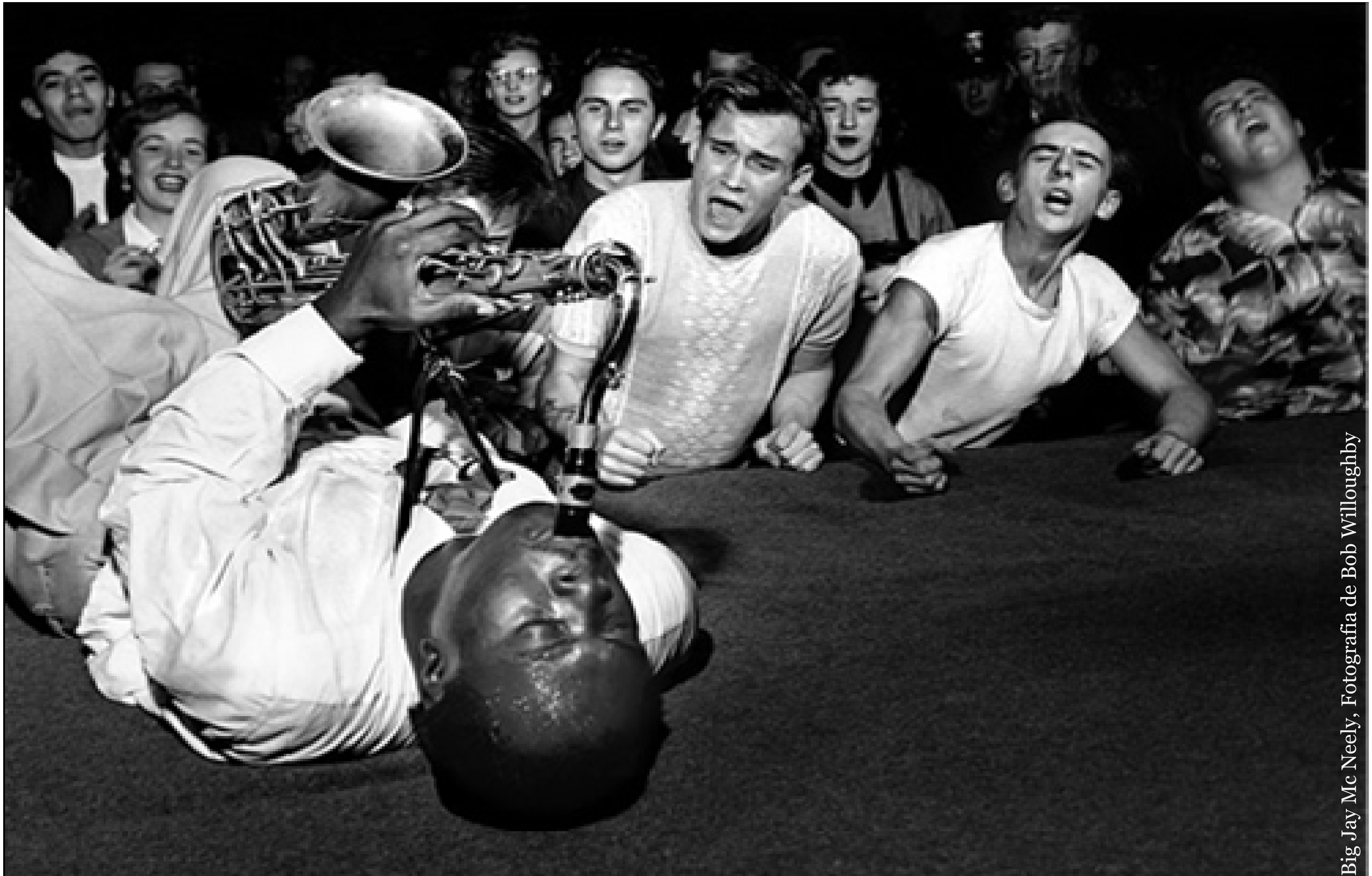


Big Jay Mc Neely, Fotografia de Bob Willoughby

“Imagine being able to see the world but you are deaf, and then suddenly someone gives you the ability to hear things as well - you get an extra dimension of perception” B. Schutz, BBC

GW physics contains 100 years of developments in

astrophysics
black hole physics
instrumentation
mathematical relativity
numerical methods
quantum mechanics
signal processing
…and more

and is the work of thousands of colleagues

• 1910 – The analysis of finite difference methods for PDEs is initiated with Richardson [649].
• 1915 – Einstein develops GR [293, 295].
• 1916 – Schwarzschild derives the first solution of Einstein's equations, describing the gravitational field generated by a point mass. Most of the subtleties and implications of this solution will only be understood many years later [688].
• 1917 – de Sitter derives a solution of Einstein's equations describing a Universe with constant, positive curvature Λ. His solution would later be generalized to the case $\Lambda < 0$ [255].
• 1921, 1926 – In order to unify electromagnetism with GR, Kaluza and Klein propose a model in which the spacetime has five dimensions, one of which is compactified on a circle [463, 476].
• 1928 – Courant, Friedrichs and Lewy use finite differences to establish existence and uniqueness results for elliptic boundary-value and eigenvalue problems, and for the initial-value problem for hyperbolic and parabolic PDEs [228].
• 1931 – Chandrasekhar derives an upper limit for white dwarf masses, above which electron degeneracy pressure cannot sustain the star [193]. The Chandrasekhar limit was subsequently extended to NSs by Oppenheimer and Volkoff [591].
• 1939 – Oppenheimer and Snyder present the first dynamical collapse solution within GR [590].
• 1944 – Lichnerowicz [515] proposes the conformal decomposition of the Hamiltonian constraint laying the foundation for the solution of the initial data problem.
• 1947 – Modern numerical analysis is considered by many to have begun with the influential work of John von Neumann and Herman Goldstine [764], which studies rounding error and includes a discussion of what one today calls scientific computing.
• 1952 – Choquet-Bruhat [327] shows that the Cauchy problem obtained from the spacetime decomposition of the Einstein equations has locally a unique solution.
• 1957 – Regge and Wheeler [642] analyze a special class of gravitational perturbations of the Schwarzschild geometry. This effectively marks the birth of BH perturbation theory, even before the birth of the BH concept itself.

Cardoso et al, Living Reviews in Relativity (2015)

<u>Almost</u> all you want to know about GWs

How to look for GWs: what are they, optimized searches, sources

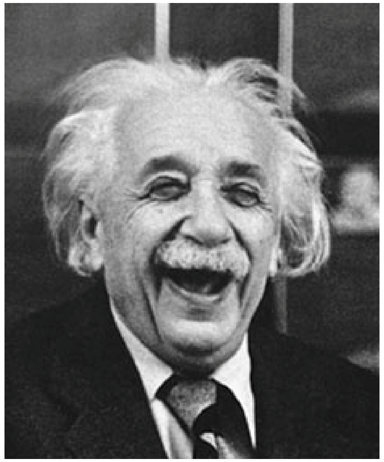
How to model GW emission I: Perturbation theory (ringdown, quasinormal modes, slow-motion, plunges)

How to model GW emission II: Numerical Relativity

What science can we learn from GWs

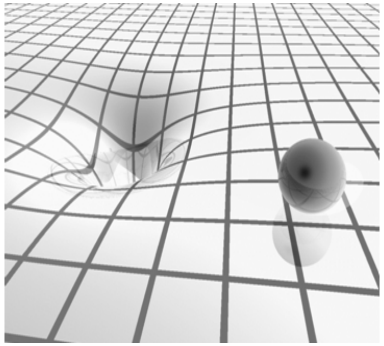
Einstein: Gravity is curvature

"Space-time tells mater how to move,
matter tells spacetime how to curve"



$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}$$

Any mass-energy curves spacetime;
free objects follow curvature



The equivalence principle

$$\ddot{X}^{\alpha} + \Gamma^{\alpha}_{\beta\gamma}\dot{X}^{\beta}\dot{X}^{\gamma} = 0$$

Far away from sources

What are GWs?

Let's kick Minkowski, using transverse and traceless (TT) gauge

$$ds^2 = -dt^2 + (1 + h_+)dx^2 + (1 - h_+)dy^2 + 2h_\times dxdy + dz^2$$

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0$$ yields $$\frac{\partial^2 h_{+,\times}}{\partial z^2} - \frac{\partial^2 h_{+,\times}}{\partial t^2} = 0$$

Fluctuations h travel at speed of light.

