Pole structure and compositeness

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1. Basic set-up

*Highly subjective presentation of the vast literature. My apologizes if you think that important works are missed*

Starting point, basic set-up:

**Bound state near a two-body threshold.**

**Non-Relativistic Dynamics**

S. Weinberg, PR130,776(1963); PR137,B672(1964)

\[ H = H_0 + V \]

Spectrum:

\[ H|\psi_\alpha\rangle = E_\alpha|\psi_\alpha\rangle \quad \text{Continuum spectrum} \]
\[ H|\psi_{B_i}\rangle = E_{B_i}|\psi_{B_i}\rangle \quad , \quad E_{B_i} < 0 \quad \text{Discrete Spectrum} \]

Bare spectrum:

\[ H_0|\varphi_\alpha\rangle = E_\alpha|\varphi_\alpha\rangle \]
\[ H_0|\varphi_n\rangle = E_n|\varphi_n\rangle \]
Elementariness: $Z$

Compositeness: $X$

\[
\langle \psi_B | \psi_B \rangle = 1 = \sum_n \left| \langle n | d \rangle \right|^2 + \int d\alpha \left| \langle \varphi_{\alpha} | d \rangle \right|^2
\]

\[
1 = Z + X
\]

\[
X = 1 - Z = \int d\alpha \frac{\left| \langle \varphi_{\alpha} | V | \psi_B \rangle \right|^2}{(E_{\alpha} - E_B)^2}
\]

Wave Function Renormalization: $Z^{1/2}$

There is only one “elementary” bare state around $E_B$  

S. Weinberg, 

PRC130,776(1963);PRC131,440(1963)

\[
\langle \varphi_0 | \psi_B \rangle = Z^{1/2}
\]
2. Different perspective

We focus our attention in the continuum spectrum
The continuum spectrum is common to $H$ and $H_0$

Let two particles of types $A$ and $B$

Creation, annihilation operators: $a_\alpha^\dagger a_\alpha, b_\beta^\dagger b_\beta$

Operator Numbers:

$$H_0 = \int dE_\alpha E_\alpha a_\alpha^\dagger a_\alpha + \int d\beta E_\beta b_\beta^\dagger b_\beta + \sum_n E_n |\varphi_n\rangle \langle \varphi_n|$$

$$N_D = \int d\alpha a_\alpha^\dagger a_\alpha + \int d\beta b_\beta^\dagger b_\beta$$

$$= \int d^3x \left[ \psi_A^\dagger(x)\psi_A(x) + \psi_B^\dagger(x)\psi_B(x) \right]$$

$$[H_0, N_D] = 0 \rightarrow N_D(t) = N_D(0) = N_H(0)$$
New definition of $X$

$$X = \frac{1}{2} \langle \psi_B | N_D | \psi_B \rangle$$

Equivalence to the previous definition

$$|\psi_B\rangle = \int d\gamma C_\gamma |AB\gamma\rangle + \sum_n C_n |\varphi_n\rangle$$

$$X = \frac{1}{2} \langle \psi_B | N_D | \psi_B \rangle = \int d\gamma |C_\gamma|^2 .$$

Specially suitable when using perturbative EFT (e.g. ChPT) with nonperturbative techniques
3. QFT-like Calculation

This new definition is suitable for a QFT treatment

Dirac or Interacting Image

\[ V \rightarrow Ve^{-\varepsilon|t|}, \quad \varepsilon \rightarrow 0^+ \]

\[ |\psi_B\rangle = |\varphi_B(0)\rangle = UD(0, -\infty)|\varphi_B(-\infty)\rangle \]

\[ |\psi_B\rangle = |\varphi_B(0)\rangle = UD(0, +\infty)|\varphi_B(+\infty)\rangle \]

\[ X = \frac{1}{2} \langle \psi_B | N_D | \psi_B \rangle \]

\[ = \frac{1}{2} \langle \varphi_B(+\infty) | UD(+\infty, 0) N_D UD(0, -\infty) | \varphi_B(-\infty) \rangle \]

\[ = \lim_{T \to \infty} \frac{1}{2T} \int_{-T/2}^{+T/2} dt \langle \varphi_B(+\infty) | UD(+\infty, t) N_D(t) UD(t, -\infty) | \varphi_B(-\infty) \rangle \]

\[ UD(t, -\infty)|\varphi_B(-\infty)\rangle = e^{iH_0t} e^{-iE_Bt} UD(0, -\infty)|\varphi_B(-\infty)\rangle \]
Basic set-up Different perspective QFT-like calculation New equation for $X$ Relativistic case Resonances $S$-matrix transformations

