Overview

Part III:

Spectral covers for F-theory

- Tate model
- ALE fibrations
- spectral covers as a local description of geometry
- gauge flux in spectral cover picture

  [Donagi,Wijnholt 0802.2969, 0904.1218]

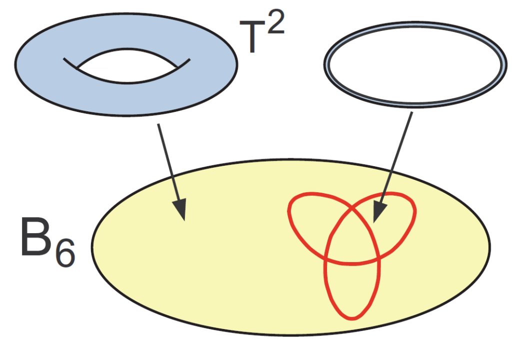
  [Hayashi,Tatar,Toda,Watari,Yamazaki 0805.1057]

  [Hayashi,Kawano,Tatar,Watari 0901.4941]
- global validity of spectral cover?

  [Blumenhagen,Grimm,Jurke,TW '09]; [Grimm,Krause,TW '09]

Summary so far

Gauge dynamics described by singular fibers in ell. fibration $Y \to B$:

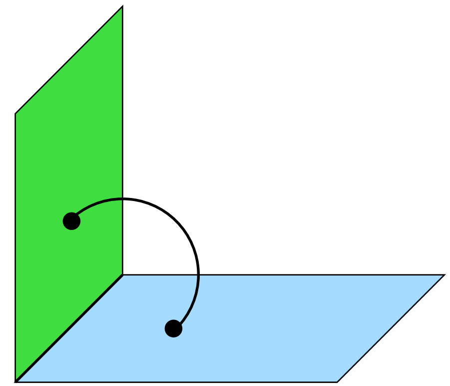


- ADE gauge group G along 4-cycle $S \subset B$

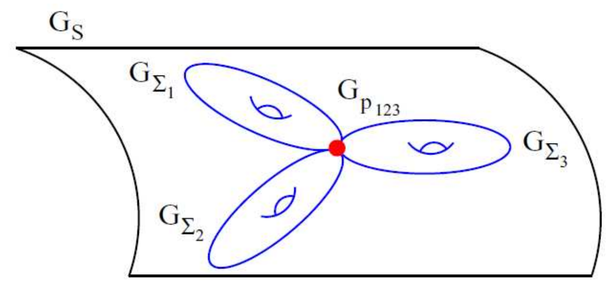
- intersection $S_a \cap S_b$ along 2-cycle C_{ab}: extra massless matter

  enhancement to G_{ab}:

$$ad(G_{ab}) \quad \to \quad ad(G_a) + ad(G_b) + \sum_i (U_i, R_i)$$



- intersection of three matter curves: Yukawa couplings



Tate model

Recall: elliptic fibration $Y : T^2 \to B$ given by Weierstrass model

$P : y^2 - x^3 - f(u_i)\, x\, z^4 - g(u_i)\, z^6 = 0 \qquad (x, y, z) \simeq (\lambda^2 x, \lambda^3 y, \lambda z)$

f, g vary over base B with coordinates u_i

- x, y, f, g are sections of a line bundle $\mathcal{L}$ on the basis
  non-triviality of $\mathcal{L}$ responsible for non-trivial fibration
- homogeneity of P: $x \leftrightarrow \mathcal{L}^2, \quad y \leftrightarrow \mathcal{L}^3, \quad f \leftrightarrow \mathcal{L}^4, \quad g \leftrightarrow \mathcal{L}^6$
- discriminant: $\Delta = 4f^3 + 27g^2 \leftrightarrow \mathcal{L}^{12}$

$\mathcal{L}$ determined by Calabi-Yau condition for Y: $\sum_i a_i \delta(\Gamma_i) = 12\, c_1(T_B)$

$\delta(\Gamma_i)$ vanishing locus of Δ with multiplicity a_i
$\sum_i a_i \delta(\Gamma_i)$ is in same class as Δ (because it gives full degenerate locus)

$$\Longrightarrow \mathcal{L} = K_B^{-1} \qquad f \leftrightarrow K_B^{-4} \qquad g \leftrightarrow K_B^{-6}$$

Tate model

More convenient: **Tate form**

$$P_{\mathrm{W}} = x^3 - y^2 + x\,y\,z\,a_1 + x^2\,z^2\,a_2 + y\,z^3\,a_3 + x\,z^4\,a_4 + \;z^6\,a_6 \;\; = 0$$

Now: $a_n \in H^0(B, K_B^{-n})$

equivalence with Weierstrass model by completing square and cubic

brane position read off from a_i:

introduce the new sections

$$\beta_2 = a_1^2 + 4a_2, \qquad \beta_4 = a_1 a_3 + 2\,a_4, \qquad \beta_6 = a_3^2 + 4a_6$$
$$f = -\frac{1}{48}(\beta_2^2 - 24\,\beta_4), \qquad g = -\frac{1}{864}(-\beta_2^3 + 36\beta_2 b_4 - 216\,\beta_6)$$
$$\Rightarrow \Delta = -\tfrac{1}{4}\beta_2^2(\beta_2\beta_6 - \beta_4^2) - 8\beta_4^3 - 27\beta_6^2 + 9\beta_2\beta_4\beta_6$$

precise characterisation of singularities and resulting gauge groups in terms of Tate algorithm