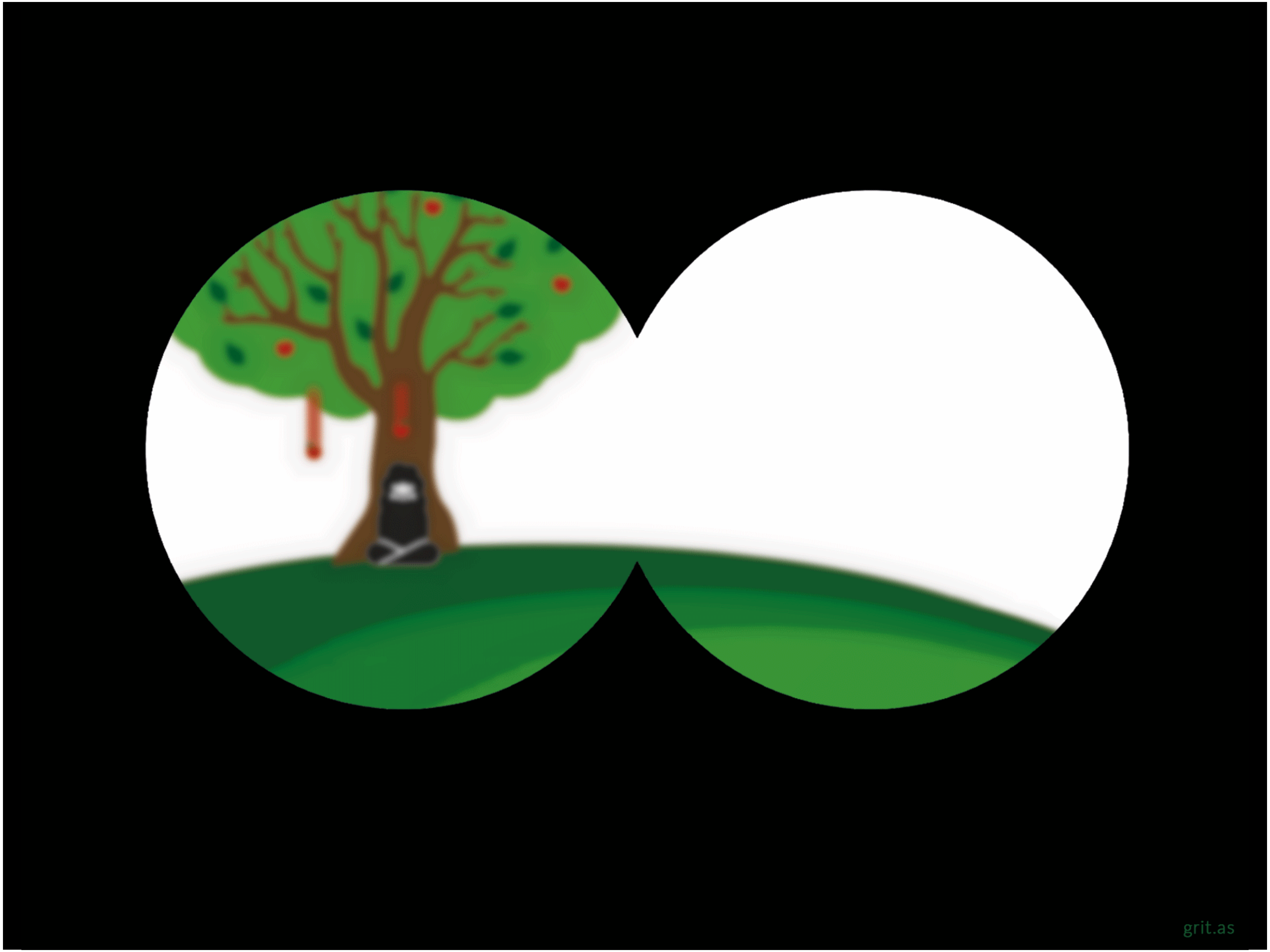
There are two independent components, two polarizations.

Effect of GWs on particles

$$\ddot{X}^{\alpha} + \Gamma^{\alpha}_{\beta\gamma} \dot{X}^{\beta} \dot{X}^{\gamma} = 0$$

$$\ddot{X}^{i}(t) = 0$$


grit.as


grit.as

Effect of GW on particles

$$\ddot{X}^i(t) = 0$$

TT gauge is freely falling (co-moving with free particles): Particles at rest initially, will be at rest after wave passes... But *relative motion* is non-trivial: compute either proper distance or light-travel time and get

$$L(t) = L_0 \left(1 + \frac{1}{2} n^i n^j h_{ij}^{\mathrm{TT}}\right)$$

$$n^i \equiv \Delta x^i / L_0$$

GWs are tidal forces

Effect of GW on detectors

In absence of noise, output of detector is difference in strain between the two arms, and this can be written

$$h = h_+ F_+(\theta_S, \phi_S, \Psi_S) + h_\times F_\times(\theta_S, \phi_S, \Psi_S)$$

Where the response functions F depend on the detector. For LIGO, their sky-averaged value is 0.447.

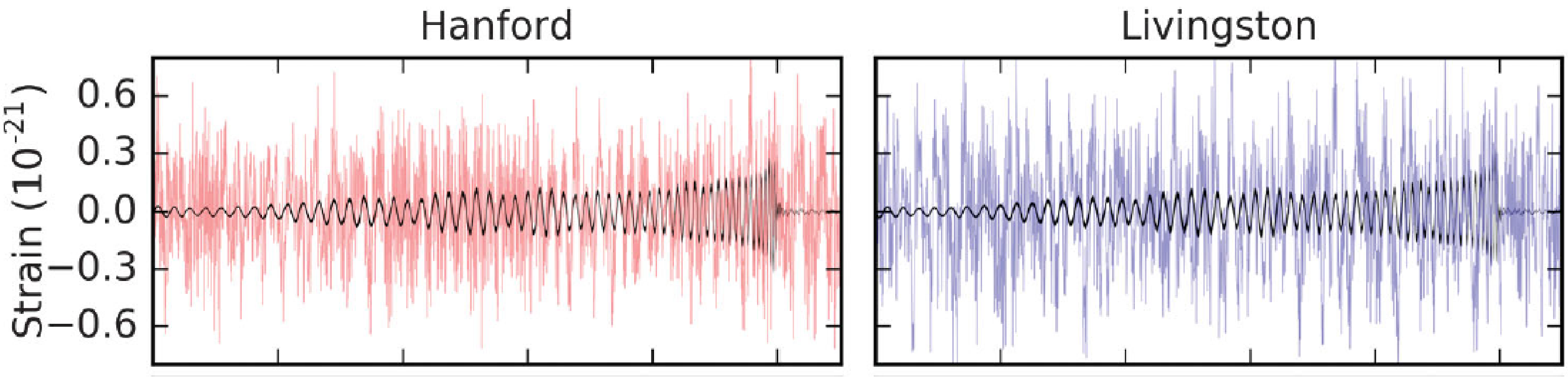
K. S. Thorne, in Three Hundred Years of Gravitation (Cambridge University Press)

If light waves are stretched by GWs, can we use light as ruler?

Yes, we can!
Light travels always at c, it *will* take longer if arm stretches
(think about wave crests in different arms)

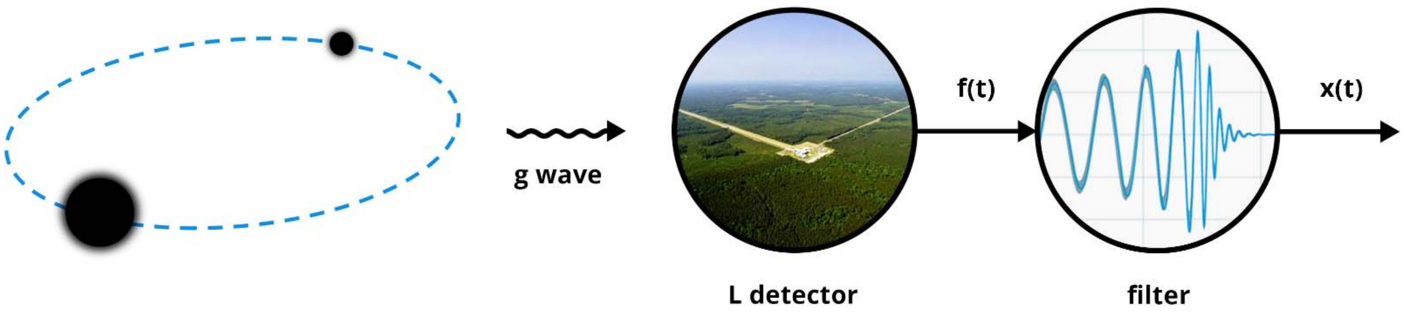
Peter Saulson, Am. J. Phys. 65: 6 (1996)
Valerio Faraoni gr-qc/0702079

The needle in the haystack problem



The LIGO Collaboration, PRL116:241103 (2016)