$\psi_A \rightarrow \psi_B$ $\psi_A \rightarrow \psi_B$

\[ X_{\ell S} = \int \frac{d^3 k}{(2\pi)^3} \frac{g_{\ell S}^2(k^2)}{(k^2/2\mu - E_B)^2} \]

\[ X = \sum_{\ell S} X_{\ell S} \]

$\ell = 0$: one has the Weinberg's equation for $1 - Z$

\[ g(k) = \frac{1}{2\pi^2} \int_0^\infty k'^2 dk' V(k, k') \frac{1}{k'^2/2\mu - E_B} g(k') \]

\[ g_{\ell S}(-k) = (-1)^\ell g_{\ell S}(k) \quad [T = V + VGT] \]
4. New equation for $X$

We rewrite symmetrically the integration in $k$ for $X$

$$X = \left( \frac{\mu}{\pi} \right)^2 \int_{-\infty}^{+\infty} dk k^2 \frac{g^2(k^2)e^{i\varepsilon k}}{(k^2 - 2\mu E_B)^2}$$

Convergent factor $e^{i\varepsilon k}$, $\varepsilon \to 0^+$

Analogous to Dimensional Regularization

Notation: $\pm i\gamma = \sqrt{2\mu E_B}$

$$X = \frac{2i\mu^2}{\pi} \frac{\partial}{\partial k} \left[ \frac{k^2 g^2(k^2)}{(k + i\gamma)^2} \right]_{k=i\gamma}$$

$$= \frac{\partial G}{\partial E_B} g^2(-\gamma^2) - \frac{\mu^2 \gamma}{\pi} \frac{\partial g^2(k^2)}{\partial k^2} \bigg|_{k=i\gamma}$$

$$G(E) = -\frac{i\mu}{2\pi} k(E) = \frac{\mu}{\pi^2} \int_0^\infty dk \frac{k^2 e^{i\varepsilon k}}{2\mu E_B - k^2}$$
The 1st term is model independent
Already a well known contribution Hyodo, Jido, Hosaka PRC85, 015201 (2012)

The 2nd term is the new one.
E.g. it takes into account that $g_{\ell S}(k^2) \propto k^{2\ell}$ for $k \to 0$
Aceti, Oset, PRD86, 014012 (2012)

The 2nd term depends on $V$

\[
g(k) = \frac{1}{2\pi^2} \int_0^\infty k'^2 dk' V(k, k') \frac{1}{k'^2/2\mu - E_B} g(k')
\]
5. Relativistic case

The attention is focused on the wave function renormalization $Z$

There is a lack of a general framework

Partial results are available:

$0 \leq Z \leq 1$: Lee model Vaughn, Aaron, Amado PRC124,1258(1961); Yukawa type interactions Salam, Nuovo Cim.25,224(1962); Lurié, Macfarlane, PR136,B816(1964)

$Z=0$ equivalence between 4-Fermi theories and Yukawa theories

Issue: In the relativistic case we can also have multiparticle states. E.g. $\pi\pi$, $4\pi$, . . . , being all of them present in the spectrum of $H_0$

Nonetheless:

$$[H_0, N_D] = 0$$
Non-Relativistic formalism:

\[ |\psi_B\rangle = \int d\gamma C_{\gamma} |AB\rangle + \sum_n C_n |\varphi_n\rangle \]

Relativistic formalism:

\[ |\psi_B\rangle = \int d\gamma C_{\gamma} |AB\rangle + \int d\eta D_{\eta} |AAB\rangle + \int d\mu \delta_{\mu} |ABB\rangle + \ldots \]

\[ + \int d\eta_{\nu} F_{\nu} |CD\rangle + \ldots + \sum_n C_n |\varphi_n\rangle + \sum_n \int d\alpha C_{n\alpha} |A_{\alpha}\varphi_n\rangle + \ldots \]

We provide a new criterion for elementariness of a relativistic bound state

Number operator for each species of particle

\[ N^A_D = \int d\alpha a^{\dagger}_{\alpha} a_{\alpha} \]

\[ \langle \psi_B | N^A_D |\psi_B\rangle = \int d\alpha |C_{\gamma}|^2 + 2 \int d\eta |D_{\eta}|^2 + \int d\mu |\delta_{\mu}|^2 + \ldots \]
Criterion for an elementary stable particle

\[ \langle N_D^A \rangle \equiv \langle \psi_B | N_D^A | \psi_B \rangle \ll 1 \]

For relativistic systems we also have to discard that

\[ \langle N_D^E \rangle \equiv \langle \psi_B | N_D^E | \psi_B \rangle > 1 \]