[Bershadsky,Intriligator,Kachru,Morrison,Sadov,Vafa 9605200]

Tate algorithm

sing. type	discr. $\deg(\Delta)$	group enhancement	coefficient vanishing degrees a_1	a_2	a_3	a_4	a_6
I_0	0	—	0	0	0	0	0
I_1	1	—	0	0	1	1	1
I_2	2	$SU(2)$	0	0	1	1	2
I_3^{ns}	3	[unconv.]	0	0	2	2	3
I_3^{s}	3	[unconv.]	0	1	1	2	3
I_{2k}^{ns}	$2k$	$SP(2k)$	0	0	k	k	$2k$
I_{2k}^{s}	$2k$	$SU(2k)$	0	1	k	k	$2k$
I_{2k+1}^{ns}	$2k+1$	[unconv.]	0	0	$k+1$	$k+1$	$2k+1$
I_{2k+1}^{s}	$2k+1$	$SU(2k+1)$	0	1	k	$k+1$	$2k+1$
II	2	—	1	1	1	1	1
III	3	$SU(2)$	1	1	1	1	2
IV^{ns}	4	[unconv.]	1	1	1	2	2
IV^{s}	4	$SU(3)$	1	1	1	2	3
$I_0^{*\,ns}$	6	G_2	1	1	2	2	3

stolen from: [Bershadsky,Intriligator,Kachru,Morrison,Sadov,Vafa 9605200]

Tate algorithm

sing. type	discr. $\deg(\Delta)$	group enhancement	coefficient vanishing degrees a_1	a_2	a_3	a_4	a_6
$I_0^{*\,ss}$	6	$SO(7)$	1	1	2	2	4
$I_0^{*\,s}$	6	$SO(8)^*$	1	1	2	2	4
$I_1^{*\,ns}$	7	$SO(9)$	1	1	2	3	4
$I_1^{*\,s}$	7	$SO(10)$	1	1	2	3	5
$I_2^{*\,ns}$	8	$SO(11)$	1	1	3	3	5
$I_2^{*\,s}$	8	$SO(12)^*$	1	1	3	3	5
$I_{2k-3}^{*\,ns}$	$2k+3$	$SO(4k+1)$	1	1	k	$k+1$	$2k$
$I_{2k-3}^{*\,s}$	$2k+3$	$SO(4k+2)$	1	1	k	$k+1$	$2k+1$
$I_{2k-2}^{*\,ns}$	$2k+4$	$SO(4k+3)$	1	1	$k+1$	$k+1$	$2k+1$
$I_{2k-2}^{*\,s}$	$2k+4$	$SO(4k+4)^*$	1	1	$k+1$	$k+1$	$2k+1$
$IV^{*\,ns}$	8	F_4	1	2	2	3	4
$IV^{*\,s}$	8	E_6	1	2	2	3	5
III^*	9	E_7	1	2	3	3	5
II^*	10	E_8	1	2	3	4	5

stolen from: [Bershadsky,Intriligator,Kachru,Morrison,Sadov,Vafa 9605200]

Tate model: Cautionary note

Existence of desired elliptic Calabi-Yau fourfold Y not guaranteed for every base B

Example:

- Let B be Kähler and $K_B^{-1} \geq 0$ for every holomorphic curve. Then B is called Fano.
- If B is Fano and, more strongly, $K_B^{-1} > 0$, then existence of a Weierstrass model with only I_1 singular fibers is guaranteed.

In general we want non-abelian singularities for model building

A good way to keep control over singular fourfold Y is by direction construction as a hypersurface/complete intersection in toric ambient space

`For concrete examples see: [Blumenhagen,Grimm,Jurke,TW '09]; [Grimm,Krause,TW '09]`

SU(5) GUT from Tate

Engineering of concrete models by suitable choice of a_i

Example of phenomenological relevance: SU(5) GUT gauge group along divisor $S : w = 0$

split base coordinates $u_i = (w, y_i)$ and read off from Tate:

$$a_1 = \mathfrak{b}_5, \quad a_2 = \mathfrak{b}_4 w, \quad a_3 = \mathfrak{b}_3 w^2, \quad a_4 = \mathfrak{b}_2 w^3, \quad a_6 = \mathfrak{b}_0 w^5,$$