Matched-Filtering



The detector output

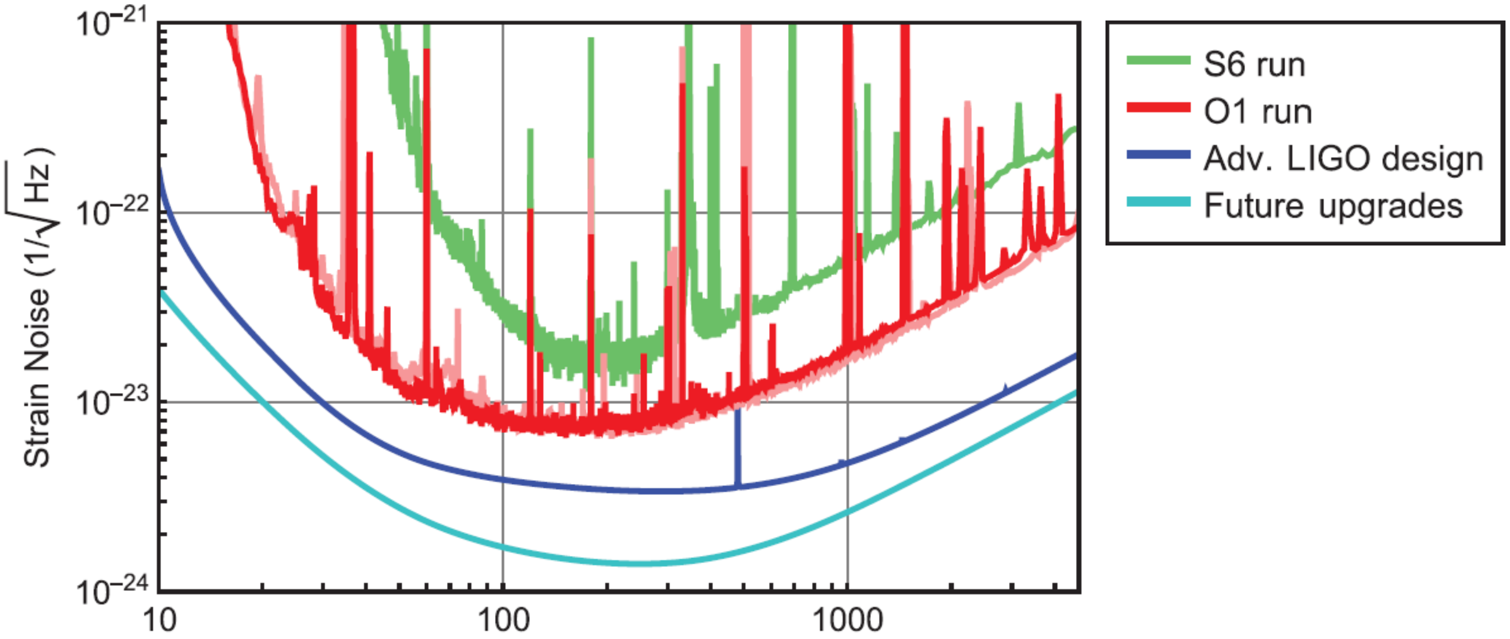
$$f(t) = h(t) + n(t)\,,$$

where n(t) is the noise. Consider stationary Gaussian noise (with zero mean), characterized by

$$< \tilde{n}(f)\tilde{n}^*(f') >= \frac{1}{2}\delta(f - f')S_n(f)\,,$$

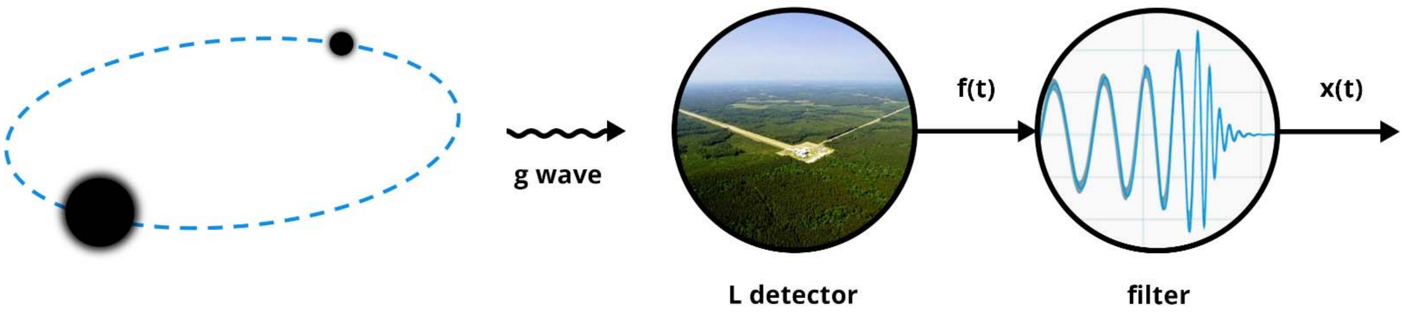
with the PSD giving the time average of detector noise

$$\overline{|n(t)|^2} = \int_0^\infty df\, S_n(f)$$



The LIGO Collaboration, PRL116:131103 (2016)

Matched-Filtering

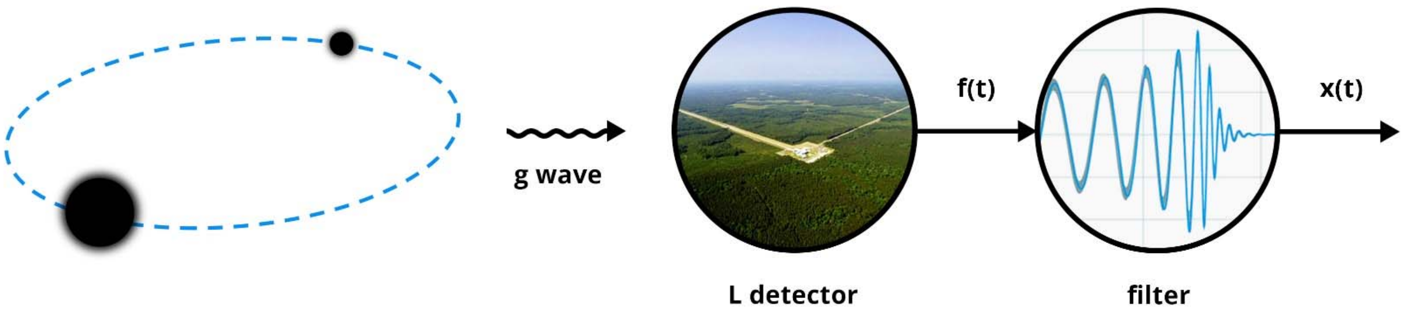


Process the signal with filter K(t) against the data stream, producing number

Standard definition of signal-to-noise ratio yields

$$\begin{aligned}\frac{S}{N} &= \frac{\text{expected value of X with signal}}{\text{rms value of X with no signal}} \\ &= \frac{\langle X \rangle}{\sqrt{\langle X^2 \rangle_{h=0}}} \\ &= \frac{4\int_0^\infty df \mathcal{R}\left(\tilde{h}(f)\tilde{K}(f)\right)}{\sqrt{4\int_0^\infty df\, |\tilde{K}|^2 S_n(f)}}\end{aligned}$$

Matched-Filtering



Optimum filter K maximizes SNR

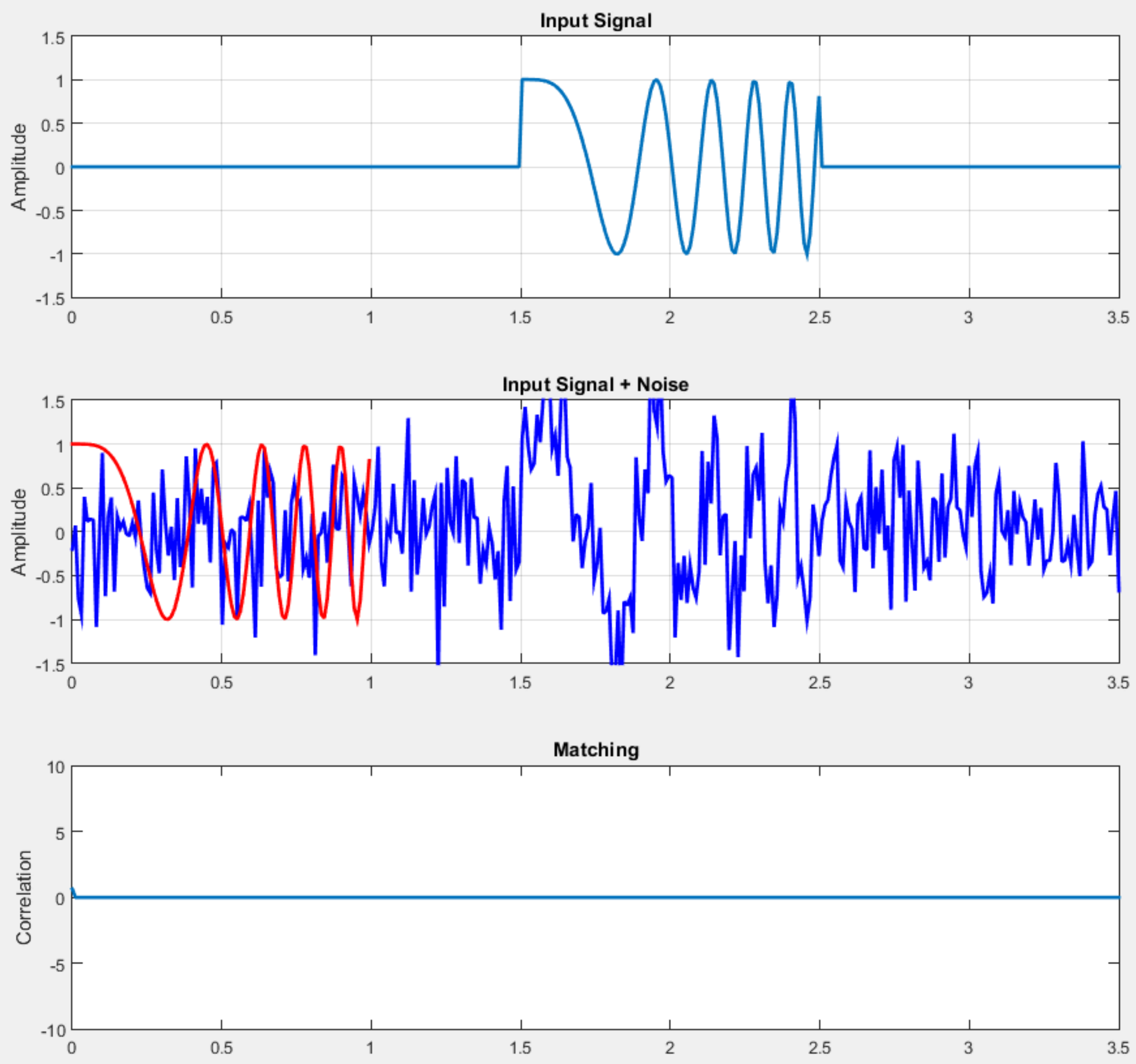
$$\tilde{K}(f) = \frac{\tilde{h}(f)}{S_n(f)}$$

$$\rho^2 = \left(\frac{S}{N}\right)^2 = 4\int_0^\infty \frac{|\tilde{h}(f)|^2}{S_n(f)}\,df$$