\( N_D^E \) is the total number operator of bare elementary states.
Criterion for an elementary stable particle

\[ \langle N_D^A \rangle \equiv \langle \psi_B | N_D^A | \psi_B \rangle \ll 1 \]

We can calculate this expectation value within QFT too

\[ V \to Ve^{-\varepsilon |t|} \]

\[ |\psi_B\rangle = |\varphi_B(0)\rangle = U_D(0, -\infty)|\varphi_B(-\infty)\rangle \]

\[ |\psi_B\rangle = |\varphi_B(0)\rangle = U_D(0, +\infty)|\varphi_B(+\infty)\rangle \]

\[ \langle N_D^A \rangle = \langle \psi_B | N_D^A | \psi_B \rangle \]

\[ = \langle \varphi_B(+\infty) | U_D(+\infty, 0) N_D^A U_D(0, -\infty) | \varphi_B(-\infty) \rangle \]

\[ = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{+T/2} dt \langle \varphi_B(+\infty) | U_D(+\infty, t) N_D^A(t) U_D(t, -\infty) | \varphi_B(-\infty) \rangle \]

\[ U_D(t, -\infty)|\varphi_B(-\infty)\rangle = e^{iH_0 t} e^{-iE_B t} U_D(0, -\infty)|\varphi_B(-\infty)\rangle \]
**Technicalities**

In general there are many more diagrams now apart from those in the NR case.

One cannot apply the trick associated with the convergent factor in the same way.

**Other consequences:**

If

\[ \sum_i \langle N_{Di}^A \rangle = 2 + m \geq 2 \]

The multiparticle components with \(2 + m\) and more particles are important.

Other similar *exclusive* conditional conclusions can be also established.

- In the case in which we are close to a two-body threshold \((AB)\) we come back to the NR case for evaluating \(X_{AB}\).
• If *there are good reasons* for dominance of two-body channels

\[ X_{AB} = \frac{1}{2} \left( \langle N_D^A \rangle + \langle N_D^B \rangle \right) \approx \int d\gamma |C_\gamma|^2 \]
6. Resonances. NR case

Bogdanova, Hale, Markushin, PRC44,1289(1991);

**Spectral density of the bare state** $|\psi_0\rangle : \omega(E)$

$$|\psi_0\rangle = \int dk c_0(k)|k\rangle$$

$$\omega(E) = 4\pi \mu k|c_0(E)|^2 \theta(E)$$

$$\int_0^\infty dE \omega(E) = \begin{cases} 
1 & \text{No bound states} \\
1 - Z & \text{With bound states}
\end{cases}$$

How to implement it? **Select** the resonant region around threshold

$$W = \int_{E_-}^{E_+} dE \omega(E)$$

Conceptually, it is not **fully** settled as a quantitative estimate of compositeness for resonances
It provides a nice smooth transition from the clear bound states and narrow resonances.

It has a clear connection with the pole-counting rule of Morgan NPA543,632(1992), with the presence of nearby CDD poles Kang, Oller, EPJC77,399(2017).
7. Operator-number interpretation

In my developments a resonance follows by analytical continuation from the physical axis

- Let \( |\psi^+_\alpha\rangle \) be a two-body in-state

\[
|\psi^+_\alpha\rangle = U_D(0, -\infty) |\varphi_\alpha\rangle \\
= |\varphi_\alpha\rangle + \int d\gamma \frac{T_{\gamma\alpha}(E + i\varepsilon)}{E - E_\gamma + i\varepsilon} |\varphi_\gamma\rangle + \sum_n \frac{T_{n\alpha}(E)}{E - E_n} |\varphi_n\rangle
\]

S. Weinberg, QFT, Vol.1

\[
\langle\psi^+_\alpha| \int d\gamma a_\gamma^\dagger a_\gamma + \int d\eta b_\eta^\dagger b_\eta |\psi^+_\alpha\rangle = 2 \langle\varphi_\alpha|\varphi_\alpha\rangle \quad \text{Fine!}
\]

There are cancellations because of unitarity
Problem: This expectation value cannot be analytically continued to the resonance pole

\[
\langle \psi_\alpha^+ \rangle = \langle \varphi_\alpha \rangle + \int d\gamma \frac{T_{\gamma\alpha}(E - i\varepsilon)}{E - i\varepsilon - E_\gamma} \langle \varphi_\gamma \rangle + \sum_n \frac{T_{n\alpha}(E - i\varepsilon)}{E - i\varepsilon - E_n} \langle \varphi_n \rangle
\]

\[T(E \pm i\varepsilon)^\dagger = T(E \mp i\varepsilon)\]