$\mathfrak{b}_i = \mathfrak{b}_i(w, y_i)$ but no overall factor of w

$$y^2 = x^3 + \mathfrak{b}_0 w^5 + \mathfrak{b}_2 x w^3 + \mathfrak{b}_3 y w^2 + \mathfrak{b}_4 x^2 w + \mathfrak{b}_5 xy$$

$$\Delta = -\underbrace{w^5}_{5[S]} \underbrace{\left(\mathfrak{b}_5^4 P + w\mathfrak{b}_5^2(8\mathfrak{b}_4 P + \mathfrak{b}_5 R) + w^2(16\mathfrak{b}_3^2\mathfrak{b}_4^2 + \mathfrak{b}_5 Q) + \mathcal{O}(w^3)\right)}_{[D_1]},$$

$$P = \mathfrak{b}_3^2\mathfrak{b}_4 - \mathfrak{b}_2\mathfrak{b}_3\mathfrak{b}_5 + \mathfrak{b}_0\mathfrak{b}_5^2, \qquad R = 4\mathfrak{b}_0\mathfrak{b}_4\mathfrak{b}_5 - \mathfrak{b}_3^3 - \mathfrak{b}_2^2\mathfrak{b}_5$$

$[\Delta] = 5[S] + [D_1]$

Generically: D_1 does not factorise further $\rightarrow$ I_1 locus

extra singular locus in addition to $SU(5)$ unavoidable in global models

SU(5) GUT from Tate

Matter curves are read off as follows:

- matter in **10**: Enhancement $SU(5) \rightarrow SO(10)$:

  $\mathbf{45} \rightarrow \mathbf{24} + 1 + \mathbf{10} + \overline{\mathbf{10}}$

  Tate: $\Delta \simeq w^7$, $\quad (a_1, a_2, a_3, a_4, a_6) = (1, 1, 2, 3, 5)$

  $$\rightarrow P_{10} : \quad w = 0 \quad \cap \quad \mathfrak{b}_5 = 0$$

  Note: fits with orientifold picture, where **10** localises on O-plane

- matter in **5**: Enhancement $SU(5) \rightarrow SU(6)$:

  $\mathbf{35} \rightarrow \mathbf{24} + 1 + \mathbf{5} + \overline{\mathbf{5}}$

  Tate: $\Delta \simeq w^6$, $\quad a_1 \simeq w^0$

  $$P_5 : \quad w = 0 \quad \cap \quad P = \mathfrak{b}_3^2 \mathfrak{b}_4 - \mathfrak{b}_2 \mathfrak{b}_3 \mathfrak{b}_5 + \mathfrak{b}_0 \mathfrak{b}_5^2 = 0$$

In phenomenological SU(5) GUT models:

$\mathbf{10} \leftrightarrow (Q_L, u_R^c, e_R^c)$

$\overline{\mathbf{5}}_m \leftrightarrow (d_R^c, L) \qquad \mathbf{5}_H \leftrightarrow (T_u, H_u),\ \overline{\mathbf{5}}_H \leftrightarrow (T_d, H_d)$

SU(5) GUT from Tate

Yukawa couplings from further enhancements at points:

- $\langle \mathbf{10}\,\overline{\mathbf{5}}\,\overline{\mathbf{5}} \rangle$ from enhancement $SU(5) \rightarrow SO(12)$:
  $\langle \mathbf{10}\,\overline{\mathbf{5}}\,\overline{\mathbf{5}} \rangle \subset \langle (\mathbf{66})^3 \rangle$ of SO(12)
  D_6 point: $w = 0 \cap \mathfrak{b}_5 = 0 \cap \mathfrak{b}_3 = 0$
  possible also perturbatively in IIB theory
- $\langle \mathbf{10}\,\mathbf{10}\,\mathbf{5} \rangle$ from enhancement $SU(5) \rightarrow E_6$:
  $\langle \mathbf{10}\,\mathbf{10}\,\mathbf{5} \rangle \subset \langle (\mathbf{78})^3 \rangle$ of E_6
  E_6 point: $w = 0 \cap \mathfrak{b}_5 = 0 \cap \mathfrak{b}_4 = 0$
  $\Longrightarrow$ **requires exceptional enhancements**
  - not possible perturbatively in IIB theory
    $\rightsquigarrow$**genuine F-theory effect**
  - D-brane instantons do generate coupling in principle also in IIB

    For review see: Blumenhagen,Cvetič,Kachru,Weigand '09

    But different effect: E_6 enhancement is not the F-theory version of the IIB instanton effect!

Local picture: ALE fibrations

Structure of matter curves and Yukawas for states charged under SU(5) determined only by local neighbourhood of S within Y

Locally can think of B as total space of normal bundle $N_{S/B} \to S$

(more precisely: canonical bundle - see later)

- view normal coordinate w as parametrizing one copy of $\mathbb{C}$
- T^2 fiber combines locally with w into local K_3 fibration over S

Globally this is not the full story since B is in general not fibered over S!