K is the Wiener filter, or *matched filter*

Vaynshtein and Zubakov, Signal Processing in Noise (1960); Flanagan & Hughes PRD57:4535 (1998)
See also Moore, Cole, Berry CQG32:015014 (2015)

Matched Filtering

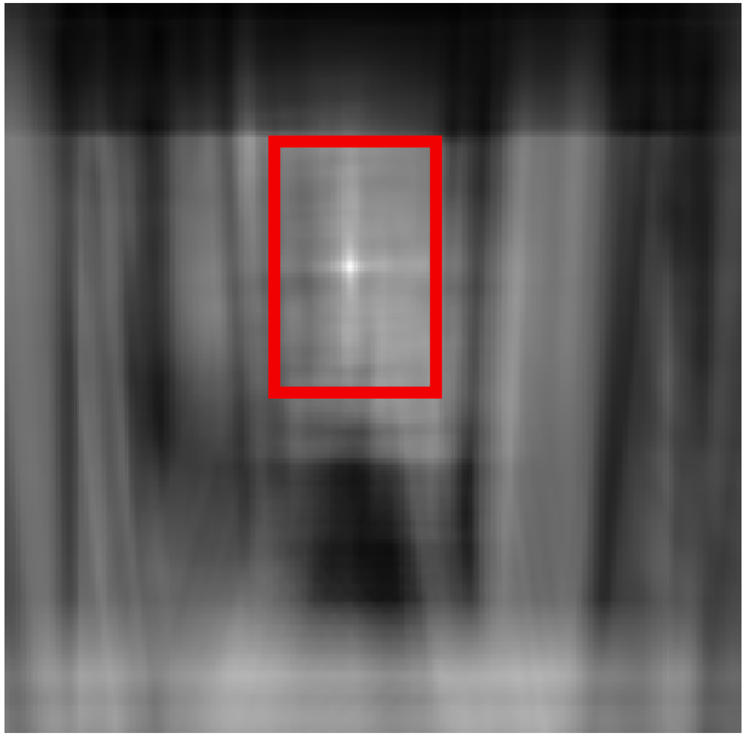


Object recognition…

This is a chair

Find the chair in this image

Output of normalized correlation



Template bank

Problem:

Do not know the intrinsic parameters of signal, *masses, spins, distances*...

Want to detect any signal in a space of possible signals, all with different phase evolution...

And of course, with a finite set of templates!

3% Mismatch: 10% lost events!

LIGO uses ~250000 templates for CBC searches

Sathyaprakash & Dhurandhar PRD44:3819 (1991); Owen, PRD53: 6749 (1996)

“Wir müssen wissen, wir werden wissen.”
(We must know, we will know)

D. Hilbert, Address to the Society of German Scientists
and Physicians, Königsberg (September 08, 1930)

We must know the waveforms,

we must know the sources

1900

Derives astronomical bounds
on curvature radius of space:

64 light years if hyperbolic

1600 light years if elliptic

1914

Volunteers for war

Belgium: weather station

France, Russia: artillery trajectories

March 1916

Sent home, ill with pemphigus.

Dies in May.

"I made at once by good luck a search for a full solution. A not too difficult calculation gave the following result: ..."

K. Schwarzschild to A. Einstein
(Letter dated 22 December 1915)

Solution re-discovered by many others:

J. Droste, May 1916 (part of PhD thesis under Lorentz): Same coordinates, more elegant

P. Painlevé, 1921, A. Gullstrand, 1922: P-G coordinates (not realize solution was the same)

...and others

Long, complex path to correct interpretation

Eddington

Lemaître

Oppenheimer

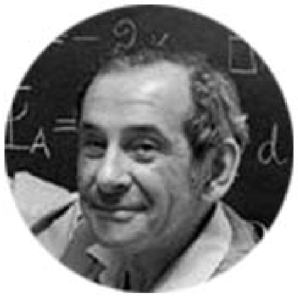
Snyder

Wheeler

Finkelstein

Kruskal

Penrose



Israel

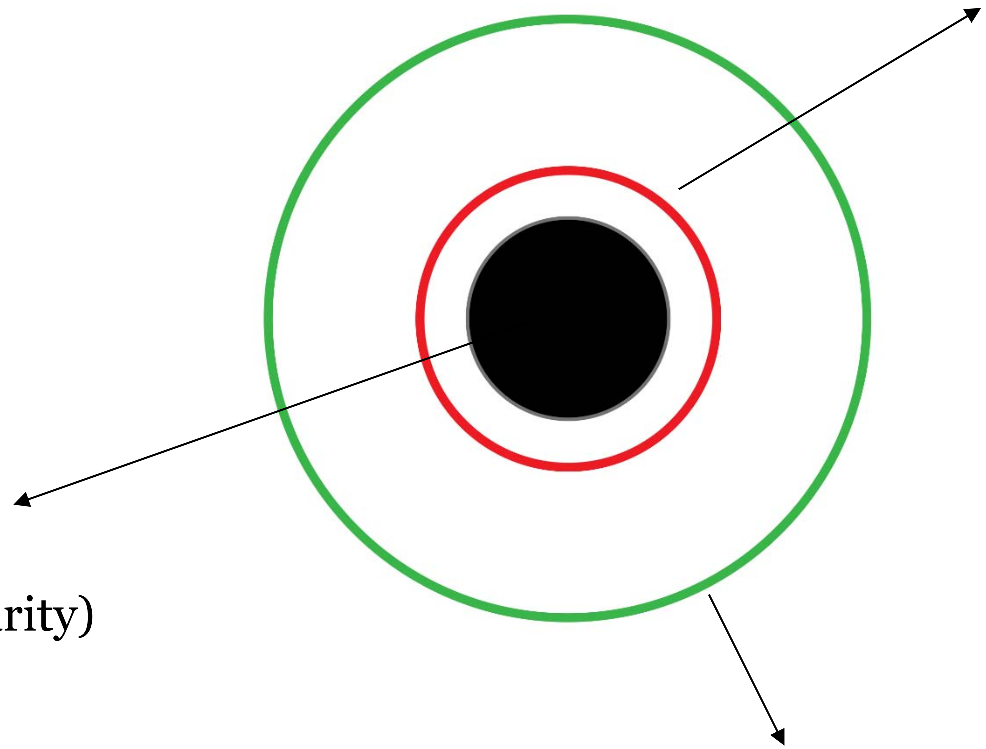
Carter

Hawking

Black holes

Light ring

(defines photosphere)



Event Horizon

(covers singularity)

Innermost Stable Circular Orbit (ISCO)

Specific energy$=\frac{2\sqrt{2}}{3} = 0.94$

Uniqueness: the Kerr solution

Theorem (Carter 1971; Robinson 1975):
A stationary, asymptotically flat, vacuum solution must be Kerr

$$
\begin{aligned}
ds^2 \quad = \quad & \frac{\Delta - a^2 \sin^2\theta}{\Sigma} dt^2 + \frac{2a(r^2 + a^2 - \Delta)\sin^2\theta}{\Sigma} dt d\phi \\
& - \frac{(r^2+a^2)^2 - \Delta a^2 \sin^2\theta}{\Sigma} \sin^2\theta d\phi^2 - \frac{\Sigma}{\Delta} dr^2 - \Sigma d\theta^2
\end{aligned}
$$

$$\Sigma = r^2 + a^2 \cos^2\theta \,, \quad \Delta = r^2 + a^2 - 2Mr$$

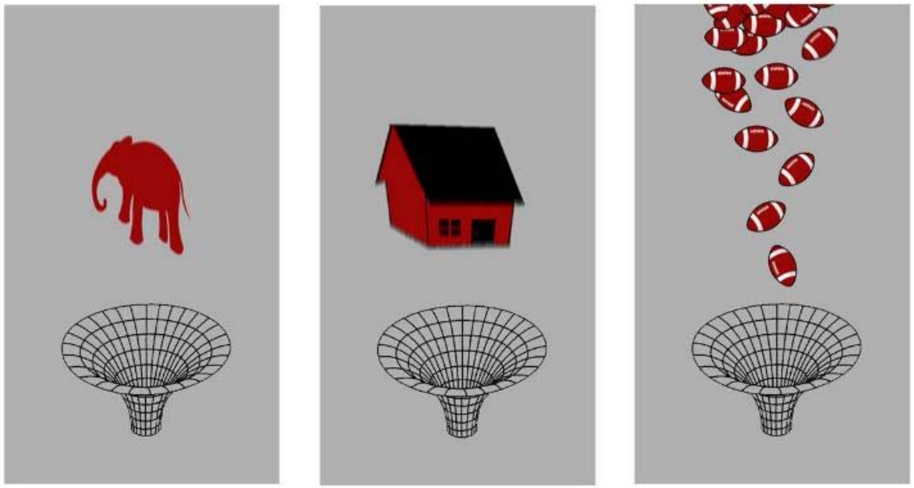
Describes a rotating BH with mass M and angular momentum J=aM

"In my entire scientific life, extending over forty-five years, the most shattering experience has been the realization that an exact solution of Einstein's equations of general relativity provides the *absolutely exact representation* of untold numbers of black holes that populate the universe."