The analytical continuation to \( E = M_R - i\Gamma/2 \) remains in the 1st or physical Riemann Sheet (RS)
No resonance pole there
The analytical continuation must be done as in the calculation of the $S$-matrix:

**out state** $|\psi_{\alpha}^-\rangle$, $E - i\epsilon$

$$
\langle \psi_{\alpha}^- | N_D^A + N_D^B | \psi_{\alpha}^+ \rangle
$$

$$
\langle \psi_{\alpha}^- | = \langle \varphi_{\alpha} | + \int d\gamma \frac{T_{\gamma\alpha}(E + i\epsilon)}{E + i\epsilon - E_{\gamma}} \langle \varphi_{\gamma} | + \sum_n \frac{T_{n\alpha}(E + i\epsilon)}{E + i\epsilon - E_n} \langle \varphi_n |
$$

When crossing the real positive energy axis

$$
T(E + i\epsilon) \rightarrow T^{II}(E - i\epsilon)
$$

The resonance pole is now reached both for the ket and the bra.
8. QFT-like calculation

Dirac or Interacting Image

\[ V \rightarrow V e^{-\varepsilon |t|} \]

\[ |\psi_R^+\rangle = U_D(0, -\infty) |\varphi_R^+\rangle \]

\[ \langle \psi_R^- | = \langle \varphi_R^- | U_D(+\infty, 0) \]

\[ X = \frac{1}{2} \langle \psi_R^- | N_D |\psi_R^+\rangle \]

\[ = \frac{1}{2} \langle \varphi_R^- | U_D(+\infty, 0) N_D U_D(0, -\infty) |\varphi_R^+\rangle \]

\[ = \lim_{T \to \infty} \frac{1}{2T} \int_{-T/2}^{+T/2} dt \langle \varphi_R^- | U_D(+\infty, t) N_D(t) U_D(t, -\infty) |\varphi_R^+\rangle \]

\[ U_D(t, -\infty) |\varphi_R^+\rangle = e^{iH_0 t} e^{-iH t} |\psi_R^+\rangle = e^{-(IM_R + \frac{\Gamma}{2}) t} e^{iH_0 t} U_D(0, -\infty) |\varphi_R^+\rangle \]

\[ \langle \varphi_R^- | U_D(+\infty, t) = \langle \psi_R^- | e^{iH t} e^{-iH_0 t} = \langle \varphi_R^- | U_D(+\infty, 0) e^{-iH_0 t} e^{(IM_R + \frac{\Gamma}{2}) t} \]

\[ \langle \psi_R^- | e^{iH t} e^{-iH_0 t} = \langle \varphi_R^- | U_D(+\infty, 0) e^{-iH_0 t} e^{(IM_R + \frac{\Gamma}{2}) t} \]
2nd Riemann Sheet: $E_R = \kappa^2/2\mu$

\[
X_{\ell S} = \int \frac{d^3k}{(2\pi)^3} \frac{g_{\ell S}^2(k^2)}{(k^2/2\mu - E_R)^2} + \frac{i\mu^2}{\pi\kappa} \frac{\partial}{\partial k} \left[ k g_{\ell S}^2(k^2) \right]_{k=\kappa}
\]

\[
X = \sum_{\ell S} X_{\ell S}
\]

\[
g(k) = \frac{\mu}{\pi^2} \int_0^{\infty} dk' k'^2 V(k, k') g(k') \frac{1}{k'^2 - \kappa^2}
\]

\[
+ \frac{i\mu\pi V(k, \kappa)}{1 - i\mu\pi V(\kappa, \kappa)} \frac{\mu}{\pi^2} \int_0^{\infty} dk' k'^2 V(\kappa, k') g(k') \frac{1}{k'^2 - \kappa^2}
\]

\[
g_{\ell S}(-k) = (-1)^\ell g_{\ell S}(k)
\]
9. New equation for $X$

We include the **convergent factor** for the 2nd RS calculation:

$$X = \frac{\mu^2}{\pi^2} \int_{-\infty}^{+\infty} dk k^2 \frac{g^2(k^2) e^{-i\varepsilon k}}{(k^2 - \kappa^2)^2} + \frac{i\mu^2}{\pi \kappa} \frac{\partial}{\partial k} \left[ k g^2(k^2) \right]_{k=\kappa}$$

$$X = \frac{i\mu^2}{2\pi \kappa} \frac{\partial}{\partial k} \left[ k g^2(k^2) \right]_{k=\kappa}$$

$$= -\frac{\partial G^{II}}{\partial E_R} g^2(\kappa^2) + \frac{i\mu^2}{2\pi} \frac{\partial}{\partial k} g^2(k^2) \bigg|_{k=\kappa}$$