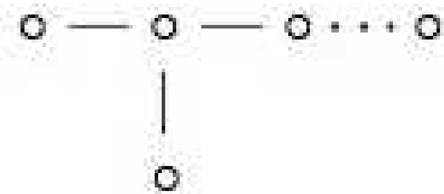
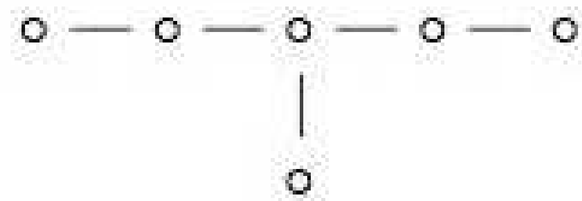
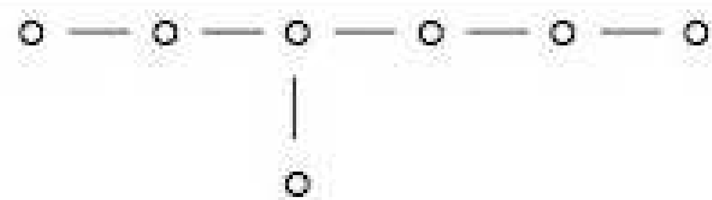
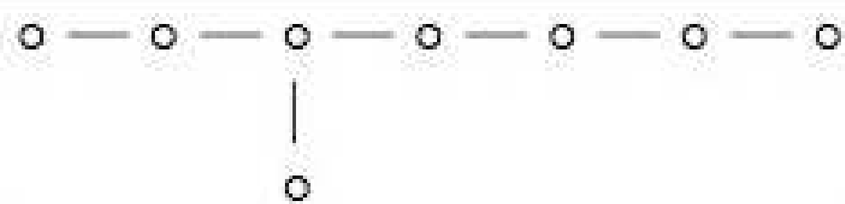
Locally replace Y by $K_3 \to S$

singular fiber over S can be understood in terms of ALE fibration over S: describe K_3 fiber as hypersurface of $\mathbb{C}^3$ spanned by (x, y, w) with ADE singularity at origin

$$A_n : y^2 = x^2 + w^{n+1} \quad D_n : y^2 = x^2 z + w^{n-1} \quad E_8 : y^2 = x^3 + w^5$$

(x, y, w) sections of appropriate powers of K_S

Local picture: ALE fibrations

Name	Equation	Resolution graph
A_n	$x^2 + y^2 + z^{n+1}$	
D_n	$x^2 + y^2 z + z^{n-1}$	
E_6	$x^2 + y^3 + z^4$	
E_7	$x^2 + y^3 + yz^3$	
E_8	$x^2 + y^3 + z^5$	

Local picture: ALE fibrations

Relation to Tate model (e.g. for SU(5)):

$$y^2 = x^3 + \mathfrak{b}_0 w^5 + \mathfrak{b}_2 x w^3 + \mathfrak{b}_3 y w^2 + \mathfrak{b}_4 x^2 w + \mathfrak{b}_5 x y, \qquad \mathfrak{b}_i = \mathfrak{b}_i(w, y_i)$$

restriction to neighbourhood of S by truncation $\mathfrak{b}_i \to b_i = \mathfrak{b}_i|_{w=0}$

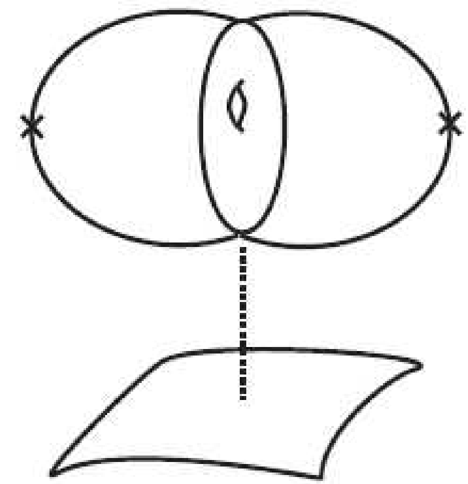
$$y^2 = x^3 + b_0 w^5 + b_2 x w^3 + b_3 y w^2 + b_4 x^2 w + b_5 x y$$

One can show:

This is a deformation of an E_8 singularity $y^2 = x^3 + w^5$ down to A_4

Local picture: ALE fibrations

Stable degeneration limit:



pic stolen from Donagi/Wijnholt 0802.2969

- view $K_3 = dP_9 \cup dP_9$ glued over common T^2
- focus on one of the two dP_9
- blow down: $dP_9 \to dP_8$

$dP_8 = \mathbb{P}^2$ with 8 points blown up to $\mathbb{P}^1$ E_i

$\Longrightarrow H_2(dP_8) = \langle l, E_i \rangle, \quad i = 1, \ldots 8 \qquad l^2 = 1, \quad E_i \cdot E_j = -\delta_{ij}$

Dynkin diagram of $E_8 \leftrightarrow l - E_1 - E_2 - E_3$, $E_i - E_j$

$\Rightarrow$ maximal singularity supported by dP_8 lattice is E_8

Only if model has heterotic dual is the 4-fold Y **globally** K_3 fibered over S

Local picture: ALE fibrations

dP_8 picture makes manifest that ALE fiber over S contains small $\mathbb{P}^1$ intersecting like the roots in E_8-Dynkin diagram:

$$H_2(\mathrm{ALE}, \mathbb{Z}) = \langle \alpha_I^{E_8} \rangle$$

- all $\alpha_I^{E_8}$ zero size $\Rightarrow$ maximal enhancement E_8 over S
- non-zero volume for subset α_i^H with Dynkin diagram $H \subset E_8$
  $\Rightarrow$ gauge group $G = E_8/H$ on S