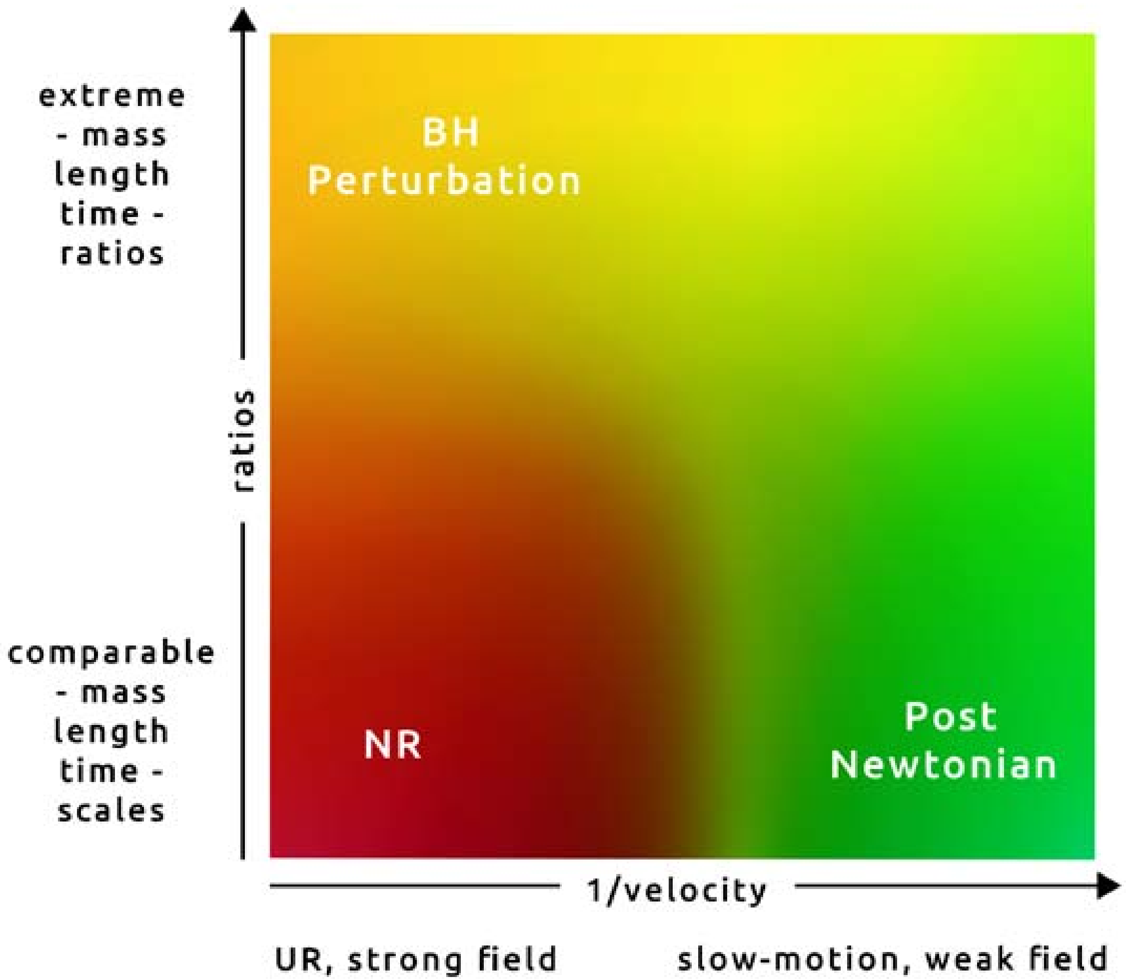
S. Chandrasekhar, The Nora and Edward Ryerson lecture, Chicago April 22 1975

"Black holes have no hair"

$$M_{2l} = (-1)^l M a^{2l}$$
$$S_{2l+1} = (-1)^l M a^{2l+1}$$

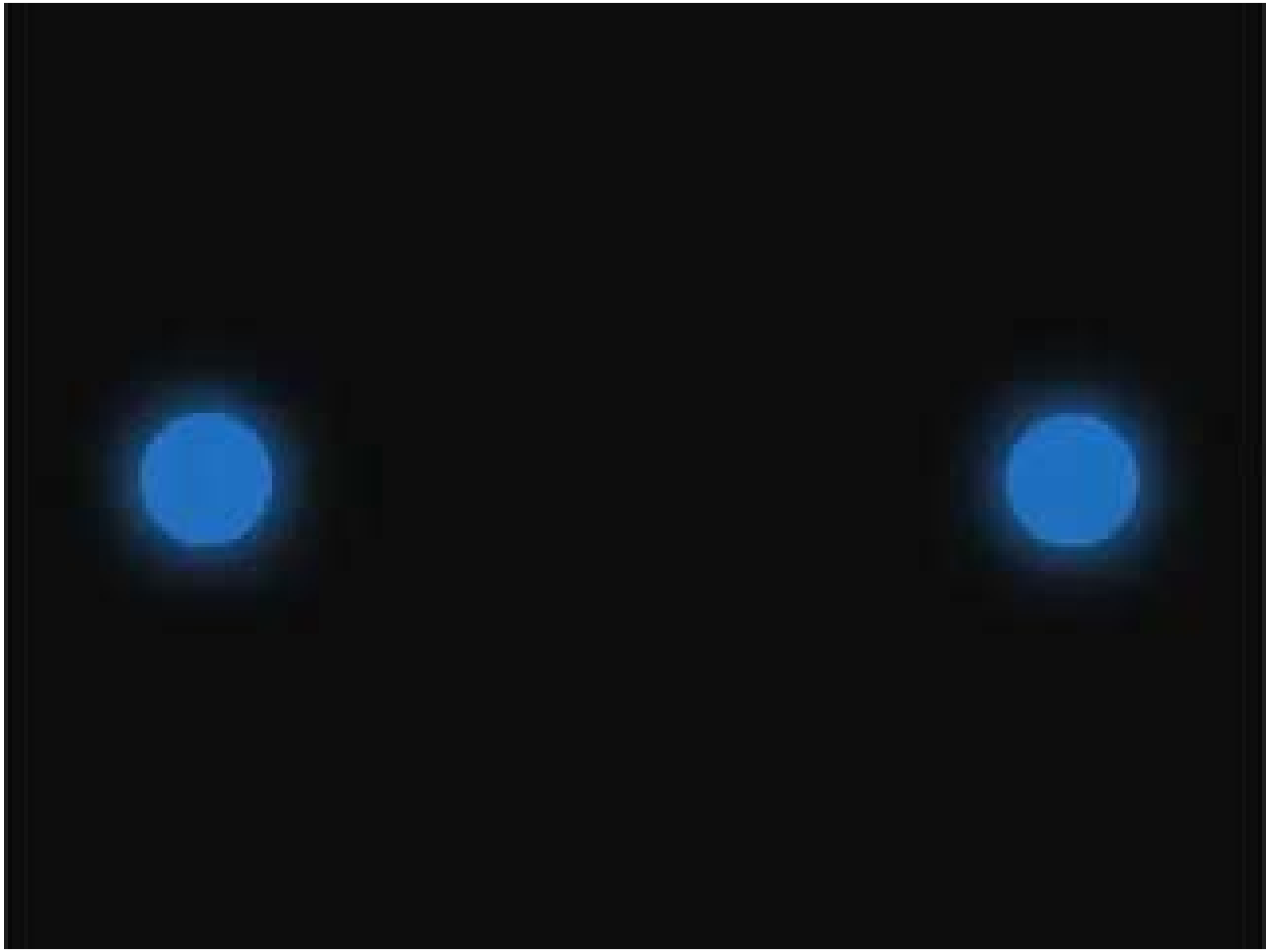


Incidentally, the first mention of the theorem was refused by PRD Editor, on the grounds of being obscene (in Kip Thorne's *Black Holes and Time Warps*)