The novel contribution is the red one

It depends on $V(k, k')$

$$X = \frac{\mu^2}{\pi^2} \int_0^{+\infty} dk k^2 \sqrt{k^2 + i\varepsilon} \frac{g^2(k^2)}{(k^2 - \kappa^2)^2}$$

**Wave function squared:**
- **resonance Gamow state**
For an energy-independent potential $X = 1$

$$V(k, k') = f(k^2)f(k'^2)V$$

In ordinary QM resonances are composite

$X$ is in general complex for a resonance

E.g. for $V = V(E)$

$$g(k^2) = V^\frac{1}{2}f(k^2) \left[ \frac{\partial (V \tilde{G}^{\Pi})}{\partial E_R} \right]^{-1}$$

$$X = \left[ \frac{\partial (V \tilde{G}^{\Pi})}{\partial E_R} \right]^{-1} \frac{-1}{(2\pi)^2} \int_0^\infty dk k^2 \frac{V f(k^2)^2}{(E_R - k^2/2\mu)^2}$$

$$= \left[ \frac{\partial (V \tilde{G}^{\Pi})}{\partial E_R} \right]^{-1} \left\{ V \frac{\partial \tilde{G}^{\Pi}}{\partial E_R} + \tilde{G}^{\Pi}(E_R) \frac{\partial V(E_R)}{\partial E_R} \right\}$$
10. Redefinition of “phases”

in-, out-states:

\( \eta(E) \) is a complex function with RHC: \( \eta(E^*) = \eta(E)^* \)

\[
\begin{align*}
|\psi^+_\alpha\rangle & \quad \longrightarrow \quad e^{\eta(E_\alpha^{} + i\varepsilon)} |\psi^+_\alpha\rangle \\
\langle \psi^-_\alpha | & \quad \longrightarrow \quad \langle \psi^-_\alpha | e^{\eta(E_\alpha^{} - i\varepsilon)^*} \\
& \quad = \quad \langle \psi^-_\alpha | e^{\eta(E_\alpha^{} + i\varepsilon)}
\end{align*}
\]

Analytical continuation \( E_\alpha \rightarrow E_R = M_R - i\Gamma/2 \)

\[
\eta(E_\alpha^{} + i\varepsilon) \rightarrow \eta^{\Pi}(E_\alpha^{} - i\varepsilon) \rightarrow \eta^{\Pi}(M_R - i\Gamma/2)
\]

An specific fact of resonances; no analogue for bound states.
These phase factors make $X_{AB}$ be positive definite.  
There could be dependence on the channel, $\eta_{AB}(E)$

$$g^2_{AB}(k^2) \to g^2_{AB}(k^2) e^{2\eta^\Pi_{AB}(E_R)}$$

$$X_{AB} \to \langle \psi_R^- | N^A_B | \psi_R^+ \rangle e^{2\eta^\Pi_{AB}(E_R)} \in \mathbb{R}^+$$

Plausible dispersion relation for $\eta(E)$
Narrow-Resonance Case:

$$\eta(E) = \frac{1}{\pi} \int_0^{\infty} dE' \frac{\text{Im} \eta(E')}{E' - M_R - i\epsilon}$$

$$= \frac{1}{\pi} \int_0^{\infty} dE' \frac{\text{Im} \eta(E')}{E' - M_R} + i\text{Im} \eta(E')$$

$\text{Im} \eta(E')$ is smooth and $\eta(M_R) \approx i\text{Im} \eta(M_R) \to e^{\eta(M_R)}$: Pure phase factor $|e^{\eta(M_R)}| \approx 1$
11. Relativistic case

\[ \langle N_D^A \rangle \equiv \langle \psi_R^{-} | N_D^A | \psi_R^{+} \rangle \]

The QFT expression can be applied in the relativistic case too

\[ V \rightarrow V e^{-\varepsilon |t|} \]

\[ \langle \psi_R^{-} | N_D^A | \psi_R^{+} \rangle = \langle \varphi_R^{-} | U_D (+\infty, 0) N_D^A U_D (0, -\infty) | \varphi_R^{+} \rangle \]

\[ = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{+T/2} dt \langle \varphi_R^{-} | U_D (+\infty, t) N_D^A (t) U_D (t, -\infty) | \varphi_R^{+} \rangle \]

**Necessary condition for a resonance to be qualified as elementary**

\[ \langle N_D^A \rangle = 0 \quad \forall A \]