To specify gauge group G:

Analyse root system of commutant $H = E_8/G$

Local picture: ALE fibrations

Field theoretic interpretation via 8D $\mathcal{N}=1$ SYM gauge theory on S

bosonic degrees of freedom in adjoint representation of E_8 on S:

- gauge field $A^{1,0}$

  Cartan $U(1) \subset E_8 \leftrightarrow$ M-theory origin: $C_3 = A_I^{1,0} \wedge \omega_I$ $(\int_{\alpha_I} \omega_J = \delta_{IJ})$

- Higgs field $=$ complex scalar field

  deformation modes of S

  due to particular twisting of SYM theory on S:

  $\Phi \in H^0(S, K_S \otimes \mathrm{ad}(E_8))$

  Cartan part $\leftrightarrow$ M-theory origin: $\delta\Omega^{4,0} = \Phi_I^{2,0} \wedge \omega_I$

geometric Higgsing of $E_8 \to G = E_8/H$ by deformation of ALE fiber

$\leftrightarrow$ non-zero VEV to H-valued components of Φ

ALE fibrations: non-zero VEV only for Cartan U(1) part of Higgs field Φ

$\rightsquigarrow$ This (un)Higgsing can happen even in absence of VEV for gauge flux

$\rightsquigarrow$ Analogy in brane language: displacement of coincident branes

Spectral covers

Idea of **spectral covers**:
describe local system by eigenvalues μ_i of Φ

Example: $G = SU(5) \rightarrow H = SU(5)_\perp \qquad \Phi \leftrightarrow ad(SU(5)_\perp)$

- μ_i are the roots $i = 1, \ldots, 5$ of $\det(s\, 1_5 - \Phi) = 0$
- Since Φ varies over S, so do the μ_i
- This is a degree 5 equation:

$$0 = s^5 + e_1 s^4 + e_2 s^3 + e_3 s^2 + e_4 s + e_5 = \prod_i (s - \mu_i) \qquad (*)$$

- $e_n(\mu_i) = \mathrm{Tr}(\Phi^n)$: symmetric polynomial of degree n
- In particular: $e_1 = \sum_i \mu_i = Tr(\phi) \equiv 0$ for $H = SU(5)$
- More generally: Higgs field might exhibit poles
  multiply by b_0 to remove these poles:

$$\boxed{0 = b_0 s^5 + b_1 s^4 + b_2 s^3 + b_3 s^2 + b_4 s + b_5, \qquad e_n = \frac{b_n}{b_0}}$$

Spectral covers

Geometric interpretation of

$$0 = b_0 s^5 + b_1 s^4 + b_2 s^3 + b_3 s^2 + b_4 s + b_5 = b_0 \prod_i (s - \mu_i) \qquad (*)$$

- since Φ is valued in bundle $K_S \to S$: degree 5 equation in $K_S \to S$
- $s = 0$ is location of S in total space $K_S \to S$
- $(*)$ is a degree 5 cover of S in K_S called the spectral cover $\mathcal{C}$
- μ_i are the 5 intersection points of $\mathcal{C}$ with fiber of K_S
  over each p in S we have 5 points in fiber over S: (p, μ_i) on $\mathcal{C} \to S$

Significance for model building:

The intersection of $\mathcal{C}$ and S determines the matter curves

Spectral covers

$\langle\phi\rangle$ makes massive components of 248 of E_8 other than $\mathrm{ad}(G)$ in:
$248 \to (\mathrm{ad}(G), 1) + (1, \mathrm{ad}(H)) + \sum_i (U_i, R_i)$

origin: interaction $[\langle\phi\rangle, \delta\phi]^2$ $\leftrightarrow$ mass terms of U_i given by weights $\lambda_i(R_i)$

$$\text{Example}: \quad \mathbf{248} \mapsto (\mathbf{24}, 1) + (1, \mathbf{24}) + [(\mathbf{10}, \mathbf{5}) + (\overline{\mathbf{5}}, \mathbf{10}) + h.c.]$$

- Generically on S: only $(\mathbf{24}, 1)$ massless, all others massive
- mass for $(\mathbf{10}, \mathbf{5})$ $\leftrightarrow \lambda_i$ weights for fundamental $\mathbf{5}$ of H
- Relation to roots:

$$\lambda_1 = \alpha_4, \quad \lambda_2 = \alpha_3 + \alpha_4, \quad \lambda_3 = \alpha_2 + \alpha_3 + \alpha_4,$$
$$\lambda_4 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \qquad \lambda_5 = \alpha_{-\theta} + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$$

- if $\lambda_i = 0$ for some $i = 1, \ldots 5$: extra $\mathbb{P}^1$ shrinks
  $\longrightarrow$ extra massless $\mathbf{10}_i$ from $(\mathbf{10}, \mathbf{5}) \subset E_8$
- location of full $\mathbf{10}$ matter curve on S: $\prod_i \lambda_i = 0$

Spectral covers

Relation to Tate picture

$y^2 = x^3 + b_0 w^5 + b_2 x w^3 + b_3 y w^2 + b_4 x^2 w + b_5 xy$:

