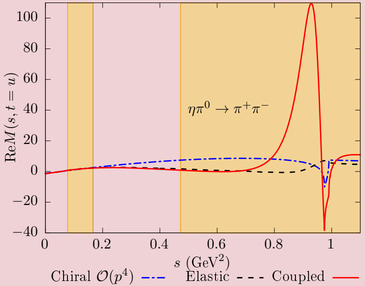
$$\begin{aligned}
\alpha_0 &= \bar{M}_0(0) + \frac{4}{3}\,\bar{M}_2(0) + 3s_0\,(\bar{M}_2'(0) - \bar{M}_1(0)) + 9s_0^2\,\bar{M}_2^{eff} \\
\beta_0 &= \bar{M}_0'(0) + 3\,\bar{M}_1(0) - \frac{5}{3}\,\bar{M}_2'(0) - 9s_0\,\bar{M}_2^{eff} - \Omega_0'(0)\,\alpha_0 \\
\beta_1 &= \bar{M}_1'(0) - \hat{I}_1(0) + \bar{M}_2^{eff} \\
\gamma_0 &= \frac{1}{2}\,\bar{M}_0''(0) - \hat{I}_0(0) + \frac{4}{3}\,\bar{M}_2^{eff} - \frac{1}{2}\Omega_0''(0)\,\alpha_0 - \Omega_0'(0)\,\beta_0 \\
\bar{M}_2^{eff} &= \frac{1}{2}\,\bar{M}_2''(0) - \hat{I}_2(0)
\end{aligned}$$

- For the **coupled channel** case, the same method is applied to fix 16 polynomial parameters (more and lengthier equations).

Outline

Results. Amplitudes





Behaviour in different regions:

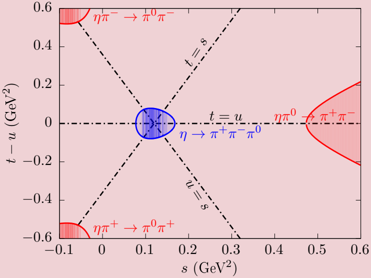
- $\boxed{s \sim 1\ \mathrm{GeV}^2}$ Very sharp energy variation,
  - $a_0(980)$ and $f_0(980)$ interference,
  - K^+K^- and $K^0\bar{K}^0$ thresholds.

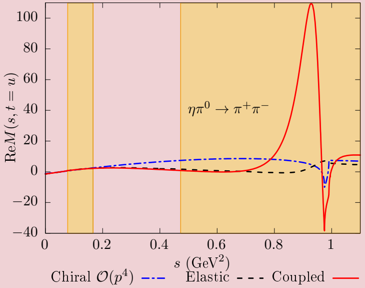
Coupled channel largely **enhanced** compared with elastic amplitude.

Effect of coupled channels is to reduce the amplitude.

Elastic and inelastic amplitudes indistinguishable.

Results. Amplitudes





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- $0.7 \lesssim s \lesssim 0.97$ GeV^2 Coupled channel largely **enhanced** compared with elastic amplitude.

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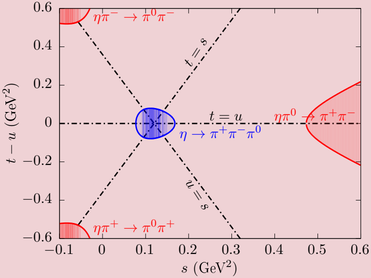
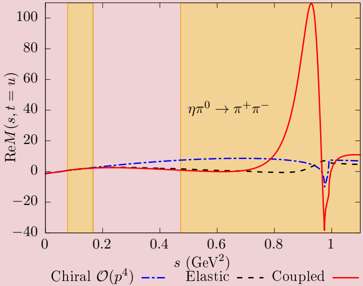
Results. Amplitudes





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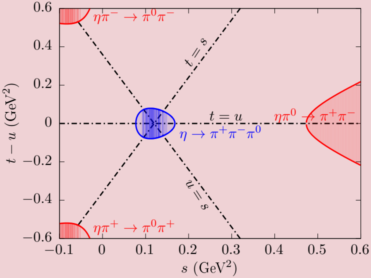

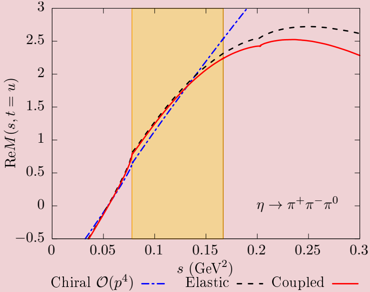

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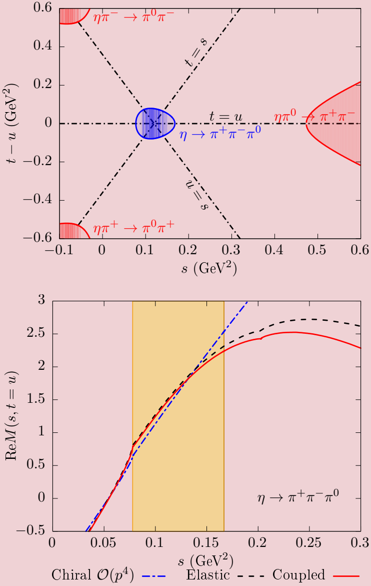