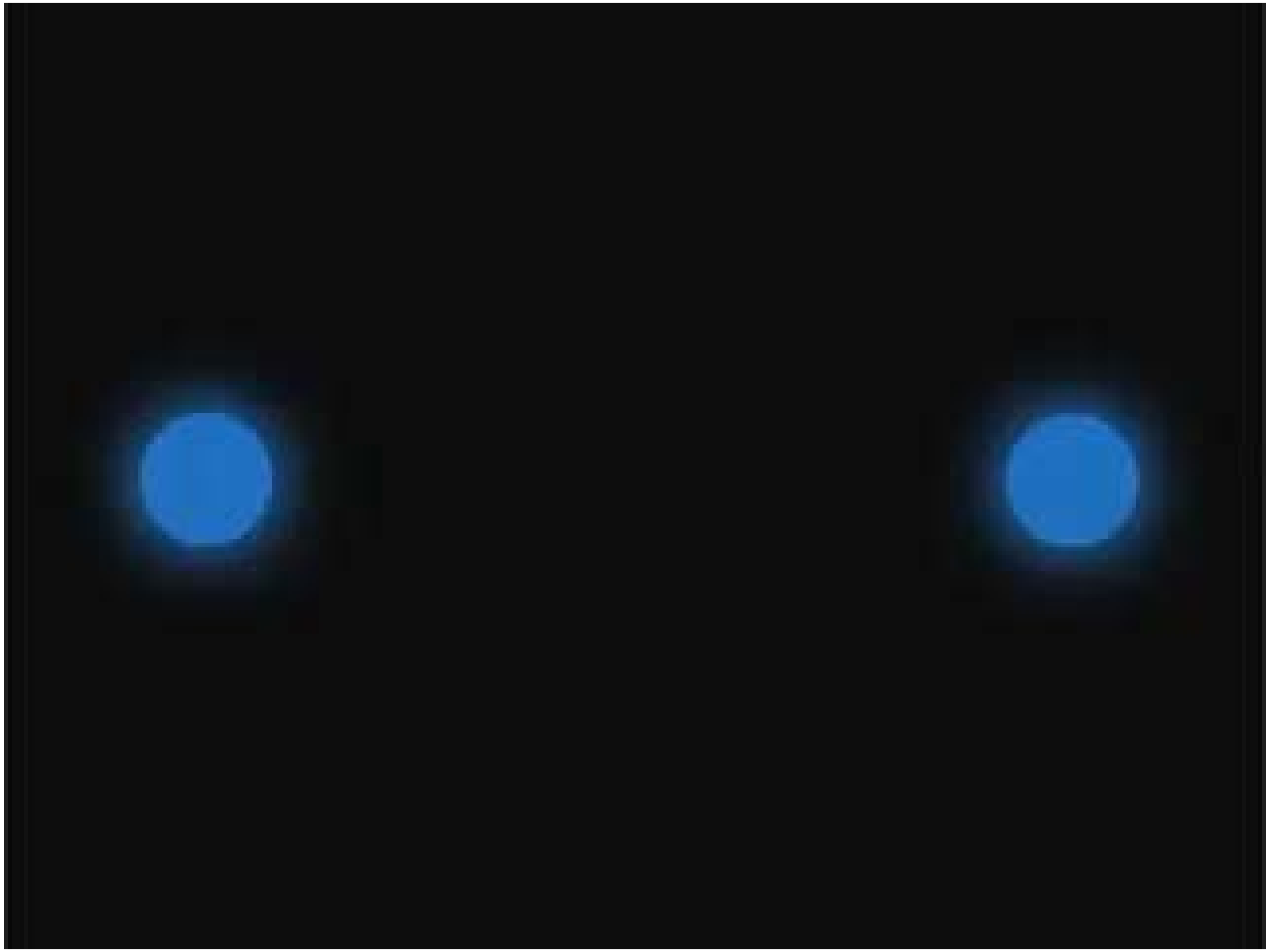


Cardoso, Gualtieri, Herdeiro, Sperhake, Living Reviews in Relativity 18:1 (2015)

Sources of gravitational waves
focus on BHs: strong and “easy” to model



Sources of gravitational waves
focus on BHs: strong and “easy” to model



Assume your spacetime is *approximately* that of a Schwarzschild black hole

$$g_{\mu\nu}(x^\nu) = {}^{(0)}g_{\mu\nu}(x^\nu) + h_{\mu\nu}(x^\nu)$$

Still too complex...second order PDEs on 4 variables...

Use background symmetries

Regge and Wheeler Phys.Rev.108: 1063 (1957)
Matthews, J. Soc. Ind. App. Math.10:768 (1962)
Zerilli J. Math. Phys.11:2203 (1970)

First "split" spacetime coordinates $x^\mu = (z^A, y^a)$,
where $z^A = (\theta,\, \phi)\,, \qquad y^a = (t,\, r)$

Introduce metric on unit sphere $ds^2 = \gamma_{AB} dz^A dz^B$

$$\gamma_{AB} \equiv \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}$$

Denote covariant derivative with respect to γ_{AB} by ∇_A.

Define Levi-Civita tensor $\varepsilon_{AB} \equiv \begin{pmatrix} 0 & \sin\theta \\ -\sin\theta & 0 \end{pmatrix}$

- Scalar harmonics $\gamma^{AB}\nabla_A\nabla_B Y^{\ell m} = -\ell(\ell+1)Y^{\ell m}$
- Polar vector harmonics $Y_A^{\ell m} \equiv \nabla_A Y^{\ell m}$
- Axial vector harmonics $S_A^{\ell m} \equiv \varepsilon_{AC}\gamma^{BC}\nabla_B Y^{\ell m}$
- Polar rank-two tensor harmonics $Z_{AB}^{\ell m} \equiv \nabla_A\nabla_B Y^{\ell m} + \frac{\ell(\ell+1)}{2}\gamma_{AB}Y^{\ell m}$
- Axial rank-two tensor harmonics $S_{AB}^{\ell m} \equiv \frac{1}{2}(\nabla_B S_A^{\ell m} + \nabla_A S_B^{\ell m})$

$$h_{\mu\nu}^{\mathrm{ax}} = \begin{pmatrix} 0 & 0 & \frac{h_0}{\sin\theta} Y^{\ell m}_{,\phi} & -h_0 \sin\theta Y^{\ell m}_{,\theta} \\ 0 & 0 & \frac{h_1}{\sin\theta} Y^{\ell m}_{,\phi} & -h_1 \sin\theta Y^{\ell m}_{,\theta} \\ \frac{h_0}{\sin\theta} Y^{\ell m}_{,\phi} & \frac{h_1}{\sin\theta} Y^{\ell m}_{,\phi} & 0 & 0 \\ -h_0 \sin\theta Y^{\ell m}_{,\theta} & -h_1 \sin\theta Y^{\ell m}_{,\theta} & 0 & 0 \end{pmatrix}$$

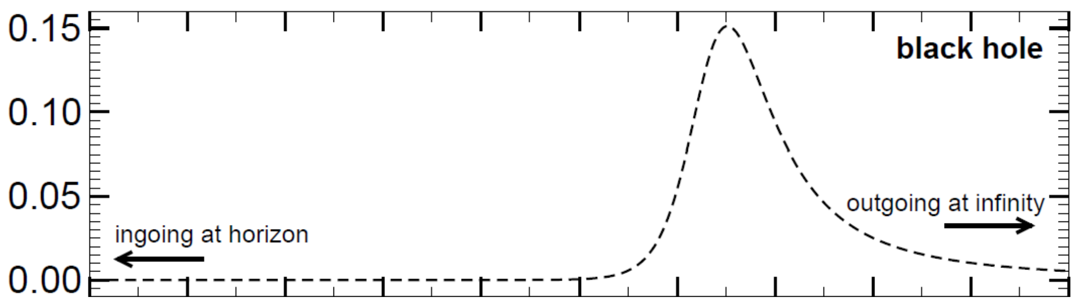
$$h_{\mu\nu}^{\mathrm{pol}} = \begin{pmatrix} -f(r)\, H_0 Y_{\ell m} & -H_1 Y_{\ell m} & 0 & 0 \\ -H_1 Y_{\ell m} & -\frac{1}{f(r)} H_2 Y_{\ell m} & 0 & 0 \\ 0 & 0 & -r^2 K Y_{\ell m} & 0 \\ 0 & 0 & 0 & -r^2 \sin^2\theta K Y_{\ell m} \end{pmatrix}$$

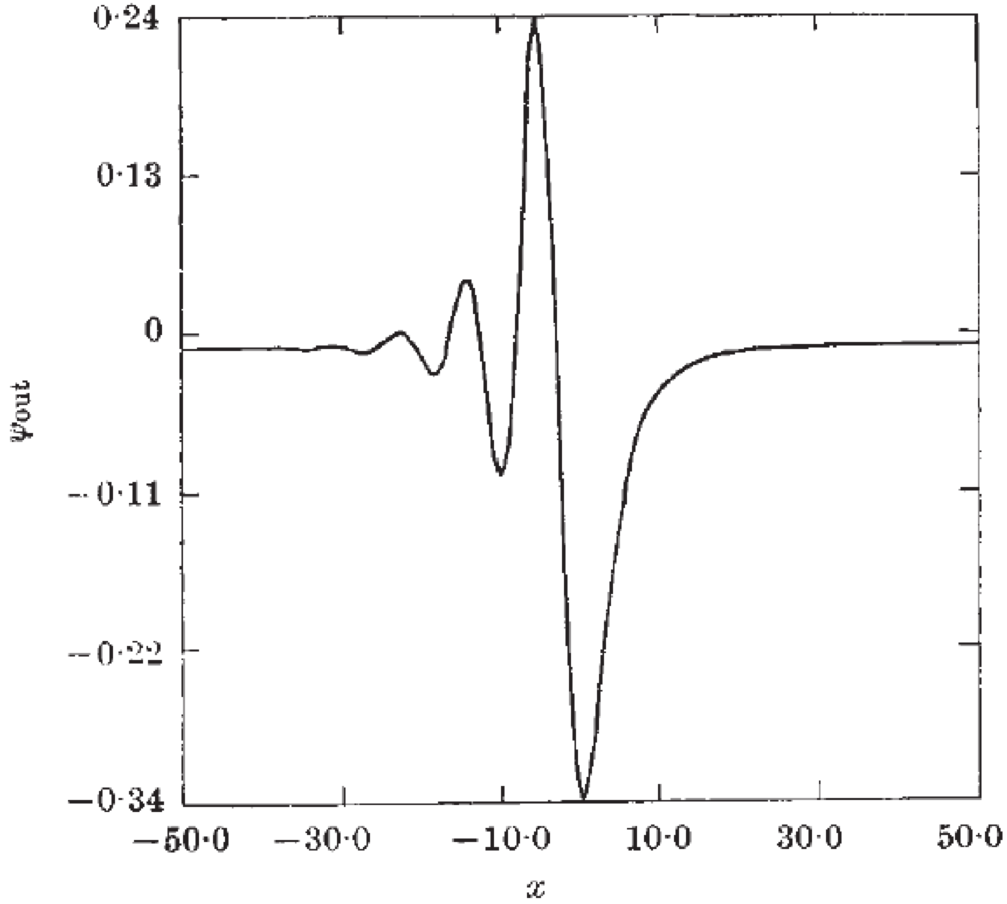
Vacuum

$$\Psi = \frac{1}{r}\left(1 - \frac{2M}{r}\right) h_1$$

$$0 = \left(1 - \frac{2M}{r}\right)^2 \frac{\partial^2 \Psi}{\partial r^2} + \frac{2M}{r^2}\left(1 - \frac{2M}{r}\right)\frac{\partial \Psi}{\partial r} - \frac{\partial^2 \Psi}{\partial t^2} - V\Psi$$

$$V = \left(1 - \frac{2M}{r}\right)\left(\frac{l(l+1)}{r^2} - \frac{6M}{r^3}\right)$$