$SO(10)$ enhancement along $b_5 = 0 \Leftrightarrow$ Identify $b_5 \simeq \prod_i \lambda_i$

Tate algorithm correctly reproduced if one identifies

$$b_n \simeq b_0 e_n(\mu_i)$$

In particular: $b_1 = \sum_i \lambda_i \equiv 0$ for $SU(n)$

Non-trivial check: $\mathbf{5}$ curve $\leftrightarrow$ weights $\lambda_i + \lambda_j$ for $\mathbf{10}$ of $SU(5)_\perp$

Full 5 curve P_5: $\prod_{i<j}(\lambda_i + \lambda_j) = 0$
together with $b_1 = 0 \Rightarrow b_3^2 b_4 - b_2 b_3 b_5 + b_0 b_5^2 = 0$ ✓

fundamental weights λ_i = eigenvalues μ_i of Higgs field

Note: For gauge groups other than $SU(n)$ Casimirs e_n replaced by corresponding group theoretic invariants

Spectral covers

Back to spectral cover: for SU(5) $\mathcal{C}^{(5)}$ = 5-fold cover of S in $K_S \to S$

$$0 = b_0 s^5 + b_2 s^3 + b_3 s^2 + b_4 s + b_5$$

Technically easier to deal with compact space instead of $K_S \to S$
Define auxiliary non-CY 3-fold X as compactification of $K_S \to S$

$$X = \mathbb{P}(\mathcal{O}_S \oplus K_S), \qquad p_X : X \to S$$

- S is the vanishing locus of the section $s = 0$ in X of class σ in X
  find: $\sigma|_S = -c_1(S)$
- defining data b_i are sections entirely on S
  $b_0 \in H^0(S, \eta) \to b_j \in H^0\big(S; \mathcal{O}(\eta - jc_1(S))\big)$
- coupling to Tate model:
  $y^2 = x^3 + b_0 s^5 + b_2 x s^3 + b_3 y s^2 + b_4 x^2 s + b_5 xy$:
  with x section of K_B^{-2} and $K_B|_S = K_S \otimes N_{S/B}^{-1}$

$$\eta = 6c_1(S) - t, \qquad -t = c_1(N_{S/B}), \qquad b_j \in H^0\big(S; \mathcal{O}(\eta - jc_1(S))\big)$$

Spectral covers: Matter curves

Finally:

$\mathcal{C}^{(5)} \subset X$ is 5-fold cover of S with projection $\pi_5 : \mathcal{C}^{(5)} \to S$

$$[\mathcal{C}^{(5)}] = 5\sigma + \pi_5^* \eta$$

Matter curves encoded in spectral cover:

- $\mathcal{C}^{(5)} \leftrightarrow$ weights λ_i of fundamental of $SU(5) \leftrightarrow$ matter $\mathbf{10}$
  define $\mathbf{10}$ curve in X: $\mathcal{P}_{10} = \mathcal{C}^{(5)} \cap \sigma \subset X$
  $\to$ actual matter curve on S is related to $\mathcal{P}_{10}$ as

$$[P_{10}] = [\mathcal{P}_{10}]|_\sigma = (5\sigma + \pi_5^* \eta)|_\sigma = \eta - 5c_1(S)$$

- for $\overline{\mathbf{5}}$ define spectral cover $\mathcal{C}_{\bigwedge^2 V}$ associated with $\mathbf{10}$
  matter curve for the $\overline{\mathbf{5}}$ on X: $P_5 = c_{\bigwedge^2 V} = \sigma \cap \mathcal{C}_{\bigwedge^2 V}$
  P_5 is singular $\to$ resolution of this curve
  details more messy....

Spectral covers: Flux

Actual power of spectral cover: **description of gauge flux G**

2 types of flux:

- F_2 on S with values in $G = SU(5) \leftrightarrow G_4 \simeq F_2 \wedge \omega_i^G$,
  $F_2 \in H^{(1,1)}(S, \mathbb{Q}) \Rightarrow$ breaks gauge group on S
- F_2 on extra branes constituting I_1 locus
  leaves G on S unaffected and allows for chiral matter

Locally, the I_1 locus is described by spectral cover $\mathcal{C}^{(5)}$

$\Rightarrow$ describe flux on I_1 by line bundle $\mathcal{N}$ on $\mathcal{C}^{(5)}$ with $c_1(\mathcal{N}) \in H^{(1,1)}(\mathcal{C}^{(5)}, \mathbb{Z})$

- Within X, fibration structure of $\mathcal{C}^{(5)} \to S$ with fiber $f = \{\lambda_i\}$ implies:
  $$H^2(\mathcal{C}^{(5)}, \mathbb{Z}) = H^2(S, H^0(f))$$
- on corresponding ALE fibration: $H^0(f) \to \langle \omega_i^H \rangle$

therefore: (1,1)-form on $\mathcal{C}^{(5)} \leftrightarrow G_4 = F_2 \wedge \omega_i^H \qquad F_2 \in H^{(1,1)}(S, \mathbb{Z})$

Spectral covers: Flux

Why is I_1 flux described as $G_4 = F_2 \wedge \omega_i^H$ with $F_2 \in H^{(1,1)}(S,\mathbb{Z})$?