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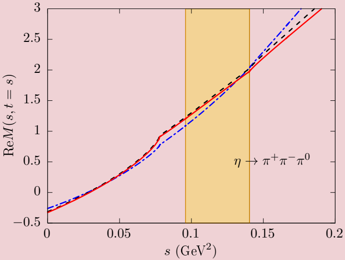
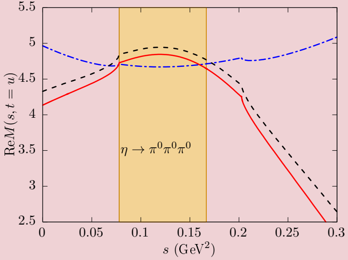
- $s \lesssim s_{th}$ Elastic and inelastic amplitudes indistinguishable.

Results. Amplitudes (cont'd)





- Subthreshold region: chiral, elastic, and coupled amplitudes **very close.**
- Adler zero:

	NLO	el.	cou.
$s_A/m_{\pi^+}^2 =$	1.42	1.45	1.49

- Substantial influence of coupled channels in the whole region,
- and there is no region in which dispersive and chiral amplitudes agree.
- $T_{\eta\to 3\pi^0} = M_0(s)+M_0(t)+M_0(u)+\dots$

Results: Dalitz plot parameters

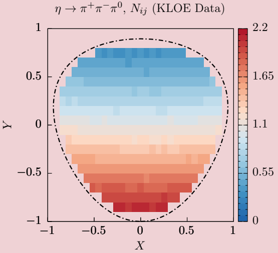
- DP variables X,Y: $X = \frac{\sqrt{3}}{2m_\eta Q_c}(u-t)$, $Y = \frac{3}{2m_\eta Q_c}\left((m_\eta - m_{\pi^0})^2 - s\right) - 1$

- Charged mode amplitude written as:

$$\frac{|M_c(X,Y)|^2}{|M_c(0,0)|^2} = \boxed{1 + a\,Y + b\,Y^2 + d\,X^2 + f\,Y^3 + g\,X^2 Y} + \cdots$$

- Neutral decay mode amplitude $[Q_c \to Q_n]$:

$$\frac{|M_n(X,Y)|^2}{|M_n(0,0)|^2} = \boxed{1 + 2\alpha\,|z|^2 + 2\beta\,\mathrm{Im}(z^3)} + \cdots$$



		$O(p^4)$	elastic	coupled	KLOE	BESIII
charged	a	-1.328	-1.156	-1.142(45)	-1.095(4)	-1.128(15)
	b	0.429	0.200	0.172(16)	0.145(6)	0.153(17)
	d	0.090	0.095	0.097(13)	0.081(7)	0.085(16)
	f	0.017	0.109	0.122(16)	0.141(10)	0.173(28)
	g	-0.081	-0.088	-0.089(10)	-0.044(16)	–
neutral					PDG	
	α	+0.0142	-0.0268	-0.0319(34)	-0.0318(15)	
	β	-0.0007	-0.0046	-0.0056	–	

BESIII Collab., Phys. Rev. D**92**,012014 (2015)
KLOE-2 Collab., JHEP **1605**, 019 (2016)

- (Theory) **uncertainty estimation**:
  1. $\eta\pi$ interaction put to zero or to "large"
  2. $10^3 L_3^r = -3.82 \to -2.65$
- General trend: **improve agreement** $[\mathcal{O}(p^4) \to$ elastic $\to$ coupled$]$
- Particularly relevant: $\boldsymbol{\alpha}$.

Results. Quark mass ratio Q

From the amplitudes $M_I(s)$ one can compute the width up to the unknown factor Q^2:

$$\Gamma = \epsilon_L^2 \int_{4m_\pi^2}^{m_-^2} ds \int_{t_-(s)}^{t_+(s)} dt\, |M_0(s) + \cdots|^2$$

$$\epsilon_L = Q^{-2} \frac{m_K^2 - m_\pi^2}{3\sqrt{3} f_\pi^2} \frac{m_K^2}{m_\pi^2}\,, \qquad Q^{-2} = \frac{m_d^2 - m_u^2}{m_s^2 - \hat{m}^2}$$

$\Gamma(\eta \to 3\pi^0)/\Gamma(\eta \to \pi^+\pi^-\pi^0)$	
PDG (fit)	1.426(26)
PDG (average)	1.48(5)
CLEO	1.496(43)(35)
chiral $\mathcal{O}(p^4)$	1.425
elastic	1.449
coupled	1.451

Decay	Q elastic	Q coupled
$\Gamma^{(\text{exp})}_{(\text{neu.})} = 299(11)$ eV	21.9(2)	21.7(2)
$\Gamma^{(\text{exp})}_{(\text{cha.})} = 427(15)$ eV	21.8(2)	21.6(2)