**Criterion for an elementary narrow resonance** with respect the open channels

\[ \langle N_D^A \rangle = \langle \psi_R^{-} | N_D^A | \psi_R^{+} \rangle e^{2i \text{Im} \eta^H_A (E_R)} = \left| \langle \psi_R^{-} | N_D^A | \psi_R^{+} \rangle \right| \ll 1 \]
12. \(S\)-matrix transformations

Introduced in Z.H.Guo, Oller, PRD93,096001(2016)

**Example: Narrow resonance case**

Laurent series around the resonance pole: \(s_P = (M_R - i\Gamma/2)^2\)

\[
S(s) = \frac{R}{s - s_P} + S_0(s)
\]

\[S(s)S(s)^\dagger = I\]

\(S_0(s) \rightarrow S_0, \text{ constant}\)

\[
(s - s_P)(s - s_P^*)S_0S_0^\dagger + (s - s_P)S_0R^\dagger + (s - s_P^*)RS_0^\dagger + RR^\dagger = (s - s_P)(s - s_P^*)
\]

\[
S_0S_0^\dagger = I
\]

\[S_0R^\dagger + RS_0^\dagger = 0\]

\[- s_P S_0 R^\dagger - s_P^* R S_0^\dagger + RR^\dagger = 0\]
Solution:

\[ S_0 = \mathcal{O} \mathcal{O}^T \]
\[ \mathcal{O} \mathcal{O}^\dagger = I \]

Rank 1 Symmetric Projection Operator \( \mathcal{A} \):

\[ R = i \lambda \mathcal{A} \mathcal{O} \mathcal{O}^T, \quad \lambda \in \mathbb{R} \]
\[ \mathcal{A}^\dagger = \mathcal{A} \]
\[ \mathcal{A}^2 = \mathcal{A} \]
\[ \lambda = 2 \text{Im} \ s_p = -2M_R \Gamma_R \]

Resonant S-matrix \( S_R(s) \):

\[ S(s) = \mathcal{O} \left( I + \frac{i\lambda \mathcal{A}}{s - s_R} \right) \mathcal{O}^T \]
\[ \underbrace{S(s)}_{S_R(s)} \]
Origin of phases: **Smooth non-resonant terms, \( \mathcal{O} \)**

E.g. Coulomb phases in nuclear physics

**In general, do not take the real part in** \( \langle \psi_R^- | N_D^A | \psi_R^+ \rangle \) **to make it real!!**

The right procedure is doing the phase or \( S \)-matrix transformations

The transformed \( S \) matrix

\[
S_{\mathcal{O}}(s) \equiv \mathcal{O} S(s) \mathcal{O}^T
\]

E.g. for the case of only one channel:

\[
g_A^2 \rightarrow g_A^2 \quad \quad S_0^{-1} \quad \quad \quad \quad e^{-i\phi}
\]

Non-Resonant terms

Resonance Propagator is Complex

\[
\langle \psi_R^- | N_D^A | \psi_R^+ \rangle \rightarrow |\langle \psi_R^- | N_D^A | \psi_R^+ \rangle|
\]
13. Finite width resonances

Necessary Condition for still interpreting $|\langle \psi_R^- | N_D^A | \psi_R^+ \rangle|$ as an average number of particles Z.H.Guo, Oller, PRD93,096001(2016)

The transformations

$$S_\Theta(s) \equiv \Theta S(s) \Theta^T$$

$$\Theta \Theta^\dagger = I$$

$$g_A^2 \rightarrow g_A^2 \Theta_{AA}^2$$

make sense only if:

▷ The Laurent expansion around $s_P$ is valid in some interval of physical (real values above threshold) for $s$

$$S(s)S(s)^\dagger = I \text{ is meaningful}$$

**Condition A:** $s_n < \text{Re} s_P < s_{n+1}$

$s_n$ is the threshold of channel $n$
Physical idea

- If this condition is fulfilled one can think of a physical process with a clear resonance contribution. E.g. the $\sigma$ and E791 data on $D^+$ and $D_s^+$ decays.
- The resonance phenomenon is physically manifest in the open channels.
- We preserve $|g_A|$ to the open channels.
A resonance is then very different

\[
\frac{g^2}{s - s_R} + \frac{g^{2*}}{s - s_R^*} = 2\text{Re} \frac{g^2}{s - s_R}
\]

Double-pole like virtual state

This could well be the case for the \( X(3872) \), at least as a double-like pole. It could also be triple-like, etc. Z.H. Guo, Oller, PRD93,096001(2016)