- Spectral cover is local "first order" description of geometry of S and the neighbouring I_1 locus
- 2 ways to break original E_8 symmetry: by gauge flux and by scalar deformations
- E_8 symmetry $\leftrightarrow$ all branes along same locus S spectral cover $\mathcal{C}^{(5)}$ = product of tilting a suitable number of copies of branes on S locally this is the I_1 locus, i.e. the remaining branes in the system
- flux on I_1 at same order of deformation as gauge flux in representation of H, but with 2 legs on S 2 legs along I_1 and in H would be "second" order

at least for all purposes of chiral matter on curves entirely S correct description of I_1-flux

Spectral covers: Flux

- line bundle $\mathcal{N}$ on $\mathcal{C}^{(n)} \equiv C \leftrightarrow$ rank n vector bundle $V = \pi_{n*}\mathcal{N}$ on S
- structure group $H = E_8/G$

$\mathrm{ch}(V) \leftrightarrow \mathrm{ch}(\mathcal{N})$ via Grothendieck-Riemann-Roch theorem:

$$\pi_* \left(\mathrm{ch}(\mathcal{N})\mathrm{Td}(C)\right) = \mathrm{ch}(\pi_*\mathcal{N})\mathrm{Td}(S)$$

From $\mathrm{Td} = 1 + \frac{1}{2}c_1 + \ldots$, $\qquad \pi_*\pi^*\omega = n\omega$ for $\omega \subset H^2(S,\mathbb{Q})$

$$\zeta \equiv c_1(V) = -\frac{n}{2}c_1(S) + \frac{1}{2}\pi_* c_1(C) + \pi_* c_1(\mathcal{N})$$

$$c_1(\mathcal{N}) = \frac{r}{2} + \gamma$$

$$r = -\,c_1(C) + \pi_n^* c_1(S), \qquad \gamma = \frac{1}{n}\,\pi_n^*\zeta + \gamma_u, \qquad \pi_{n*}\gamma_u = 0$$

most general form of γ_u that is generically in $H^{(1,1)}(C)$:

$\gamma_u = \lambda\left(n\sigma - \pi_n^*\eta + n\pi_n^* c_1(S)\right)$, $\qquad \lambda \in \mathbb{Q}$

$c_1(\mathcal{N}) \in H^{1,1}(C,\mathbb{Z})$ gives constraints on quantisation of λ, n, η

Spectral covers

Example $G = SU(5)$:

for generic spectral cover $H = SU(5) \Longrightarrow c_1(V) = 0$

more generally:

H can contain abelian factors

- $\mathcal{C}^{(5)}$ splits into 2 or more connected components
- In Tate model: I_1 splits - at least in neighbourhood of S!

Example:

$H = S[U(4) \times U(1)] \leftrightarrow$ split spectral cover $C_5 = C_4 \cup C_1$

$\Longrightarrow$ D-term SUSY condition $\int_S J \wedge c_1(V) = 0$
(in absence of VEVs for charged fields)

Will become relevant for model building:

1) More flexibility for flux solutions

2) resulting U(1) may forbid unwanted couplings

Spectral covers and chiral matter

Matter curve $\mathcal{P}_{10} = \sigma \cap \mathcal{C}^{(5)}$ for $\mathbf{10}$ is a curve in auxiliary 3-fold X

view this as system of intersecting B-type branes:

- S with trivial bundle $\mathcal{O}$ $\cap$ $\mathcal{C}^{(5)}$ with line bundle $\mathcal{N}$
- number of zero modes given by cohomology of twisted Dirac operator

chiral multiplets in $\mathbf{10}$ counted by

$$H^{(p-1)}(\mathcal{C}^{(5)} \cap \sigma; \mathcal{N} \otimes K_S|_{\mathcal{C}\cap\sigma}),$$

$$p = 1 \leftrightarrow \text{chiral}, \qquad p = 2 \leftrightarrow \text{anti} - \text{chiral}$$

hard to compute in general, but chiral index easy via Hirzebruch:

$$\begin{aligned}\chi_{\mathbf{10}} &= \chi(\mathcal{P}_{10}, \mathcal{N} \otimes K_S|_{\mathcal{P}_{10}}) = \int_{\mathcal{P}_{10}} c_1(\mathcal{N}) + c_1(K_S) + \tfrac{1}{2}c_1(\mathcal{P}_{10})|_{\mathcal{P}_{10}} \\ &= \int_{\mathcal{P}_{10}} \gamma - \tfrac{1}{2}c_1(\mathcal{C}^{(5)}) - \tfrac{1}{2}\pi_5^*(c_1(S)) + \tfrac{1}{2}c_1(\mathcal{P}_{10})|_{\mathcal{P}_{10}} = \int_{\mathcal{P}_{10}} \gamma\end{aligned}$$

$\mathbf{5}$ matter curve more intricate, but chiral index turns out similar:

$$\chi_{\overline{\mathbf{5}}} = \int_{\mathcal{P}_5} \gamma$$

Global aspects

How exact is the spectral cover construction (SCC)?