- Effect of inelastic channels $\sim 1\%$ (decreasing)
- Theoretical error on Q:
  - Phase shifts [$s \leqslant 1$ GeV^2]: $\sim 1\%$
  - $\mathcal{O}(p^4)$ ampl. [L_3]: $\sim 1\%$
  - NNLO ampl.: $\Delta Q_{\text{th.}} = \pm 2.2$

$$\boxed{Q = 21.6 \pm 0.2 \pm 2.2}$$

- Fitted (not matched) polynomial parameters:

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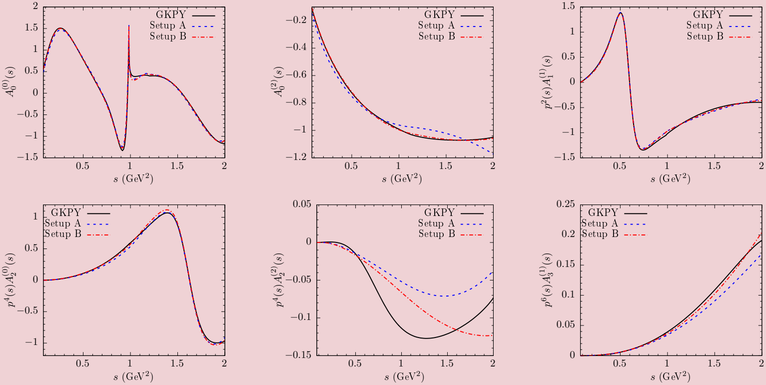
$$\boxed{Q_{\text{fit}} = 21.50 \pm 0.67 \pm 0.70}$$

Khuri-Treiman and Roy equations for $\pi\pi$

MA *et al.* [JPAC Collaboration], *in preparation*

- What happens if you apply KT equations to $\pi\pi$ scattering?
- KT equations can be cast as integral equations, just as Roy equations for $\pi\pi$



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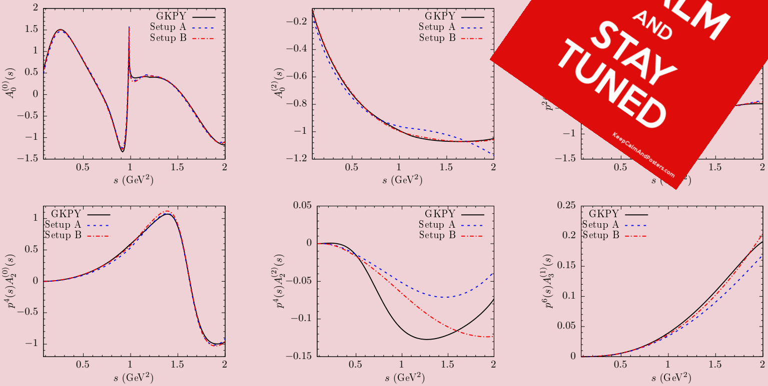

Outline

Summary

- $\eta \to 3\pi$ is not well described by the perturbative chiral amplitudes. Rescattering.
- **Khuri-Treiman equations** are a good approach to this problem. They include $\pi\pi$ unitarity effects in $s-$, $t-$, and u channels.
- We have presented an **extension** of this approach to **coupled channels**. The extension is quite **general**.
- In particular, we have included the effects of $K\bar{K}$ and $\eta\pi$ amplitudes [$f_0(980)$, $a_0(980)$]
- The effect on the amplitudes is very large around 1 GeV, but small in the decay region. Nonetheless, the effects on the DP parameters (in particular for the $\eta \to 3\pi^0$ parameter α) tend to improve the agreement with the experimental ones.
- This extension could be used in **other 3-body decays** as well.

$\eta \to 3\pi$ in coupled-channels Khuri-Treimann formalism

Based on: Eur. Phys. J. C**77**, 508 (2017)
[arXiv:1702.04931]

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In collaboration with:
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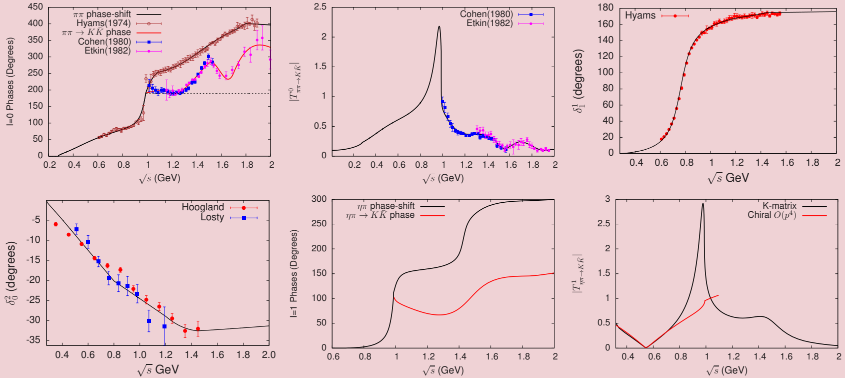
Isospin conserving T-matrices



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Fitting