\( \bar{D}^0 D^{*0} \) threshold. Tiny width
Basic set-up  Different perspective  QFT-like calculation  New equation for $X$  Relativistic case  Resonances  \textit{S}-matrix transformations

\textbf{An example: \textit{S}-wave Effective Range Expansion}

X.W.Kang, Z.H. Guo, Oller, PRD94, 014012 (2016)

$$T(k) = \frac{1}{-\frac{1}{a} + \frac{1}{2} r k^2 - i k}$$

$$G(k) = -i k$$

$$\tan \phi = \frac{\Gamma}{2 M_R} \quad \longrightarrow \quad 0 \leq \phi \leq \pi/2 \text{ for } M_R \geq 0$$

$$k_R = k_r - i k_i = \sqrt{2 \mu(M_R - i\Gamma/2)} = |k_R| \left( \cos \frac{\phi}{2} - i \sin \frac{\phi}{2} \right)$$

$$X = -\gamma^2 \frac{dG}{ds} = -\gamma^2 \frac{dG}{dk} = i \frac{k_i}{k_r} = i \tan \frac{\phi}{2}$$

$$|X| \leq 1 \iff k_r \geq k_i \iff M_R \geq 0$$

($|X| = 1 \text{ for } M_R = 0 \text{ and } \Gamma > 0$)

If the real part is taken then ALWAYS $X = 0$!
14. Scattering Amplitude $t(E)$

Dispersion Relation for the inverse of $t(E)$

$$\text{Im} t(E)^{-1} = -ik$$

One subtraction is needed

$$\oint dz \frac{t(z)^{-1}}{(z - E)(z - C)}$$

The only other structure apart from the threshold that can give rise to a strong distortion in $t(E)^{-1}$ is a pole at $M_Z$

$$t(E) = \frac{1}{\lambda} \frac{1}{E - M_Z} + \beta - ik$$

CDD pole Castillejo, Dalitz, Dyson, PR, 101, 453 (1956)

The general formula for a partial-wave without crossed-channel dynamics was deduced in: Oller, Oset PRD60, 074023 (1999)
Contact interaction plus s-channel exchange of bare resonances

Kang, Oller, EPJC77, 399 (2017) study of the $X(3872)$

Interplay of quark and meson degrees of freedom in a near-threshold resonance

$[ABK]$ Artoisenet, Braaten, Kang, PRD, 82, 014013 (2010) Using line shapes to discriminate between binding mechanisms for the $X(3872)$
Basic set-up  Different perspective  QFT-like calculation  New equation for \( X \)  Relativistic case  Resonances  \( S \)-matrix transformations.

\[
[BHKKN]
\]

\[
D_F(E) = E - E_f - \frac{(E - E_f)^2}{(E - M_Z)^2} + \frac{i}{2} g_f k
\]

\[
t(E) = \frac{g_f}{8\pi^2 \mu D_F(E)}
\]

\[
t(E) = \frac{1}{4\pi^2 \mu} \frac{E - E_f + \frac{1}{2} g_f \gamma_V}{(E - E_f)(\gamma_V + i k) + \frac{i}{2} g_f \gamma_V k}
\]

\[
g_f = \frac{2\lambda}{\beta^2}
\]

\[
E_f = M_Z - \frac{\lambda}{\beta}
\]

\[
\gamma_V = -\beta
\]

\[
\gamma_V = 1/a_V, \ a_V \text{ scattering length in pure contact-interaction theory.}
\]

For \( |M_Z| \gg |E_f| \) one recovers the standard Flatté approximation.
Limitation of [BHKKN] and [ABK]

- They predict only $\lambda \geq 0$

$$\begin{align*}
[BHKKN] & \quad [ABK] \\
\lambda &= \frac{\gamma^2}{2} g_f & \lambda &= \frac{2g^2\gamma_0^2(\gamma_1-\kappa_2)^2}{(\gamma_0+\gamma_1-2\kappa_2)^2}
\end{align*}$$

- Positive effective range $r$, $v_3$, $v_5$, etc, cannot be reproduced with $\lambda \geq 0$:

$$
r = -\frac{\lambda}{\mu M_Z^2} < 0
$$

$$
v_3 = -\frac{\lambda}{8\mu^3 M_Z^4} < 0
$$

- $\omega(E) \geq 0 \rightarrow \lambda \geq 0$:

$$
\omega(E) = \theta(E) \frac{\lambda k/\pi}{|\lambda + (\beta - ik)(E - M_Z)|^2}
$$

Constant contact term plus one $s$-channel bare-pole exchange picture collapses for $\lambda < 0$
14. Conclusions