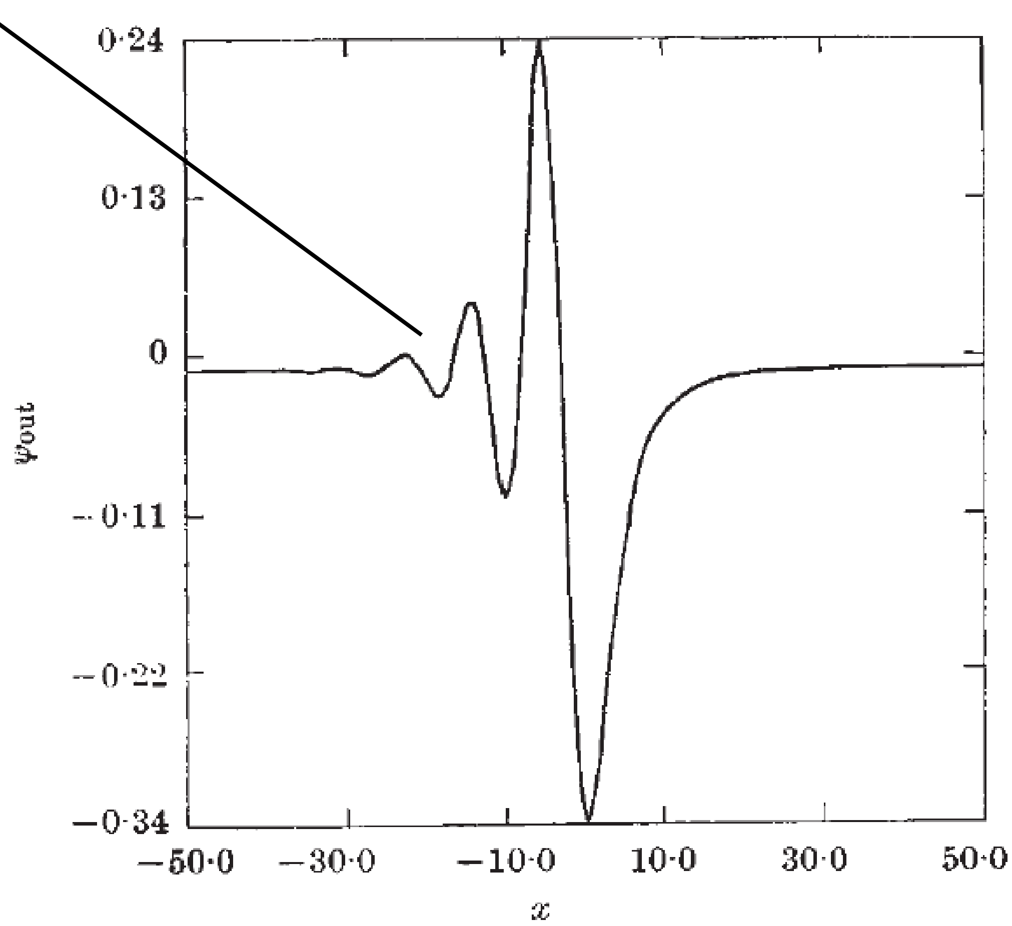




C.V.Vishveshwara, Nature 227: 938 (1970)
Data and routines at blackholes.ist.utl.pt

Experiment repeated: same decay timescale and ringing for different initial conditions;

Universal ringdown



C.V.Vishveshwara Nature 227: 938 (1970)

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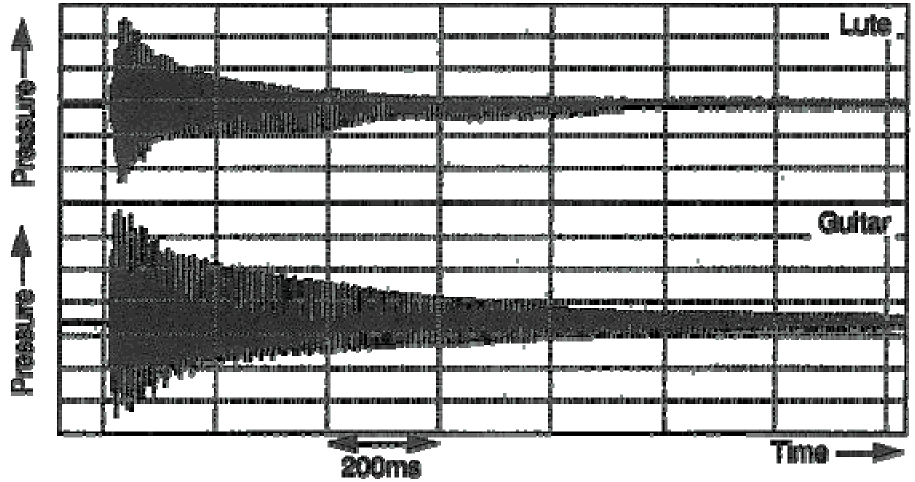
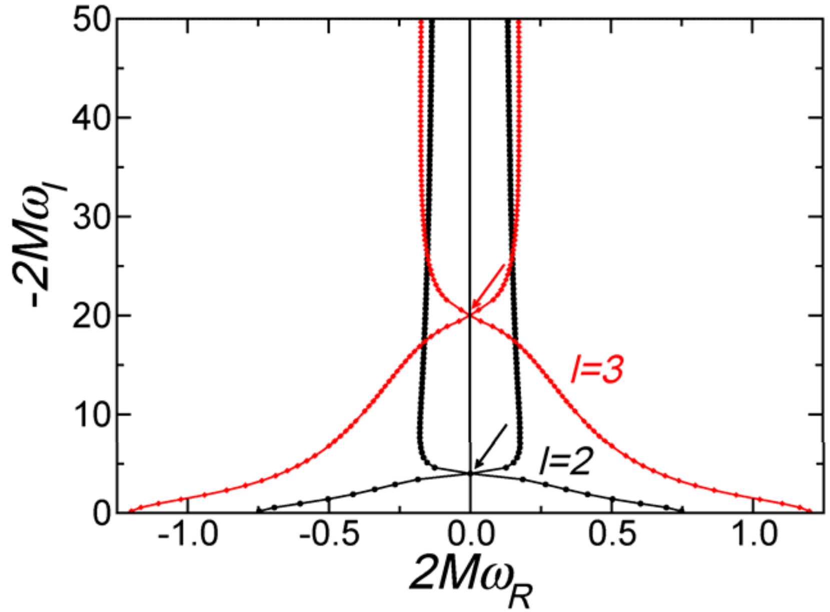


(http://www.feilding.net/sfuad/musi3012-01/html/lectures/014_instruments_I.htm)

$$\Psi(t,r) = \psi(r)e^{-i\omega t}$$

$$0 = \left(1 - \frac{2M}{r}\right)^2 \frac{\partial^2\psi}{\partial r^2} + \frac{2M}{r^2}\left(1 - \frac{2M}{r}\right)\frac{\partial\psi}{\partial r} + \left(\omega^2 - V\right)\psi$$

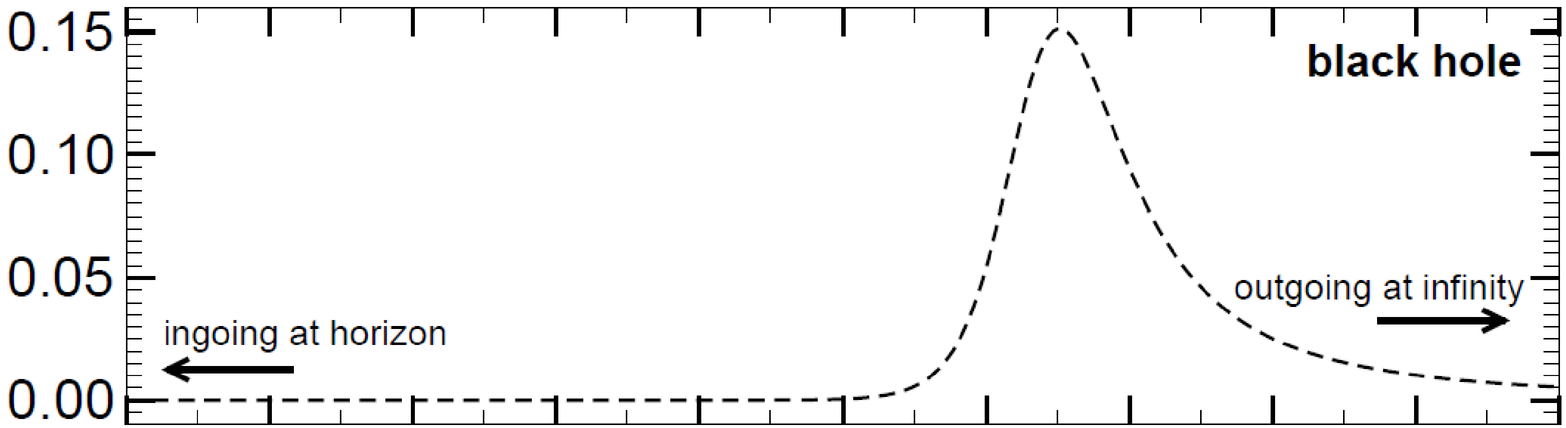
Chandrasekhar and Detweiler, Proc. R. Soc. Lond. 344 (1975)



$$f = \omega_R/2\pi = 1.207 \left(\frac{10\ M_\odot}{M}\right) \mathrm{kHz}$$

$$\tau = 1/|\omega_I| = 0.5537 \left(\frac{M}{10\ M_\odot}\right) \mathrm{ms}\,.$$