- A priori designed to capture only local neighbourhood of S within B
- based on a local $\mathbb{P}^1$ fibration over S: expect correct description of matter curves and Yukawas on S

But: at least in certain cases, SCC captures also global information

- For models with heterotic dual, SCC allows us to compute $\frac{\chi}{24}$ exactly
  Method: Comparison of $D3$-tadpole with M_5 tadpole in heterotic M-theory
- This formula turns out correct - under certain assumptions - also in cases without heterotic dual!

Blumenhagen,Jurke,Grimm,T.W.; Grimm,Krause,T.W. '09

In models with heterotic dual: 4-fold $Y \longrightarrow B$ is globally K_3-fibered over S
$\Rightarrow B$ is globally $\mathbb{P}^1$ fibration over $S \equiv B_2$
Heterotic dual: CY_3 Z is T^2 fibered over B_2

χ from SCC

Ingredients of heterotic $E_8 \times E_8$ on elliptic Z:

- VEV to internal curvature $F_{i\bar{j}}$ in subgroup $H_1 \times H_2 \subset E_8 \times E_8$
  $\leftrightarrow$ holomorphic vector bundle $V_1 \times V_2$ with structure group $H_1 \times H_2$
  gauge group: $E_8 \times E_8 \to G_1 \times G_2, \qquad G_i = E_8/H_i$

  3-form $H_3 = dB_2 + \omega_{CS}(F) - \omega_{CS}(F)$: $dH_3 = \frac{\alpha'}{4}(trF^2 - trR^2)$
- M_5 branes along 2-cycle γ with dual 4-form Γ

Tadpole condition: $\boxed{\sum_i N_i[\Gamma_i] = c_2(Z) + ch_2(V_1) + ch_2(V_2)}$

Special case: M_5 wraps T^2 fiber of $Z \to \Gamma$ supported on B_2:

$$\to N_{\rm M5} = \int_{B_2} c_2(Z) + ch_2(V_1) + ch_2(V_2)$$

- $c_2(Z) = 12\sigma c_1(B_2) + 11c_1^2(B_2) + c_2(B_2)$
- $ch_2(V_i)$ depends on concrete bundle

χ from SCC

On elliptic CY Z a class of such bundles can be described via SCC

For structure group $H_i = SU(N_i)$:

- $[C_i] = n_i \sigma + \pi^*(\eta_i)$, line bundle $\mathcal{N}$
- $\int_{B_2} c_2(V_1) = \int_{B_2} \eta_1 \sigma - \frac{1}{24} \chi_{SU(n)} - \frac{1}{2} \int_{B_2} \pi_{n*}(\gamma^2)$

  $\rightsquigarrow \chi_{SU(n)} = \int_S c_1^2(S)(n^3 - n) + 3n\,\eta\big(\eta - n c_1(S)\big) \leftrightarrow$ info of $[C_i]$

  $\rightsquigarrow \int_{B_2} \pi_{n*}(\gamma^2) \leftrightarrow$ information of line bundle $c_1(\mathcal{N})$
- $\Rightarrow \eta_{(1)} = 6c_1(B_2) - t$ $\quad\eta_{(2)} = 6c_1(B_2) + t$ from σ part in N_5

Special case: $E_8^{(2)}$ broken completely via $H_2 = E_8$

$\chi_{E_8}^{(2)} = 120 \int_S \big(3\eta_{(2)}^2 - 27\eta c_1(S) + 62 c_1^2(S)\big)$ $\qquad \int_{B_2} \pi_{n*}(\gamma^2) = 0$

$$N_5 = \int_{B_2} \big(11 c_1^2(B_2) + c_2(B_2)\big) + \frac{1}{24}\Big(\chi_{SU^{(1)}(n)} + \chi_{E_8^{(2)}}\Big) + \frac{1}{2}\int_{B_2} \pi_{n*}(\gamma^2)$$

χ from SCC

Dual F-theory model:

- gauge group G_1 along S $\qquad t = c_1(N_{S/B})$
- M5-brane along $T^2 \longrightarrow D_3$ brane at point on B
- $N_5 = N_3 = \frac{\chi(Y)}{24} - \frac{1}{2}\int_Y G \wedge G$

$$\frac{\chi(Y)}{24} = \int_S \left(11c_1^2(B_2) + c_2(B_2)\right) + \frac{1}{24}\left(\chi_{SU^{(1)}(n)} + \chi_{E_8^{(2)}}\right)$$

Important observation:

for $H_1 = E_8$ no non-ab. gauge group along S

$\rightarrow$ Y is smooth and $\chi(Y)^* = 360\int_B c_1^3(B) + 12\int_B c_1(B)\, c_2(B)$

Eliminate $\int_S \left(11c_1^2(B_2) + c_2(B_2)\right)$ and find:

$$\chi(\mathbf{Y}) = \chi^*(\mathbf{Y}) + \chi_{\mathbf{SU^{(1)}(n)}} - \chi_{\mathbf{E_8^{(1)}}}$$

This formula holds true even in absence of heterotic dual!