- A new perspective on compositeness based on the number operators
- Amenable to calculations employing QFT
- New equation of compositeness for NR systems
- It can be also extended to relativistic systems
- Generalization to resonances
- Phase-factor transformations
- $S$-matrix transformations
- Universal criterion for a relativistic or non-relativistic bound state to be qualified as elementary
- Necessary condition for a resonance to be elementary
- More work is needed for finite-width resonances.
- CDD & including bare state explicitly
Other methods to study the nature of resonances

- **Study of form factors and determination of the corresponding quadratic radius** Sekihara, Hyodo, Jido, PRC83, 055202 (2011); Albaladejo, Oller, PRD86, 034003 (2012)

Other methods to study the nature of resonances

- **Study of form factors and determination of the corresponding quadratic radius** Sekihara, Hyodo, Jido, PRC83, 055202 (2011); Albaladejo, Oller, PRD86, 034003 (2012)


- **Evolution of the pole positions with the increase in the number of color of QCD.** E.g. for a $q\bar{q} \ M = \mathcal{O}(N_C^0)$ and $\Gamma = \mathcal{O}(N_C^{-1})$. Pioneer works Oset, Oller, PRD60, 074023 (1999); Peláez, PRL92, 102001 (2004); Hyodo, Jido, Hosaka, PRL97, 192002 (2006)

- **Regge trajectories** Londergan, Nebreda, Peláez, Szczepaniak, PLB729, 9 (2014)

- **Dependence on the mass under quark mass variations.** Lattice QCD. Ruiz de Elvira, Meißner, Rusetsky, Schierholz, arXiv:1706.09015

- **Compare predictions within specific models with experiment**, e.g. spectrum, decay properties, etc
<table>
<thead>
<tr>
<th>Name</th>
<th>$\sqrt{s_{\text{p}}}$ [MeV]</th>
<th>$X^R_{\pi\pi}$</th>
<th>$X^R_{KK}$</th>
<th>$X^R_{\eta\eta}$</th>
<th>$X^R_{\eta\eta'}$</th>
<th>$X^R_{\rho\pi}$</th>
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<tr>
<td>$f_0(500)$</td>
<td>$442^{+4}<em>{-4} - i246^{+7}</em>{-5}$</td>
<td>$0.40^{+0.01}_{-0.01}$</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>$0.40^{+0.01}_{-0.01}$</td>
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<tr>
<td>$f_0(980)$</td>
<td>$978^{+17}<em>{-11} - i29^{+9}</em>{-11}$</td>
<td>$0.02^{+0.01}_{-0.01}$</td>
<td>$0.68^{+0.10}_{-0.16}$</td>
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<td>...</td>
<td>$0.67^{+0.11}_{-0.17}$</td>
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<td>$f_0(1710)$</td>
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<td>$0.03^{+0.02}_{-0.02}$</td>
<td>$0.02^{+0.03}_{-0.03}$</td>
<td>$0.25^{+0.16}_{-0.16}$</td>
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<tr>
<td>$\rho(770)$</td>
<td>$760^{+7}<em>{-5} - i71^{+4}</em>{-5}$</td>
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<tr>
<td>$K^*_0(800)$</td>
<td>$643^{+75}<em>{-30} - i303^{+25}</em>{-75}$</td>
<td>$0.94^{+0.19}_{-0.39}$</td>
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<td>$0.94^{+0.19}_{-0.39}$</td>
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<tr>
<td>$K^*(892)$</td>
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<td>$0.05^{+0.01}_{-0.01}$</td>
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<td>$0.23^{+0.35}_{-0.17}$</td>
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<tr>
<td>$a_1(1260)$</td>
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<tr>
<td>Hyperon with $l = 0$</td>
<td>$\Lambda(1405)$ broad</td>
<td>$1388^{+9}<em>{-9} - i114^{+24}</em>{-25}$</td>
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<td>$0.73^{+0.16}_{-0.07}$</td>
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<tr>
<td>Hyperon with $l = 1$</td>
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<td>$D_s^*(2317)$</td>
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<td>$0.17^{+0.03}_{-0.03}$</td>
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<td>$Y(4260)$</td>
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<td>$0.02$</td>
<td>$0.02$</td>
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<td>$0.17$</td>
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<tr>
<td>$\Lambda_c(2595)$</td>
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<td>...</td>
<td>...</td>
<td>...</td>
<td>$0.11^{+0.02}_{-0.02}$</td>
</tr>
</tbody>
</table>

Table: Z.H. Guo, Oller, PRD93,096001(2016)