Berti, Cardoso and Starinets, Class.Quant. Gravity 26: 163001 (2009)
Data and routines at blackholes.ist.utl.pt



Ferrari and Mashhoon, PRD30:295(1984)
Cardoso et al, PRD79: 064016 (2009)

Point Particles: circular

E. Poisson, PRD47: 1497 (1993)

For low-velocities, equation can be solved in terms of Bessel functions

$$h_+ = -\frac{2G\mu}{c^2 r}\left(\frac{GM\Omega}{c^3}\right)^{2/3}(1+\cos^2\theta)\cos 2\Psi$$

$$h_\times = -\frac{4G\mu}{c^2 r}\left(\frac{GM\Omega}{c^3}\right)^{2/3}\cos\theta\sin 2\Psi$$

$$\Psi = \Omega(t-r)-\phi$$

Equal amplitudes for face-on

Only plus for edge-on (mimicks motion)

Naive extrapolation: h=1 close to BH for PP close to horizon

Extrapolate to generic case: promote mu to reduced mass!

Point Particles: circular

E. Poisson, PRD47: 1497 (1993)

For low-velocities, equation can be solved in terms of Bessel functions

$$\frac{dE}{dt} = \frac{32}{5}\frac{G}{c^5}\mu^2 L^4 \Omega^6 = \frac{32}{5}\frac{c^5}{G}\frac{\mu^2}{M^2}\left(\frac{GM\Omega}{c^3}\right)^{10/3}$$

Extremely relativistic systems: c^5/G...Dyson bound?

Circular stays circular

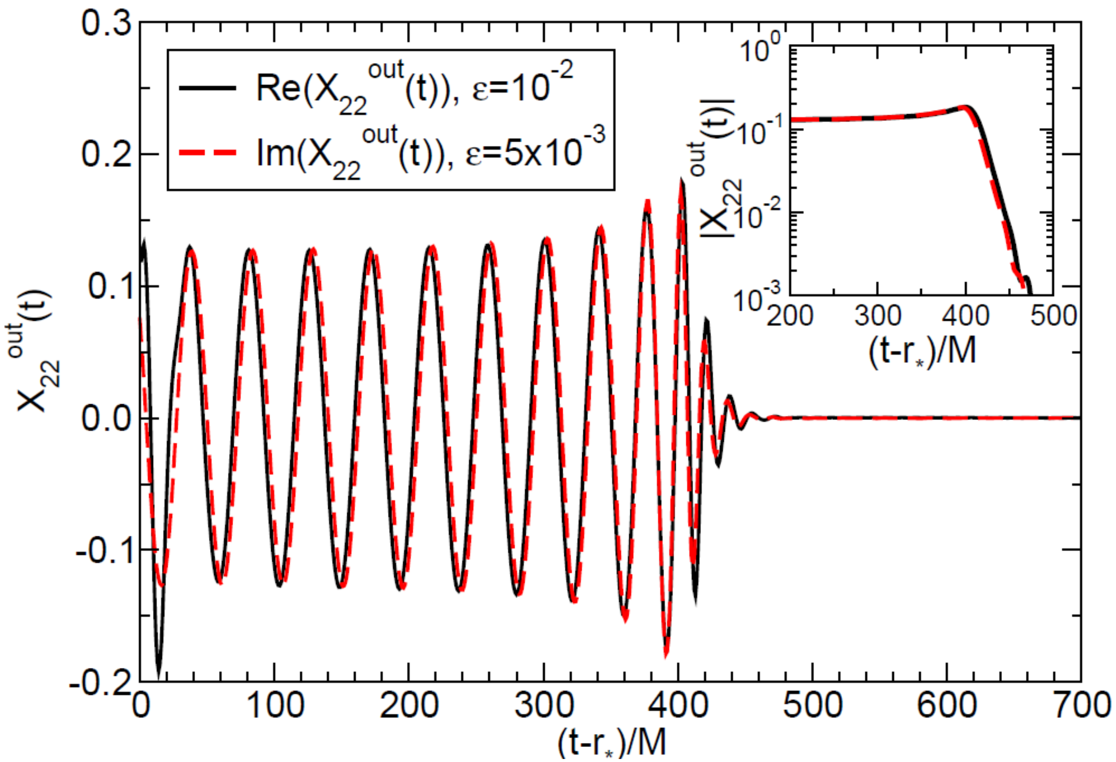
P.C. Peters PR136:B1224 (1964); E. Poisson, PRD47: 1497 (1993)

$$\frac{de}{dt} = -e\,\frac{304}{15}\,\frac{G^3}{c^5}\,\frac{M^2\mu}{a^4(1-e^2)^{5/2}}\left(1+\frac{121}{304}e^2\right)$$

Orbits evolve, under GW radiation reaction, to circular

Become unstable at r=6.68 M (outside ISCO)...and plunge

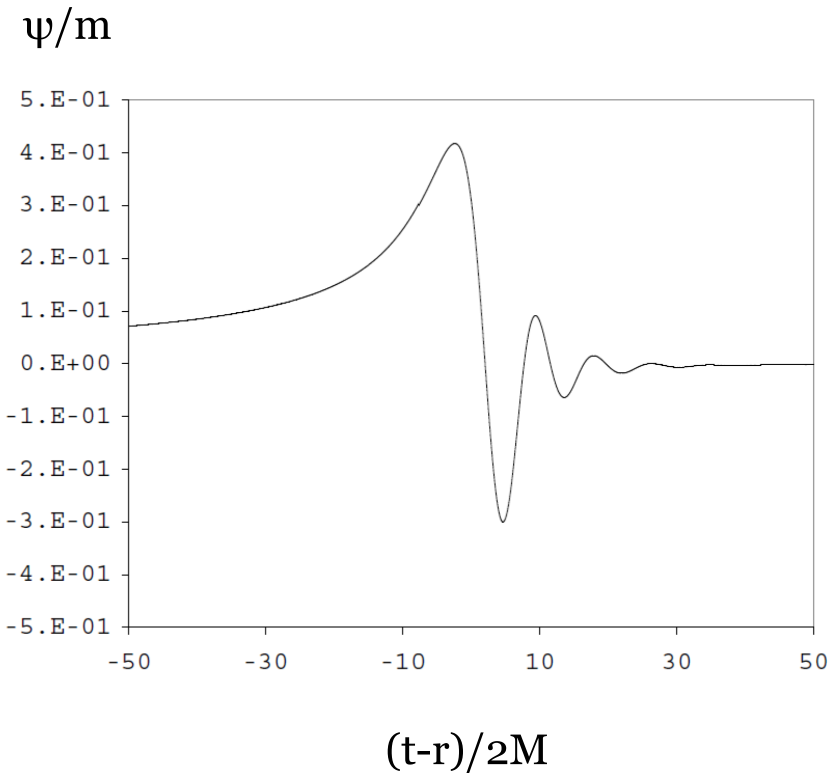
Point Particles: mergers



Hadar, Kol, Berti, Cardoso, PRD84: 047501 (2011)
Data and routines at blackholes.ist.utl.pt

End of part I

Point Particles: head-ons



Davis, Ruffini, Press, Price, PRL27: 1466 (1971)
Berti et al, PRD81:104048 (2010)
Data and routines at blackholes.ist.utl.pt

Configuration		Radiated energy (M is total ADM mass of spacetime)	NR
Head-on	Point particle	$E_{\rm rad}=0.010\frac{\mu^2}{M}$	$E_{\rm rad}=0.010\frac{\mu^2}{M}$
	Equal mass	$E_{\rm rad}=0.00065M$	$E_{\rm rad}=0.00057M$
Q-Circular	Point particle	$E_{\rm rad}=0.057\mu$	Not done
	Equal mass	$E_{\rm rad}=0.014M$	$E_{\rm rad}=0.048M$

"*the agreement is so remarkable that something deep must be at work*"

(Larry Smarr)

Berti et al, PRD76: 064034 (2007)
Sperhake et al, PRD84: 084038 (2011)
Hemberger et al, PRD88:064014 (2013)

Point Particles: finite-mass effects

Can, in principle, use energy balance arguments to determine corrections to orbit and construct approximate waveforms

S. Babak et al, PRD75: 024005 (2007)

Actual motion will deviate from this.
Compute conservative self-force effects.

Poisson, Pound, Vega, Living Reviews in Relativity 14:7 (2007)